

“Anticoherent” Spin States via the Majorana Representation

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Abstract: In this article we define and exhibit “anticoherent” spin states, which are in a sense “the opposite” of the familiar coherent spin states. Since the familiar coherent states are as “classical” as spin states can be, the anticoherent states may turn out to be better candidates for applications involving non-classical behaviors such as quantum entanglement. Thanks to the Majorana representation of spinors as $2s$ -tuples of points on the Riemann sphere, classes of anticoherent states are easy to find; the development of such examples also leads us into some curious geometry involving the perfect solids.

If we create a universe, let it not be abstract or vague but rather let it concretely represent recognizable things. M.C. Escher[1]

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1. Introduction

In 1992, I first learned about the Majorana[2] representation of spinors by reading the fascinating paper of Roger Penrose, “On Bell nonlocality without probabilities: Some curious geometry.”[3] I was astounded to learn that a complex spinor could be fully visualized in a way that preserved rotational structure. In the context of spin-3/2 states, it was as if I were being offered a glimpse of four-dimensional complex space. And poised in that veritable hall of mirrors was a beautiful set of 40 complex rays that Penrose had defined, based on the geometry of the dodecahedron.

At that time I was a research student in Professor Penrose’s group, and over the course

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of my two years at Oxford, I would come to admire his almost uncanny insight into the geometry of four dimensions. Undoubtedly seeing a hint of some additional structure in the crystalline set he had devised, he asked me to search it out. The results proved interesting and led to my M.Sc. thesis,[4] a combined project in mathematics and the foundations of quantum mechanