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Fractional Unstable Euclidean Universe

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Abstract: Despite common acceptance of Big Bang hypothesis among most cosmologists, nonetheless there are criticisms from a small number of theorists partly supported by astronomy observation suggesting that redshift data could not always be attributed to cosmological expansion. In this paper, a new approach to cosmology "*fractional calculus*" has been developed that we hope will attract attention from astrophysicists and cosmologists because of the way it challenges the conventional big bang framework.

Keywords: Fractional Derivatives and Integrals, Static Universe, Redshift.
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1. Motivations

Testing the universe expansion scenario starts namely with its major assumption that general relativistic processes operate to expand wavelengths only while light (*photons*) are in motion, that is light travels at photon's velocity. As a result, one assumes *expansion-stop* effects during *emission/absorption* in order to insure agreement with the astronomical requirement of a fixed emission wavelength. However, a detailed analysis of the many relativistic gravitational experiments performed over the last few decades reveal conflicts with the cosmic expansion paradigm's basic assumptions, and are completely in accord with the predictions of the static-spacetime which is the first cosmological model developed by Einstein.

Whatever is the situation, it will not be an easy task to talk about a static universe in a research center where all its scientists, in particular, cosmologist, astrophysicists and astronomers, are convinced that the universe is expanding. Let us summarize the

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situation (*we refer to* [**1**] *in this summary*):

- In **1916**, in that seminal paper [**2**] Einstein predicted that gravity should cause a perfect clock to go.
- In **1954**, J. W. Brault confirmed he magnitude of the gravitational redshift predicted by Einstein after performing redshift measurement of the sodium D line emanating from the sun's spectrum without clarifying in details and with clarity its origin [**3**]. Probably Einstein was correct in explaining *that the origin of the gravitational redshift came from the fact that different gravitational potentials at source and observer meant that clocks at these locations should run at intrinsically different rates.* But one may ask differently: *did the measured redshift instead have its origin in photons experiencing an in-flight energy exchange with gravity as they moved in a changing gravitational potential in their transit from a star to the Earth?*
- In **1965**, R.V. Pound and J.L. Sinder performed an experiment predicting a fractional frequency between ^{57}Fe gammas emitted at the top and received at the bottom of a tower of certain height, confirmed precisely the magnitude of the Einstein redshift without justifying its origin [**4**].
- In **1987**, J.H. Taylor compared atomic clock time with pulsar timing data by taking into account the change of local atomic clock time due to the monthly variation in the sun's gravitational potential at Earth [**5**].
- In **1997-98**, a new interpretation of the cosmological Big Bang redshift (**NIR**) had been proposed by R. V. Gentry asserted that its origin might be the gravitational potential rather than the cosmic or space-time expansion [**6**]. After reexamination of general relativistic experimental results, the same author states that *the universe is governed by Einstein's static-spacetime general relativity instead of Friedmann-Lemaitre expanding-spacetime general relativity.* In his conclusion, R. V. Gentry states that the absence of expansion redshifts in his static-spacetime universe model suggests a reevaluation of the present cosmology. In fact, recent reports of cosmological observations strongly suggest the existence of a repulsive force in the outermost reaches of the universe [**7,8**]. An important question is whether these observations may reasonably be interpreted to be a remarkable confirmation of the prediction that ours is a universe dominated by a repulsive force due to vacuum gravity modeled by the Einstein's cosmological constant [**9**]. In another work, R. V. Gentry, show how the **NRI** and a static-spacetime Einstein universe lead to new possibilities for quasar redshifts [**1**]. The latter may be of considerable interest, in particular for those researchers who have long contended that certain quasars provide strong evidence of intrinsic redshifts. The **NRI** framework's can interpret a variety of cosmological observations with Einstein's gravitational and Doppler redshifts, without the addition of any dynamical spacetime and the Cosmological Principle. The end result of R. V. Gentry's model is the fact that the **NRI** framework is an alternate interpretation of the Hubble relation and the 2.7 K Cosmic Blackbody Radiation (**CBR**).

Following these arguments, one can suggest that the big bang theory might not be correct and its foundations must be revisited.

- In **1998-99**, S. Carlip and R. Scranton [**10**], criticized R. V. Gentry **NRI** model, by showing that although supposedly based on general relativity, is inconsistent with Einstein General Relatively, that is with the Einstein field equations; *that it requires delicate fine tuning of initial conditions; that it is highly unstable, both gravitationally and thermodynamically; and that its predictions disagree clearly with observation.* In addition, the authors state that **NRI** starts from unclear theoretical foundations, requiring delicate fine tuning, including a simultaneous specification of the initial velocity of each galaxy in the universe [**10**]. In [**11**], Gentry showed that the **NRI** very definitely encompasses an expanding universe wherein galaxies are undergoing Doppler recession according to the Hubble yielding the correct form of the Hubble magnitude-redshift relation.
- T. Shimizu and K. Watanabe [**12**], give a relativistic description of Gentry's **NRI**. They show that Gentry's idea can be partially applied, in a relativistic manner, to some class of cosmological models, e.g., de Sitter space-time to obtain the distance-redshift relation again. This later is coordinate-independent, gauge invariant [**13**], however, depends on both the observer and sources. In particular, they obtain a new expression of the Friedmann-Robertson-Walker (**FRW**) metric, which is an analogue of a static chart of the de Sitter space-time. Two important functions appeared in their cosmological reduced metric model: the mass function and the gravitational potential. They find that, near the coordinate origin, the reduced metric can be approximated in a static form and that the approximated metric function, satisfies the Poisson equation. Moreover, with suitable model parameters, the approximated metric coincides with exact solutions of the Einstein equation with the perfect fluid matter. By solving the radial geodesics on the approximated space-time, they obtain the distance-redshift relation of geodesic sources observed by the comoving observer at the origin. As a result, the redshift is found to be expressed in terms of a peculiar velocity of the source and the metric function, evaluated at the source position, bringing in mind Gentry's mode, thinking that this is a new interpretation of his **NRI**.

All these important remarks and notes, as well as the general interest in stationary universe represent our motivation in this work.

In what follows, we want to present a pedagogical approach to the fractional equations governing the evolution of a stationary universe, namely the fractional Friedmann Equations (**FFE**). In general, the derivation of standard dynamical equations is intrinsically relativistic. Although in Newtonian theory, the universe must be static, the Friedmann equations can be derived from the simpler Newtonian theory. In fact, it may be considered as puzzling that Newtonian theory and general relativity give the same results. In this paper, a Newtonian derivation of the Friedmann equation is done but where the time derivative is replaced by fractional order.

2. The Importance of Fractional Calculus

In recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. It is well believed today that fractional calculus is a quite irreplaceable mean for description and investigation of classical and quantum complex dynamical system [**14,15,16,17**]. In simple words, the fractional derivatives and integrals describe more accurately the complex physical systems and at the same time, investigate more about simple dynamical systems. Dealing with fractional derivatives is not more complex than with usual differential operators.

Cosmological applications are discussed for the first time in the works of H.C. Rosu [**18,19**]. Our approach is totally different from all those approaches modeling inhomogeneous fractal universe or those using self-similar scaling of density perturbations with scale free initial conditions in a spatially flat universe as a theoretical tool to study structure formation.

As we are interested on a Newtonian derivation of the cosmological dynamical equations, we follow [**20,21,22**] and we consider a sphere of radius σ filled with a pressure fluid of uniform density ρ free-falling under its own gravitational field in an otherwise Euclidean space. The coordinate of any particle of the fluid is written as $\vec{x} = R(t)\,\vec{r}$ where $\vec{r}$ is a (r *is dimensionless*) constant vector referred to as the *commoving coordinate*, t is the time coordinate and R the scale factor (*having the dimension of a length*). The edge of the sphere is also moving as $\sigma(t) = R(t)\,\sigma_0$. Assuming that while sitting on a particle labelled i we are observing another one labelled j then the relative velocity is given by $\vec{v}_{ij} = H\vec{x}_{ij}$ where $H \equiv \dot{R}\big/R$ and $\vec{x}_{ij} \equiv \Delta\vec{r} = \vec{r}_j - \vec{r}_i$, implying the presence of a radial velocity proportional to the particle displacement so that the expansion is isotropic. This does not imply that all the particles are identical [**23**].

To determine the equation for the evolution of the scale factor R, we first compute the gravitational potential energy (E_G) of a particle of mass m by applying Gauss's law, then evaluate its kinetic energy (E_K). Because the gravitational potential on any particle inside the sphere is proportional to the distance x^2 , any other energy (*or potential*) deriving from a potential proportional to x^2 will mimic a gravitational effect. This energy is $E_\Lambda = -m\Lambda c^2 x^2/6$, where Λ is a constant (*Einstein's cosmological constant having the dimension of an inverse squared length*) and "c" being the celerity of light.

The total energy is then given by:

$$E = E_G + E_K + E_\Lambda \tag{1}$$

$$= -\frac{GM(<x)\,m}{x} + \frac{1}{2}m\dot{x}^2 - \frac{m\Lambda c^2}{6}x^2 \tag{2}$$

where G is Newton's constant and $M(<x)$ is the mass within the sphere of radius x given by $M(<x) = 4/3\pi\rho x^3$. Using the fact that the mass within any commoving volume is constant, that is $\rho(t) \propto R^{-3}(t)$ and the conservation of the total energy, using $\vec{x} = R(t)\,\vec{r}$ yields after simple manipulation:

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{3R^2} + \frac{\Lambda c^2}{3} \tag{3}$$

Here $k = -6E/mc^2r^2$ a dimensionless constant (*curvature*). In fact, the last term in equation (3) is equivalent to adding a constant amount of energy to the universe. This does not violate energy conservation. According to the Copernican principle [**9**], equation (3) can be applied to any regions of the universe. It relates and constraints how the scale factor $R(t)$ evolves given the total density, the curvature and the cosmological constant of the universe.

To obtain the acceleration equation or second Friedman equation, we model the matter and radiation of the universe as a perfect fluid. One then can use the first law of thermodynamics $dQ = dU + pdV = TdS$ (T *being the temperature and* S *the entropy*) applied to $U = mc^2 = 4/3\pi R^3\rho c^2$ from which one can easily evaluate the change of U in time dt, that is dU/dt. Assuming $dQ = dS = 0$ (*no bulk heat flow* (*adiabatic*)) and making use of $dV/dt = 4\pi R^2 dR/dt$, the first law gives the fluid equation (*energy conservation law*):

$$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{4}$$

The first term $\dot{\rho}$ inform us how fast density changes (*dilutes*) and the second term is the lost of kinetic energy from fluid into gravitational potential energy.

If we differentiate the first Friedman equation (3) with respect to time and making use of equation (4), we find the acceleration equation (*assuming* Λ *constant*):

$$\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{5}$$

The curvature k cancelled out (*a good feature*), so that we can use this equation regardless the geometry of the universe. In fact, this equation tells us how rate of expansion of the universe changes (*slowing down or speeding up*).

3. Fractional Friedman Dynamical Equations

Let us retreat now the same model but where the derivative operators are replaced with fractional one. There are about two dozens of different definitions for fractional derivatives that are in one way or another adapted to various features of classes of functions for which they are defined. The most comprehensive description of the mathematical aspects of the issue is given in the monograph [**24**]. In general, fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^{\alpha}$ where a and t are the limits of the operation. The continuous integro-differential operator is defined as

$${}_aD_t^{\alpha} = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}, \Re(\alpha) > 0 \\ 1, \Re(\alpha) = 0 \\ \int\limits_a^t (d\tau)^{-\alpha}, \Re(\alpha) < 0 \end{cases} \tag{6}$$

where $\Re(\alpha)$ is the real part of α. The two definitions for the general fractional differintegral are the Grünwald-Letnokov (**GL**) definition and the Riemann-Liouville (**RL**) definition [**25**]. The **GL** is given by

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \sum_{j=0}^{[t-a/n]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t-jh) \tag{7} $$

where $[x]$means the integer part of x.

In what follows, as fractional operators, we use the Riemann-Liouville fractional derivative and integrals defined as [**24,25**]

$$ {}_aD_t^\alpha R(t) \equiv \frac{d^\alpha R(t)}{dt^\alpha} = \frac{d^n}{dt^n} {}_aI_t^{n-\alpha} R(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{R(t')}{(t-t')^{1+\alpha-n}} dt', n-1 \le \alpha < n \tag{8} $$

$$ {}_aI_t^\alpha R(t) = \frac{1}{\Gamma(\alpha)} \int_a^t dt' \frac{R(t')}{(t-t')^{1-\alpha}} \tag{9} $$

$\Gamma(\cdot)$ is the *Gamma* function. An interesting consequence of this definition is the non-vanishing fractional differentiation of a constant:

$$ \frac{d^\alpha R_0}{dt^\alpha} = \frac{R_0 t^{-\alpha}}{\Gamma(1-\alpha)}, 0 < \alpha < 1 \tag{10} $$

where $R_0 = cte$. In general,

$$ \lim_{\alpha\to 1} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \delta(t), t \ge 0, 0 < \alpha < 1 \tag{11} $$

In the physical world, equation (11) presents a substantial problem. While today we are well familiar with the interpretation of the physical world in integer order equations, we do not (*currently*) have a practical understanding of the world in a fractional order. Our mathematical tools go beyond the practical limitations of our understanding. In general, equations (6) and (7) could be written in terms of the left-sided Riemann-Liouville fractional integral of order $0 < \alpha \le 1$:

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau = \int_0^t f(\tau) dg_t(\tau) \tag{12} $$

where

$$ g_t(\tau) = \frac{1}{\Gamma(\alpha+1)} \{t^\alpha - (t-\tau)^\alpha\} \tag{13} $$

with the interesting scaling properties

$$ g_{t_1}(\tau_1) = g_{kt}(k\tau) = k^\alpha g_t(\tau) \tag{14} $$

where $t_1 = kt$ and $\tau_1 = k\tau$.

The basic idea now is to replace the normal derivative with a fractional one, that is $\dot{R} \equiv dR/dt \rightarrow D_t^{\alpha} R \equiv (d^{\alpha}/dt^{\alpha})\, R$ and $\ddot{R} \equiv d/dt\,(dR/dt) \rightarrow D_t^{\alpha}\,(D_t^{\alpha} R)\,, 0 < \alpha < 1$.

The fractional Friedman equations for zero pressure in the presence of the cosmological constant and zero spatial curvature are written as:

$$\left(D_t^{\alpha} R\right)^2 = \left(A_1\,(G\rho)^{\alpha} + B_1\left(\Lambda c^2\right)^{\alpha}\right) R^2 \tag{15}$$

$$D_t^{\alpha}\left(D_t^{\alpha} R\right) = -\left(A_2\,(G\rho)^{\alpha} - B_2\left(\Lambda c^2\right)^{\alpha}\right) R \tag{16}$$

for solutions with positive, negative and zero spatial curvature, ρ is the density, c the velocity of light and $0 < \alpha < 1$, $A_i, B_i, i = 1, 2$ are constants. In principle, $B_1 = B_2$.

In this work, we are interested in the static models, in particular the Einstein one. That is, when $R\,(t) = R_0 = cte$, equations (8), (9),(10), (15) and (16) gives for zero pressure

$$(G\rho)^{\alpha} = \frac{C_1}{t^{2\alpha}} \tag{17}$$

$$\left(\Lambda c^2\right)^{\alpha} = \frac{C_2}{t^{2\alpha}} \tag{18}$$

where $C_i, i = 1, 2$ are positive constants depending on α. It is clear in such model, that the density and the cosmological constant decreases as $1/t^2$, whatever $0 < \alpha < 1$, unless G and c are kept constant [**26,27**]. If the density is assumed not to vary with time, than from equation (17), it is clear that the gravitational constant decreases as $1/t^2$. In contrast, if the cosmological constant is stable with time, then from (18), the velocity of light decreases as $1/t$ (*see for example* [**28-30**]). In fact, recent observations of the cosmological effects of including a time-variation of the velocity of light, into the gravitational field equations have revealed a number of important features. If the speed of light falls sufficiently rapidly over an interval of time then it is possible to solve the **FRW** horizon and flatness problems in a way that differs from the inflationary universe [**31,32**]. By adopting the variation of the speed of light (**VSL**) frame, the redshift $z = (\lambda_0 - \lambda_1)/\lambda_1$ is given by:

$$z = \frac{c\,(t_1)}{c\,(t_0)}\frac{R\,(t_0)}{R\,(t_1)} - 1 \tag{19}$$

where t_1 denotes the time a light wave leaves a galaxy and t_0 denotes the time when it reaches us on earth [**33**]. In our model, $R\,(t_0) = R\,(t_1)$, so that $z = (c\,(t_1)/c\,(t_0)) - 1$. If the velocity of light decreases as $1/t$, then the redshift is $z = t_0/t_1 - 1$. This means that the observed redshift will appear to be larger due to the speed of light in the past being bigger than the presently observed speed, all this in a static universe, reviving again Gentry's **NRI**. In summary, we have developed within the framework of fractional calculus, a homogenous, isotropic, *static,* but *unstable* Euclidean universe with vacuum energy, decreasing light velocity and presence of Redshift. The static Euclidean model describes in this paper have important feature. It differs from the static Einstein elliptic universe where the density and the cosmological constant are constants. In fact, the possibility of variation of the fundamental constants of physics in the static universe was discussed in [**34,35**]. It was shown when the velocity of light increases/decreases, the Planck's constant increases/decreases and mass of bodies decreases/increases. In order to keep the density and Λ constants, the gravitation G and the velocity of light c must decrease with time, as it is clear from equations (17) and (18) [**36,37,38**]. The Hubble constant in our model is defined as $H^{\alpha} \equiv \dot{R}\Big/R$ or $Ht = \left(1/\Gamma\left(1-\alpha\right)\right)^{1/\alpha}$. The deceleration parameter is defined by $q = -1 - \dot{H}\Big/H^2 = -1 + \left(\Gamma\left(1-\alpha\right)\right)^{-1/\alpha} > -1$ in accord with observations. One can fix the value of α from observations (*not an easy task*). This is another important feature in the model, not existing in Einstein static universe. Roughly speaking, the behavior of the deceleration equation for $0 < \alpha < 1$ described in our model is shown in the following figure:

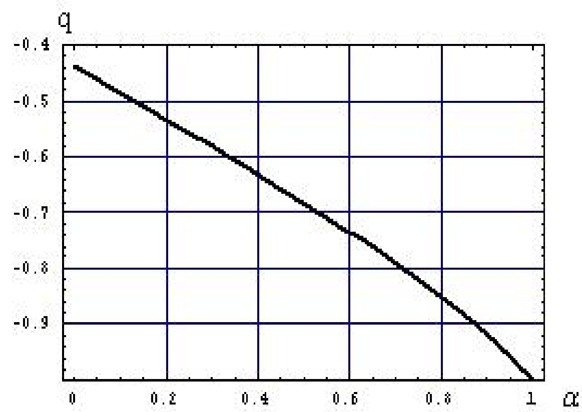

Fig. 1 The behavior of the deceleration parameter for $0 < \alpha < 1$

There exists also another important feature concerning the applications of fractional calculus on cosmology. We also mention it here only to show the reader the benefit of the fractional calculus. The first of the fractional Friedman equation (15) represents a fractional differential equation when the scale radius is considered depending on time, that is not constant.

Applying the fractional integral operator to both left-hand and right-hand sides of equation (3), and solving, one finds:

$$R\left(t\right) = R_0 E_{1,\alpha}\left(Ct^{\alpha}\right) \tag{20}$$

where

$$E_{\alpha,\beta}\left(t\right) = \sum_{k=0}^{n} \frac{t^k}{\Gamma\left(\alpha+\beta k\right)} \tag{21}$$

is so called Mittag-Leffeur function [**21**] and C is a constant depending on G, ρ, c if these later are assumed to be constants with time.

Note that at the origin of time, the universe in this case, have no singularity, that is $R\left(t\right) = R_0$ when $t = 0$. In addition, for $t > 0$, the scale factor increase with time, a characteristic that pleases those who favor an accelerated universe, (*of course if we are not in favor with a static universe*) [**39**].

If $\alpha = 1$, then the solution is given by $R(t) = R_0 e^{Ct}$ corresponding to the standard inflation case [**32**].

4. Conclusions

In conclusion, we have tried in this work to show the reader the importance of fractional calculus and its potential application in cosmology, in particular the Gentry's **NRI** of a static universe. The static universe provides an interesting candidate to explore whether it could play an important role in the evolution of the universe (*on the stability of Einstein static universe, see* [**39,40**]). We have employed a new approach to cosmology "*fractional calculus*" that we hope will certainly attract attention from astrophysicists and cosmologists because of the way it challenges the conventional big bang framework. We would like to prove in a future work definitely that our fractional cosmological differential equations agree completely with redshift data to support static universe model. In addition, some important questions arise:

(1) Does "*homogeneous*" remains a required condition?
(2) Furthermore, fractional differential equations imply fractality, so doesn't it mean that Einstein's cosmological principle cannot be applicable anymore?
(3) How the redshift could be measured in terms of gravitational redshift, instead of "*cosmic*" origin?
(4) How this "*gravitational shift*" arguments could explain "*Hubble constant*" without cosmic expansion, which is indeed the crux of the issue concerning Big Bang [**41**].
(5) If we indeed explain that redshift comes from (*Einstein's*) gravitation redshift instead of cosmic expansion, it does not mean to reject cosmic expansion itself automatically. Could it be that both types of redshift (*gravitational and cosmic expansion*) co-exit or neatly linked?
(6) If solar system expands, then it could be that galaxies are also expanding, albeit not exactly in the same way that Big Bang theory teaches. In this regard, we will explore in the future the possibility of "*expanding solar system*".
(7) What about inflation, cosmic microwave background radiation (**CMBR**) and its temperature dependence, primordial nucleosynthesis [**42,43**] and how the present universe is brought?
(8) Is there any simple variation of the full Einstein field equations with fractional derivatives? Due to the non-locality property of fractional derivatives, one would probably need to replace the Einstein-Hilbert action by some sort of complicated nonlocal expression.
(9) What about the relation of distance and surface brightness [**44**] and the time dilation of supernova light curves [**45**]?

However, further analysis and implications of all these are required in order to make any definite statement.

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Parametric Relationships Among Some Phenomenological Non-Relativistic Hadronic Potentials

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Abstract: In recent years, parametric relationships between interatomic potential energy functions have been developed in the realm of molecular chemistry and condensed matter physics. However, no parametric relationships have been developed so far among intra-atomic potentials. As an extension of previous works into the realm of intra-atomic potentials, we herein consider the possibility that hadronic potentials can be interrelated via their parameters. Hadronic potentials give quantitative description of interquark energy in terms of interquark distance, hence understanding how each potential function influences the theoretical modeling can be sought via knowledge of interrelationship amongst the potentials' parameters. Phenomenological non-relativistic hadronic potentials are related amongst the mixed-powerlaw potential themselves, and with the Logarithmic potentials using calculus. Exact nonlinear relationships were obtained between the parameters whereby the interquark distance is included as one of the variables. It is also demonstrated that, when the interquark distance in the parametric relationships is assigned a fixed value of unity, the parametric relationships remain valid from the plotted potential energy curves.

Keywords: Hadrons, non-relativistic, parametric relationship, potentials.
PACS (2003): 12.39.Jh, 12.39.Pn, 21.30.Fe

1. Introduction

It is well-known that non-relativistic potential models have proven extremely successful for the description of hadrons as bound states of quarks [1], especially if the hamiltonian is interpreted as an effective hamiltonian with the coefficients normalized by the

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relativistic corrections [2]. In this paper, we take interest in the relationship amongst some purely phenomenological potentials and some Quantum Chromodynamics (QCD) motivated phenomenological potentials. The study of QCD is a highly challenging topic in elementary particle physics. As a result of the deficiency of the analytical methods of quantum field theory, numerical methods have been introduced for QCD. Hence the continuum space-time model has been substituted with the 4-D grid points [3]. Interested readers are directed to the recent progress of QCD with the aid of high performance and parallel computing [4,5]. The motivation for establishing relationships between these potentials is to show how these potentials are related to one another since theoretical modeling of interquark potentials is chiefly dependent on the choice of potentials. Relationships between potentials have recently been developed for relating molecular force fields which is of interest to the chemical community [6-14], and for relating many-body interatomic potentials which is of concern to the condensed matter community [15-23], by means of establishing connections between parameters from different interatomic potential energy functions. Unlike the recently completed works, this paper focuses on the "intra-", rather than the "inter-", atomic potentials. It is instructive to note that there are two broad methods to mathematically relate the parameters – (a) the series expansion approach [8-12], and (b) the derivative approach [13-23]. In the foregoing analysis, we adopt the latter approach for hadronic potentials due to its versatility.

2. Analysis

Limiting our scope of phenomenological potentials to the form

$$V(r) = Ar^{a} - Br^{-b} - C \tag{1}$$

where A, B, C, a, b are non-negative constants, we have included two broad categories whereby $a = b$ [24-29] and $a \neq b$ [30-34], as listed in Table 1. For convenience, we shall refer to the potential of the form described by Eq.(1) as the mixed-powerlaw potential. Due to its ease of usage, we also consider the Logarithmic potential

$$V_L = A_L \ln r - B_L, \tag{2}$$

which is also a purely phenomenological potential [35-37] whereby A_L and B_L are non-negative constants. A relationship between the logarithmic potential and the mixed-powerlaw potential can be obtained by equating these potentials and their first derivatives

$$\frac{d^n V_L}{dr^n} = \frac{d^n V}{dr^n} \quad ; \quad (n = 0, 1) \tag{3}$$

which, upon solving simultaneously, gives

$$A_L = aAr^{+a} + bBr^{-b} \tag{4}$$

and

$$B_L = A(a \ln r - 1)r^{+a} + B(b \ln r + 1)r^{-b} + C. \tag{5}$$

Alternatively, taking

$$\frac{d^nV}{dr^n} = \frac{d^nV_L}{dr^n} \quad ; \quad (n = 0, 1, 2) \tag{6}$$

and solving simultaneously yields

$$\left\{ \begin{matrix} A \\ B \end{matrix} \right\} = \frac{A_L}{a+b} \left\{ \begin{matrix} (ar^a/b)^{-1} \\ (ar^b/b)^{+1} \end{matrix} \right\} \tag{7}$$

and

$$C = B_L - \left(\frac{a-b}{ab} + \ln r \right) A_L. \tag{8}$$

As such, Eqs.(4) and (5) give the exact non-constant coefficients of the Logarithmic potential in terms of mixed-powerlaw parameters and the interquark distance, r, whilst Eqs.(7) and (8) express the exact non-constant coefficients of the mixed-powerlaw potential in terms of Logarithmic parameters and the interquark distance. For the special case whereby $a = b \equiv x$ as in references [24-29], Eqs.(7) and (8) simplify to

$$\left\{ \begin{matrix} A \\ B \end{matrix} \right\} = \frac{A_L}{2x} \left\{ \begin{matrix} r^{-x} \\ r^{+x} \end{matrix} \right\} \tag{9}$$

and

$$C = B_L - A_L \ln r \tag{10}$$

respectively. To relate parameters among mixed-powerlaw potentials

$$V_1(r) = A_1 r^{a_1} - B_1 r^{-b_1} - C_1 \tag{11}$$

and

$$V_2(r) = A_2 r^{a_2} - B_2 r^{-b_2} - C_2, \tag{12}$$

we solve the derivatives

$$\frac{d^nV_1}{dr^n} = \frac{d^nV_2}{dr^n} \quad ; \quad (n = 0, 1, 2) \tag{13}$$

simultaneously. This leads to

$$A_1 = \frac{a_2}{a_1} \left(\frac{a_2 + b_1}{a_1 + b_1} \right) A_2 r^{-(a_1 - a_2)} + \frac{b_2}{a_1} \left(\frac{b_1 - b_2}{a_1 + b_1} \right) B_2 r^{-(b_2 + a_1)}, \tag{14}$$

$$B_1 = \frac{b_2}{b_1} \left(\frac{b_2 + a_1}{b_1 + a_1} \right) B_2 r^{+(b_1 - b_2)} + \frac{a_2}{b_1} \left(\frac{a_1 - a_2}{b_1 + a_1} \right) A_2 r^{+(a_2 + b_1)} \tag{15}$$

$$C_1 = \left[\frac{a_2 b_1 (a_2 + b_1) - a_1 a_2 (a_1 - a_2)}{a_1 b_1 (a_1 + b_1)} - 1 \right] A_2 r^{+a_2} - \left[\frac{b_2 a_1 (b_2 + a_1) - b_1 b_2 (b_1 - b_2)}{b_1 a_1 (b_1 + a_1)} - 1 \right] B_2 r^{-b_2} + C_2 \tag{16}$$

Equations (14)-(16) show that parameters of V_1 can be exactly expressed in terms of parameters of V_2 and the interquark distance. For the converted parameters to be constant, there is a need to assign the interquark distance to a fixed value. As a result, potentials related in this manner coincide only at the fixed interquark distance. Selecting $r = 1$, we have the Logarithmic constants in terms of mixed-powerlaw parameters

$$\begin{Bmatrix} A_L \\ B_L \end{Bmatrix} = \begin{bmatrix} a & b & 0 \\ -1 & +1 & +1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix}, \tag{17}$$

and vice versa

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \frac{1}{(a+b)} \begin{bmatrix} (b/a) & 0 \\ (a/b) & 0 \\ (b/a)-(a/b) & (a+b) \end{bmatrix} \begin{Bmatrix} A_L \\ B_L \end{Bmatrix}, \tag{18}$$

whilst parametric relationships among mixed-powerlaw potentials are

$$\begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \left(\frac{1}{a_1+b_1}\right) \begin{bmatrix} \left(\frac{a_2+b_1}{a_1}\right) & \left(\frac{b_1-b_2}{a_1}\right) \\ \left(\frac{a_1-a_2}{b_1}\right) & \left(\frac{b_2+a_1}{b_1}\right) \end{bmatrix} \begin{Bmatrix} a_2 A_2 \\ b_2 B_2 \end{Bmatrix} \tag{19}$$

and

$$C_1 = \left[\frac{a_2 b_1 (a_2+b_1) - a_1 a_2 (a_1-a_2)}{a_1 b_1 (a_1+b_1)} - 1\right] A_2 - \left[\frac{b_2 a_1 (b_2+a_1) - b_1 b_2 (b_1-b_2)}{b_1 a_1 (b_1+a_1)} - 1\right] B_2 + C_2. \tag{20}$$

As such, the correlation is theoretically valid only for r close to 1. For the special case whereby $a = b = x$, Eq.(18) reduces to

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \frac{1}{2x} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2x \end{bmatrix} \begin{Bmatrix} A_L \\ B_L \end{Bmatrix} \tag{21}$$

whilst Eqs.(19) and (20) collapse into

$$\begin{Bmatrix} A_1 \\ B_1 \\ C_1 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{x_2}{x_1}\left(1+\frac{x_2}{x_1}\right) & \frac{1}{2}\frac{x_2}{x_1}\left(1-\frac{x_2}{x_1}\right) & 0 \\ \frac{1}{2}\frac{x_2}{x_1}\left(1-\frac{x_2}{x_1}\right) & \frac{1}{2}\frac{x_2}{x_1}\left(1+\frac{x_2}{x_1}\right) & 0 \\ \left(\frac{x_2^2}{x_1^2}-1\right) & \left(1-\frac{x_2^2}{x_1^2}\right) & 1 \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix}. \tag{22}$$

3. Results and Discussion

While the nonlinear parametric relationships described by Eqs.(4),(5),(7),(8) and (14) to (16) are exact, the assignment of a fixed interquark distance value ($r = 1$) calls for verification to be made to examine the validity of the developed parametric relationship. A relationship between the Logarithmic and the mixed-powerlaw potentials is of interest since the 5-parameter mixed-powerlaw enables better fit while the 2-parameter Logarithmic potential is more convenient as shown in Eq.(17). As such, the precision of mixed-powerlaw can be converted for use in the more easily executable Logarithmic potential. As an illustration, however, we plot the mixed-powerlaw potentials based on Logarithmic parameters $A_L = 0.733\ GeV$ and $B_L = 0.6631\ GeV$ [35] using Eq.(21) so that the influence of parameters $a = b = x$ can be observed, as shown in Fig.1. Fig.2 shows the plots of Cornell potential [24-26] and Turin potential [27], and the plot of Cornell function using Turin parameters using Eq.(22). We observe that within the interquark distance of $0 < r < 2.5$ the simplifying assumption of $r = 1$ is valid. An understanding of the relation amongst mixed-powerlaw potentials would be useful for plotting convenient potentials (such as the Cornell potential whereby $a = b = 1$) using the more elaborate potential parameters (such as $a, b \neq 1$).

4. Conclusion

Two sets of relationship have been developed:
(1) between the Logarithmic and the mixed-powerlaw potentials, and
(2) among the mixed-powerlaw potentials.
This was attained by relating their coefficients. Plotted results reveal good agreement, and hence the validity of the currently developed parametric relationships among the presently considered phenomenological non-relativistic hadronic potentials. The parametric relations developed herein reveals the difference between parameters from each potentials, thereby shedding certain insights on how each potential differ from one another in an analytical manner. More importantly, the parametric relationships enable convenient execution of simpler potentials based on parameters corresponding to the more complicated but more accurate potentials.

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Table 1. Phenomenological non-relativistic potentials of the form

$$V(r) = Ar^a - Br^{-b} - C.$$

Potential	**Indices**	**References**
Cornell	$a = b = 1$	[24-26]
Turin	$a = b = \frac{3}{4}$	[27]
Song	$a = b = \frac{2}{3}$	[28]
Song-Lin	$a = b = \frac{1}{2}$	[29]
Heikkila-Tornquist-Ono	$a = \frac{2}{3}$, $b = 1$	[30]
Martin	$a = \frac{1}{10}$, $b = 0$	[31-33]
Grant-Rosner-Rynes	$a = 0$, $b = 0.045$	[34]

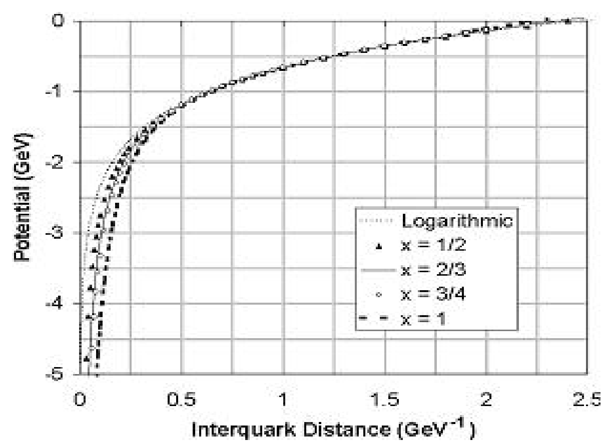

Fig. 1 Plots of mixed-powerlaw potentials using Logarithmic potential parameters [35].

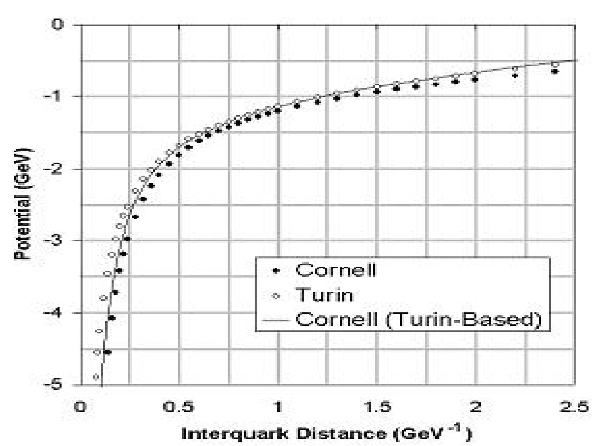

Fig. 2 Plots of Cornell [24-26] and Turin [27] potentials, and the Turin-based Cornell functional.

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Non Linear Assessment of Musical Consonance

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Abstract: The position of intervals and the degree of musical consonance can be objectively explained by temporal series formed by mixing two pure sounds covering an octave. This result is achieved by means of Recurrence Quantification Analysis (RQA) without considering neither overtones nor physiological hypotheses. The obtained prediction of a consonance can be considered a novel solution to Galileo's conjecture on the nature of consonance. It constitutes an objective link between musical performance and listeners' hearing activity.

Keywords: Musical Consonance, Recurrence Quantification Analysis
PACS (2003): 43.25.+y, 43.60.+d, 43.75.+a, 43.66.+y

1. Introduction

According to a current definition, *consonance* is the "intervallic relationships in sound frequencies producing sounds of repose" [1], and a basic problem in the field of musical acoustics concerns the fact that since musical consonance is a graduated criterion, in principle it should be possible: i) to produce an order of merit (ranking) among intervals, and ii) to find an algorithmic derivation of consonance ranking [2].

An excellent example of the consonance-dissonance issue is due to Galileo Galilei [3],

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who imagined two identical pendula, starting to move together. If both of them have the same period, corresponding to the unison interval, the simplest possible event occurs: the two pendula oscillate in perfect synchrony. If the period of one of the pendula is doubled, configuring an octave interval, a momentary event of perfect synchrony will occur periodically. Changing the ratio between the two periods from 2/1 to 3/2, 4/5 or 11/12, the recurrence of the synchronization event becomes less frequent in time, and the inter-period length changes in a complex, nonlinear fashion giving rise to less consonant combinations. This is known as Galileo's "simple-frequency ratio" theory.

Galileo's idea is qualitative and metaphoric. However, by using a rather simple non-linear algorithm, it is possible to show that Galileo's formulation is surprisingly accurate, being able to catch the prominent features of a standard psychological ranking of musical consonance.

Nonlinear mechanisms have been recently found at the root of the cochlear response in mammals and a correct coupling between non-linearities has been proposed [4-5-6].

In this Letter the recurrence (non-linear) approach is used in a "key experiment": acoustic sample containing all possible intervals between two pure sounds in the span of an octave were generated and analyzed. We show that the recurrence structure of these samples is strongly related to the musician's use of the word consonance (psychoacoustic consonance).

2. Methods

2.1 Sound generation.

A pair of pure sinusoidal tones are generated by independent channels and then mixed together by means of the sound editor Cool-Edit Pro (Sintrillium Software Corporation, Phoenix, AZ), see Fig.1(a). The first tone is a constant pure tone at fixed frequency f_1 and the second tone is a *glissando* tone, with a variable, monotonically growing, frequency, f_2. The sum of the two tones accounts for all the intervals in the span of the octave. The signal was generated, with f_1=264 Hz (C4) as root frequency and f_2 going from 200 to 600 Hz, with both f_1 and f_2 being pure sinusoidal waveforms. The corresponding signal was recorded with a sampling rate of 6000 Hz and lasted four seconds (24000 points). Furthermore, each point of the resulting sample can be labeled with an interval ratio, that is, the ratio between the changing frequency and the constant one. In table I the ratios that characterize significant signal points, and the specific name in the scale of "just intonation", are indicated [7]. A constant interval ratio is obtained mixing two pure tones at fixed f_1 and f_2 frequencies.

2.2 Estimation recurrences

Recurrence Quantification Analysis (RQA) is a relatively new nonlinear technique originally developed by Eckmann et al [8] as a purely graphical method and then made quantitative by Webber and Zbilut [9]. The technique was successfully applied to a number of different fields ranging from physiology [10] to molecular dynamics [11]. Recently, we exploited RQA also for its ability to provide a synthetic description of the otoacoustic emissions [12].

The notion of recurrence is simple: for any ordered time series, a recurrence is a value which repeats itself within an assigned tolerance (= Radius). Thus, given a reference point, X_0, and a sphere of radius r centered on it, a point X is said to recur if

$$B_r(X_0) = \{X : ||X - X_0|| \leq r\} \quad (1)$$

The application of this computation produces a Recurrence Plot (RP) which, according to the Webber and Zbilut algorithm [9], is obtained from the initial measured signal by means of the following steps:

- an *embedding matrix* of dimension 'd' is built, where the first column is the time series of the signal and the following d-1 columns are time-lagged (according to a "*lag*" parameter) copies of it;

- a *distance matrix*, where an element in the i,j position corresponds to the Euclidean distance between the i_{th} and j_{th}rows of the embedding matrix, is derived.

Thus, the Recurrence Plot is simply a graphical representation of the distance matrix, namely a square array where each element is represented as a black dot if the corresponding element in the distance matrix is lower than a fixed cut-off value (see Fig.1 (b-c-d)).

In this work the RQA working parameters are: *Embedding Dimension* $m= 8$, *Radius* = 20% of the mean distance value (two epochs are considered as recurrent if their euclidean distance is below the 20% of average distance between all the epoch pairs) and *Line* = 35 (scoring of at least 35 consecutive recurrent points is needed to consider a diagonal line as deterministic).

Webber and Zbilut [9] developed several strategies to quantify the features of the recurrence plots originally pointed out by Eckmann et al. [8]. Recurrence analysis was performed using the appropriate subprogram of the RQA suite, called Recurrence Quantification Epochs (RQE), in which the recurrence variables are computed within a moving window (*epoch*) shifted by a given number of points (delay) throughout the whole sample. This implies the setting of two other working parameters namely *window length* = 500 and *windows shifting* = 10. The RQE subprogram was used for the glissando samples providing, for each window, a set of RQA variables calculated on the basis of the number and disposition of dots in the recurrence plot. Such variables were:

- Percent Recurrence (% REC), the fraction of the plot occupied by recurrent points, i.e. by epoch pairs whose distance is lower than a threshold (Radius). This is a measure of the recurrent (both periodic and auto-similar) features of the signal.
- Percent Determinism (%DET), the fraction of recurrent points aligned into upward diagonal segments (deterministic lines). This indicates the degree of deterministic structuring due to the presence of "quasi-attractors", i.e. portions of the phase space in which the system lies for a longer time than expected by chance alone.
- Entropy (ENT), a Shannon entropy estimated over the length distribution of deter-

ministic lines and linked to the richness of deterministic structuring.

- MAXLINE, the maximum number of points in diagonal lines.

It is actually possible to define other descriptors in RQA [9]; however, we checked that the Principal Component Analysis (PCA) carried out on %REC, %DET, ENT and 1/MAXLINE, produces the same result as if performed on the whole set of descriptors. For this reason, only the four above mentioned RQA descriptors were dimensionally reduced by agency of PCA [13].

RQA was computed by means of a public domain suite of programs [14-15], and PCA by the appropriate subroutines of the SAS^{TM} statistical package.

2.3 Results and Discussion

One prominent feature of a recurrence plot [15] is related to the typical representation of the recurrences picked up in a signal (see for example Fig. 1(b)). Fig. 1(c) and 1(d) show recurrence plots of the consonant *perfect fifth* and dissonant *diminished fifth*, respectively (for the interval name see Table I). This corresponds to a preliminary, still qualitative, proof of Galileo's conjecture: the consonant pairing gives rise to the most regular (simple) recurrence pattern.

A windowing procedure (RQE) carried out on the acoustic sample shown in Fig.1(a), provides for each window the set of recurrence variables to be submitted to PCA. The first component (PC1), explaining 68% of the total variance, was plotted in Fig. 2 vs. the interval ratio. The good agreement between the peaks and the position of most of the musically relevant intervals is noticeable. Moreover, the area of the peaks is proportional to the accepted order of consonance of the musical intervals, with a square correlation coefficient (R-square) equal to 0.86 (see Fig. 3). This correlation proves the feasibility of the reconstruction of a perceptive consonance rank, like the empirical Malmberg's order of merit [16], by means of some mathematical treatment of the acoustic signal, thus opening the way for a general paradigm about the nature of consonance.

The consonance dimension of music is an objective link between musical performance and listeners' hearing activity. Thus, it is possible to consider the results depicted in figure 2 as a quantitative assessment of the consonance profile of musical fragments.

Moreover, the link between consonance and the recurrent structures of the superposition of two pure tones can be taken as an empirical evidence of Galileo's conjecture based on the consideration of the purely mechanical model provided by two oscillating pendula.

Another conclusion of our work is that to deal with the relative psychoacoustic merit of different sounds, consideration of *overtones* and, in general, of complex harmonics, is not required. For the same reason, even the timbral characteristics of sound are not necessary to predict musical consonance.

These results were tested for several different frequencies;however, it is worth noting that in the bass range (C2 frequency-65.406 Hz) the "simple-frequency ratio" theory does not hold for pure tones. In some instances, in fact, more complex ratios are more consonant than simpler ones. This behavior is actually not managed by our extension of Galileo's conjecture, and can be the result of the lack of sensitivity of the auditory system in a frequency range quite far from the speaking frequencies, and hence difficult to estimate by a purely psychoacoustical scale.

All in all, it may be safely stated that the recurrence approach is both intuitive and powerful as a paradigm to understand consonance.

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Legend to Table and Figures

Table I. General features of Musical Intervals.

The table reports the synthetic name (key) of "musically relevant" intervals, their name, the two frequency ratio ("just intonation" scale) and a psychoacoustic scale of consonance (Malmberg's order of merit)[18].

Figure 1. Sound Sample and recurrence plots.

(a) and (b) panels show the first 7500 points of a sound sample used in this work and the corresponding recurrence plot, respectively; in the lower left corner of panel (b), notice the framed critical bandwidth centered at "unison". The sound sample was generated, with f1=264 Hz (C4) as root frequency and f2 going from 200 to 600 Hz, both f1 and f2 being pure sinusoidal waveforms. The signal was recorded with a sample rate of 6000 Hz and lasted four seconds (24000 points).

Panels (c) and (d) refer to a consonant (perfect fifth) and a dissonant (diminished fifth) interval, respectively, showing four beats along the diagonal (corresponding to a plot of 2000*2000 points). Notice the much higher regularity of the beat shapes in (c) as compared to (d). The working parameters used in the generation of the recurrence plot are explained in the text.

Figure 2. First principal component (PC1) scores against interval ratios.

The scores of the first principal component (PC1) extracted from the RQE variables on sound sample of Fig.1(a) are reported as a function of the interval ratio. The most significant peaks are labeled with the interval ratio names in the scale of just intonation (see Table I).

Figure 3. PC1 scores and consonance ranking.

The peaks' areas in Figure 3, corresponding to PC1 scores vs. interval ratio, are reported vs. Malmberg's interval order of merit [18]. Notice the good linear relation as indicated by the R-square (Rsq) = 0.86.

Table I

KEY	**Name**	**Interval Ratio**	**Malmberg's order of merit**
U	unison	1:1	-
ST	semitone	16:15	1
MaT	major tone	9:8	2.5
Mi3	minor third	6:5	5.5
Ma3	major third	5:4	7.2
P4	perfect fourth	4:3	7.2
Tt	diminished fifth (tritone)	64:45	4.2
P5	perfect fifth	3:2	9.8
Mi6	minor sixth	8:5	6.5
Ma6	major sixth	5:3	8
HMi7	harmonic minor seventh	7:4	-
Mi7	minor seventh	9:5	3.5
Ma7	major seventh	15:8	2.2
P8	octave	2:1	11

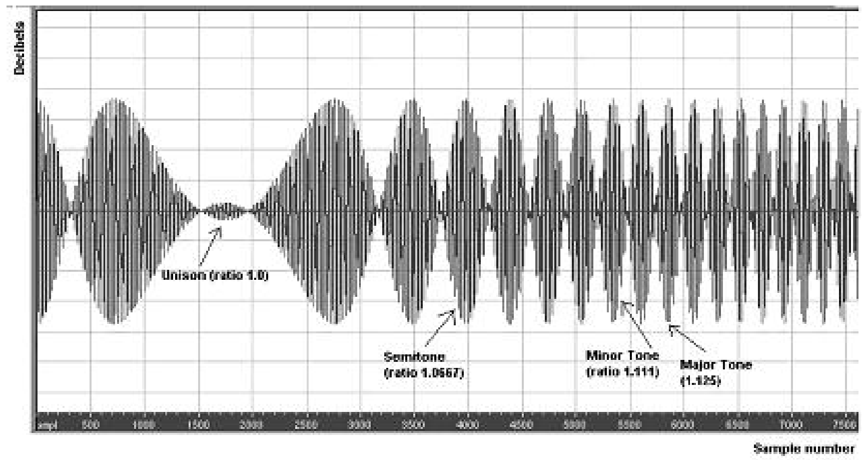
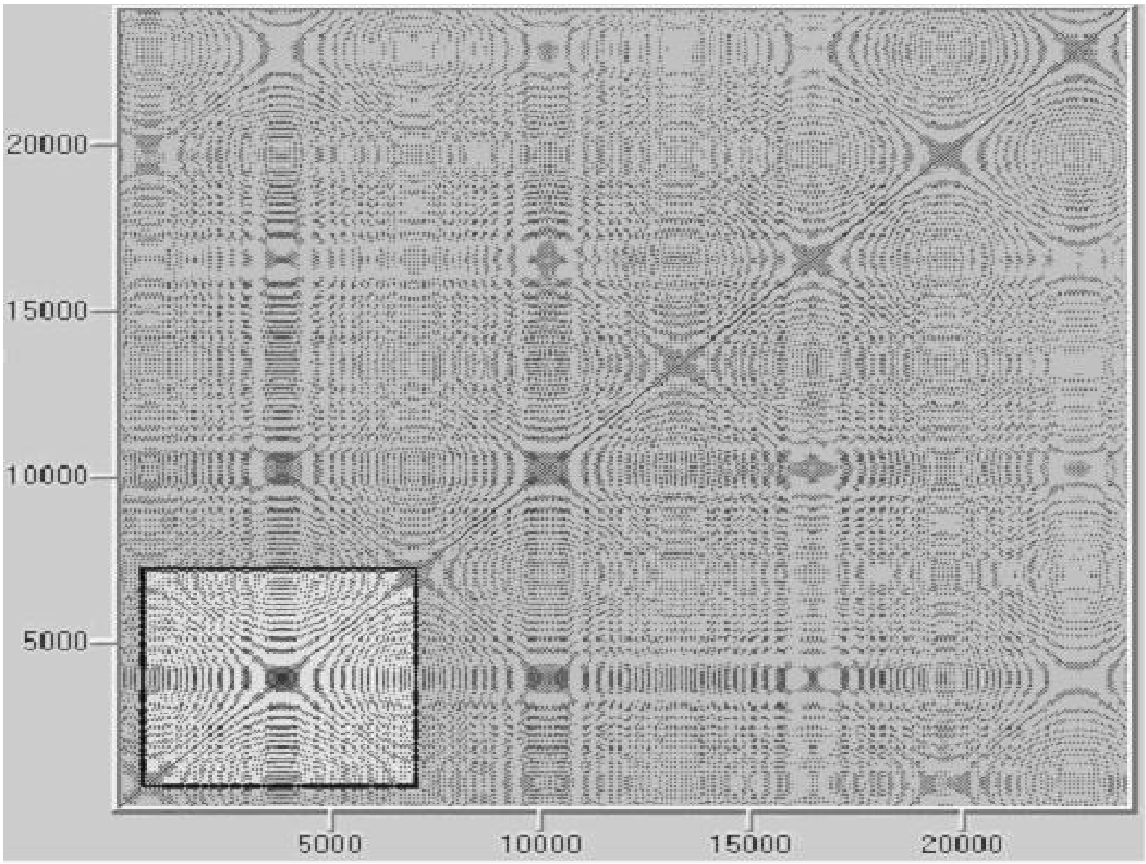
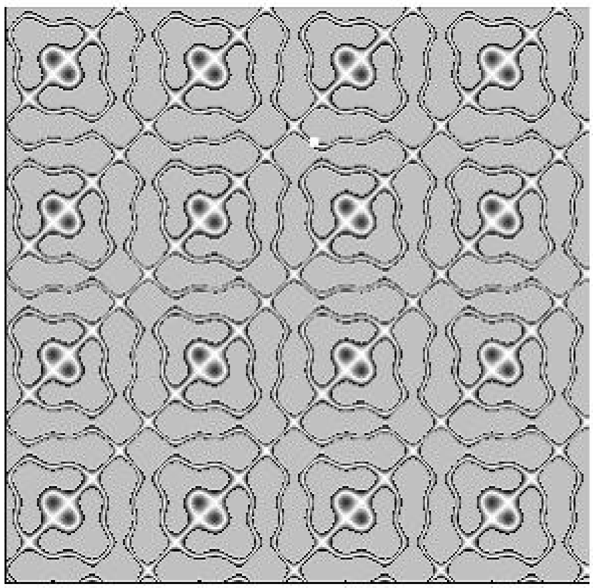


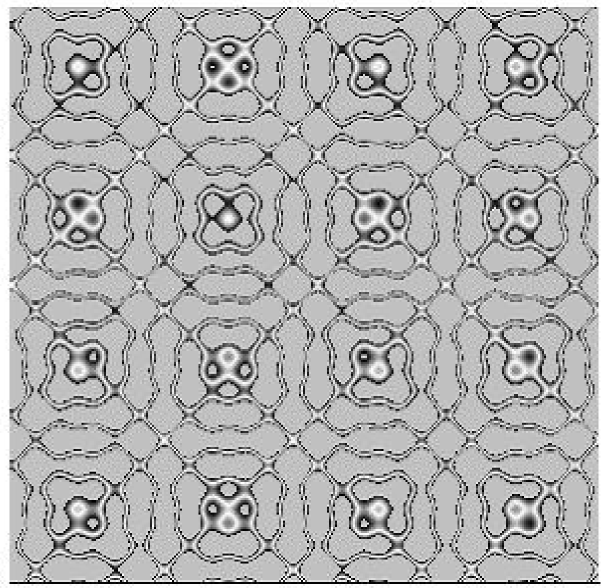
Figure 1 (a) Sound sample's waveform (7500 points)



(b) Full view of the recurrence plot (24000*24000 points)



(c) consonant interval (3/2)



(d) dissonant interval (64/45)

(c)-(d) Detailed view of the recurrence plot for different interval ratios (2000*2000 points)

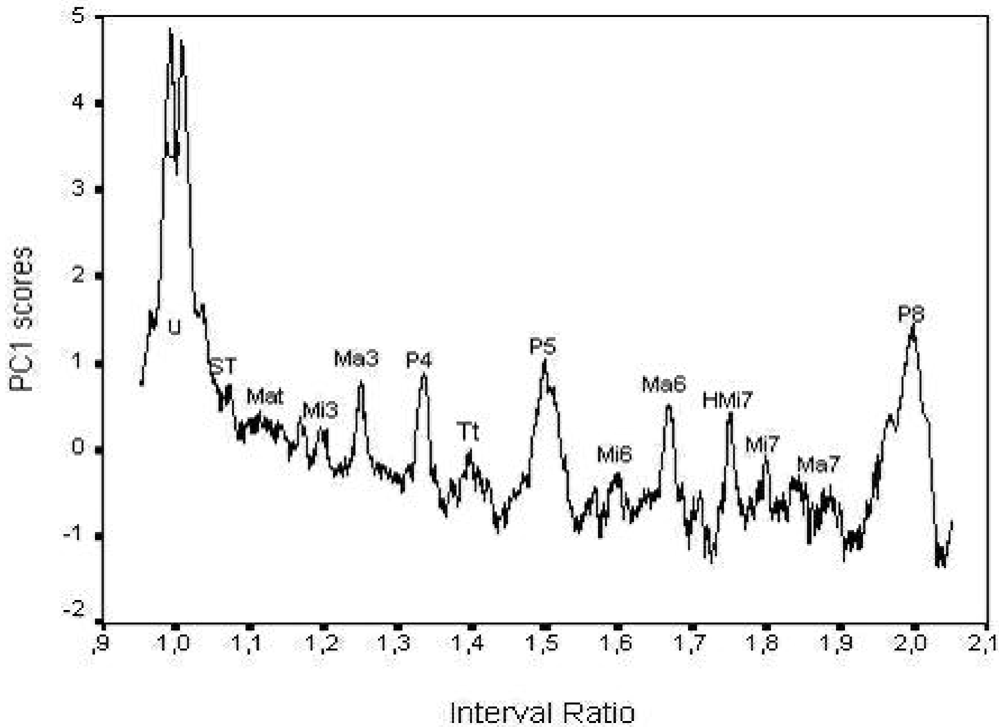


Figure 2. First principal component (PC1) scores against interval ratios.

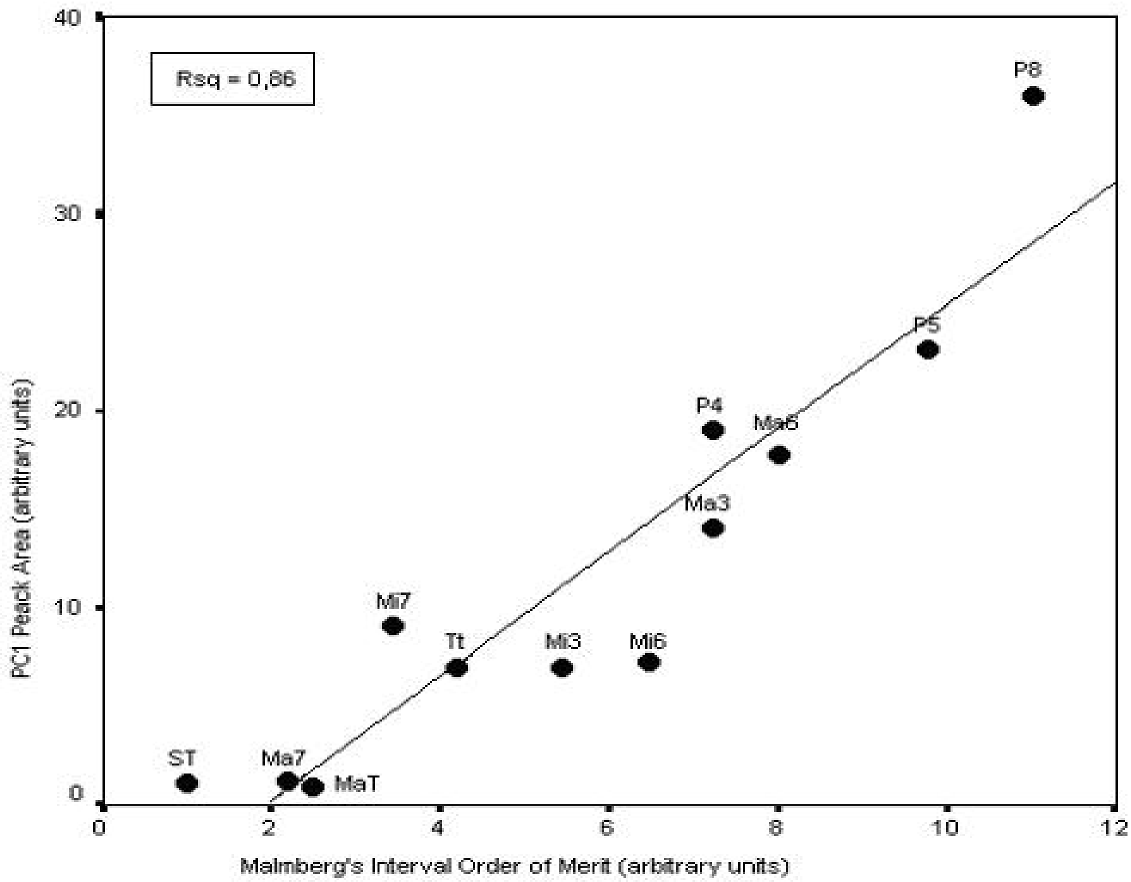


Figure 3. PC1 scores and consonance ranking.

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Conditions for the Generation of Causal Paradoxes from Superluminal Signals

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Abstract: We introduce a simple method to illustrate how the Lorentz transformation lead to causal loop paradoxes when they are applied to superluminal velocities.

Keywords: Special relativity - Superluminal signals - Causal paradoxes
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1. Introduction

In the framework of the special theory of relativity (STR), recent theoretical [1] and experimental [2], [3],[4],[5], evidencies of superluminal[2] motions necessarily lead to unacceptable causal loop paradoxes. The problem arises because while all observers agree about the time ordering of events linked by a subluminal signal, for a superluminal signal different observers disagree on whether the signal was received after or before it was emitted. In other words, viewed in a certain class of inertial frames, a superluminal signal travels backward in time. Thus, the Tolman's paradox [6], namely the communication with the past, in principle, would became possible. Although the causal paradoxes of the STR are reported in the main introductory tests on relativity only a very limited space is dedicated to them. Because of the intriguing connection between causality violation and time machines construction known from the science fiction, such an argument still continues to captivate physicist' interest.

The aim of this paper is to show the connection between the superluminal signal existence and the causality violation by means of a simple mathematical derivation including

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2 Hereafter, by "superluminal" we always mean "with a speed larger than the speed c of light in vacuum".

also an appropriate graphic representation.

2. Causal loop paradoxes

Before discussing causal paradoxes, we shall remind the reader of the Lorentz transformation relating space and time coordinates in different frames of reference. Let us consider two arbitrary inertial frames which will denote with Σ and Σ′ and having a common origin at t=t'=0, but with the origin of Σ′ moving along the x-axis of Σ at a relative speed v. Then, from the two Einstein's postulates of STR, the transformation equations between the two sets of space-time coordinates are

$$
\begin{aligned}
x' &= \gamma(x - \beta ct) \\
y' &= y \\
z' &= z \\
ct' &= \gamma(ct - \beta x)
\end{aligned}
\tag{1}
$$

with the usual meaning for $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. It is important to stress that the equations (1) are necessary consequences of the relativity postulates. In other words, any shift, however small, of the spatial and/or time coefficient away from its relativistic value necessarily implies the existence of a "preferred" frame. Within the STR one can show that, for the speed of particles and of all physical "signals", c should be an upper limit if we insist on the invariance of causality. This, in a certain sense, represents a further Einstein's postulate to always preserve the causal connections. Let us suppose now that exists any process in which an event A causes an event B at a "superlight" speed $u > c$ relative to some frame Σ. Choose coordinates in Σ so that these events both occur on the x-axis and let their spatial and time separations be $\Delta x > 0$ and $\Delta t > 0$. Then, in another inertial frame we have from the fourth of equations (1),

$$
\Delta t' = \gamma(\Delta t - \frac{\beta}{c}\Delta x) = \gamma \Delta t(1 - \frac{\beta}{c}u) \tag{2}
$$

For c/u< β we would have $\Delta t' < 0$. This means that would exist a class of inertial frames Σ′ in which B precedes A, i.e. in which cause and effect are reversed or in which information flow from receiver to transmitter. Thus, we could have foreknowledge of future events and if we decide deliberately foil them, by manipulating the past, we would incur grave contradictions. In a bidimensional Minkowski diagram having space in abscissae and time in ordinates, the line of arguments as the use of superluminal signal velocities leads to a violation of the Einstein causality is illustrated in Figure 1. There are two arbitrary inertial frames of reference displayed.

At the time t'(A_1), a superluminal signal relatively to Σ′ (the question), is emitted from its origin toward the origin of Σ. B_1 is the event associated to its arrival in Σ.

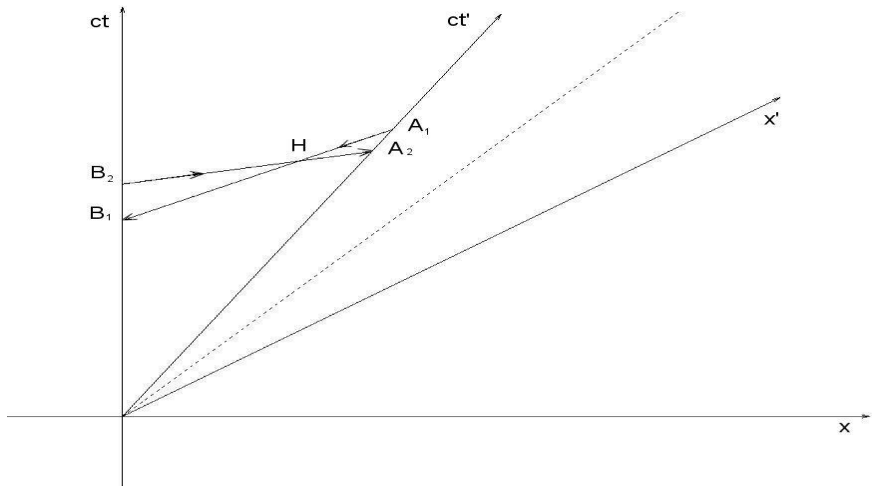

Fig. 1 Spatial and time coordinates of two arbitrary inertial observers moving with a relative velocity v. The lines indicated by arrows represent two signals which are declared superluminal and propagate in the future with respect to the reference frame in which each signal is emitted.

The observer in Σ, after waiting for a time Δt>0, decide to send, toward the origin of Σ', a superluminal signal, relatively to Σ (event labelled with B_2 in figure 1). A_2 is the event associated with the arrival of the signal (the answer) to the origin of Σ'. The only constraint is that each signal should travel in the future with respect to the observer which has emitted it. Well, now we shall demonstrate that under both the conditions:

$$\tilde{\beta} > \frac{1}{\beta} \quad and \quad \beta^* > \frac{1}{\beta} \tag{3}$$

($\tilde{\beta}$ and β^* being the velocities, in c units, of the signals in Σ' and Σ respectively), certainly exists a non-negative solution for the waiting time Δt for which the event labelled with A_2 occurs before the A_1 one with respect both the Σ and Σ' reference frames. Indeed, we have for the signal starting from A_1

$$x'(B_1) = -\tilde{\beta}[ct'(B_1) - ct'(A_1)] \tag{4}$$

and also

$$x'(B_1) = -\beta ct'(B_1) \tag{5}$$

being B_1 an event which happens on the x=0 axis. Thus, being $\tilde{\beta} > \beta$, we get

$$ct'(B_1) = \frac{\tilde{\beta}}{\tilde{\beta} - \beta} ct'(A_1) \tag{6}$$

and using the fourth equation of the Lorentz transformation (1), we find the result:

$$ct(B_1) = \frac{\tilde{\beta}}{\gamma(\tilde{\beta} - \beta)} ct'(A_1) \tag{7}$$

After a time interval Δt $\geq$0, the observer in the Σ frame decides to send his "answer". Let B_2 be the event associated with the departure, from the origin of Σ of the superluminal signal having a velocity which is β^* (in c units) with respect to the observer at rest in Σ. Thus, if A_2 denote the event associated with the arrival of such a signal to the origin of Σ', we have

$$x(A_2) = \beta^*[ct(A_2) - ct(B_2)] \tag{8}$$

$$x(A_2) = \beta ct(A_2) \tag{9}$$

Then, being also $\beta^* > \beta$, we get

$$ct(A_2) = \frac{\beta^*}{\beta^* - \beta} ct(B_2) \tag{10}$$

and using the fourth equation of the inverse Lorentz transformation we find the result:

$$ct'(A_2) = \frac{\beta^*}{\gamma(\beta^* - \beta)} ct(B_2) \tag{11}$$

By using the (7), the equation (11) provides:

$$ct'(A_2) = \frac{\beta^*}{\gamma(\beta^* - \beta)} [c\Delta t + \frac{\tilde{\beta}}{\gamma(\tilde{\beta} - \beta)} ct'(A_1)] \tag{12}$$

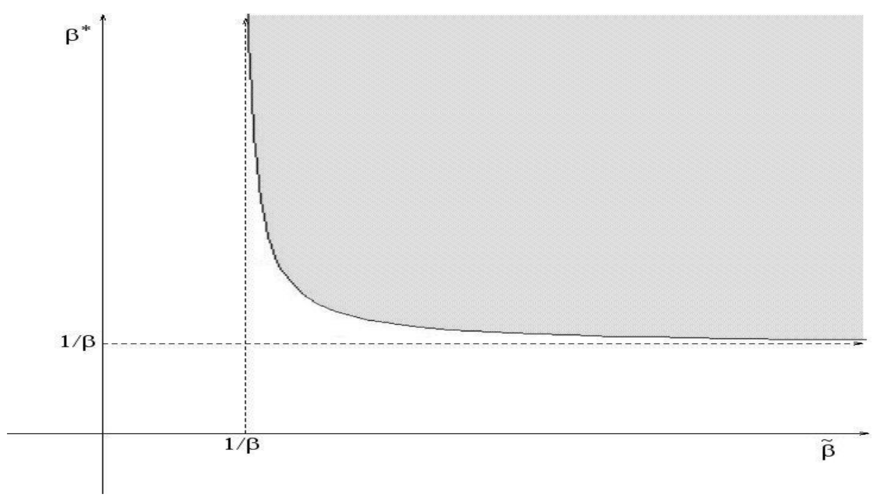

Fig. 2 Graphic representation of the inequality (17) in the $\tilde{\beta} - \beta^*$ plane. The shadowed indicates the region of values for $\tilde{\beta}$ and β^* in which the causal paradoxes became possible.

Now, for non-negative values of Δt, the $ct'(A_2) < ct'(A_1)$ inequality necessarily requires that the important condition

$$\frac{1}{\gamma^2} \frac{\tilde{\beta}\beta^*}{(\tilde{\beta} - \beta)(\beta^* - \beta)} < 1 \tag{13}$$

should be, at least, satisfied. This condition (13) has an interesting geometrical representation in the $\tilde{\beta} - \beta^*$ plane. In fact, the inequality (13) also writes,

$$\tilde{\beta} + \beta^* - \beta\tilde{\beta}\beta^* - \beta < 0 \tag{14}$$

representing a region delimited by an hyperbola which, by means the of translation:

$$\tilde{\beta} = \tilde{\Lambda} + \frac{1}{\beta} \tag{15}$$

$$\beta^* = \Lambda^* + \frac{1}{\beta} \tag{16}$$

becomes:

$$\tilde{\Lambda}\Lambda^* > \frac{1}{(\gamma\beta)^2} \tag{17}$$

In figure 2 we displayed the region characterized from both $\tilde{\Lambda} > 0$ and $\Lambda^* > 0$ inequalities in which the (17) is satisfied, namely it is shown the region (shadowed) where the causality can be violated. It is worth noting that in such a region the superluminals signal have slopes which are strictly less than β. Finally, under the condition (17), from equation (12) we find also an interval of non-negative values for Δt which is limited by a maximum value given by:

$$\Delta t^{max} = \frac{\gamma\beta^2}{\beta^*(\tilde{\beta} - \beta)} t'(A_1)[(\tilde{\beta} - \frac{1}{\beta})(\beta^* - \frac{1}{\beta}) - \frac{1}{(\gamma\beta)^2}] = \tag{18}$$

$$= \frac{\gamma^2\beta^2}{\beta^*\tilde{\beta}} t(B_1)[\tilde{\Lambda}\Lambda^* - \frac{1}{(\gamma\beta)^2}]$$

As outlined at the beginning of this section (see equation (2)), the signals A_1-B_1 and B_2-A_2 travel backwards in time for the Σ and Σ' frames respectively. Thus, we have a double meaning for the world lines of figure 1. In fact, under the conditions which assure the existence of the common H event, the H-B_1 and H-A_2 world lines represent two time-reversed return paths leading to the B_1 and A_2 events which lie in the past light-cones of the B_2 and A_1 respectively.

3. Conclusion and outlook

In conclusion we have shown, by means of elementary mathematical tools and an appropriate graphic representation, how the use of signals declared "superluminal" with respect to each inertial observer can lead to a causal loop paradox when the velocity of any signal is greater than c^2/v.

In order to avoid the causal paradoxes, different solutions are suggested in the literature. The first is based on the so-called Extended STR proposed by Recami and Mignani [7]. Within such an approach the causal paradoxes are avoided through "the switching procedure" also known as the "reinterpretation principle" [8]: a negative-energy superluminal particle moving backwards in time is actually a positive-energy superluminal antiparticle propagating forwards in time and viceversa. According to the "switching rule" the causality violation is avoided, assuming of course that negative-energy objects travelling forward in time do not exist.

Concerning the compatibility between superluminal velocities and STR, another solution was discussed by Nimtz and Haibel [9]. They have shown that, experimentally, the principle of causality could not be violated in consequence of the narrow frequency band width of all signals.

Finally, in the context of a more general theory, the causal paradoxes seem to be typical of a set of transformations: those so-called equivalent (ET) to the STR [10], [11] depending on the synchronization parameter, the coefficient of x in the transformation of time, which is in large part conventional and often indicated as e_1. One can also show [12] that for the subset of transformations characterized by $e_1 \geq 0$, the causal paradoxes are not permitted. Within such a class that corresponding to a particular e_1=0 value, provides the so-called "inertial transformation" which transform time independently of space coordinates [11], [13]. Although these agree with the microscopic and macroscopic experimental evidences for time dilation, they are based on the absolute synchronization concept.

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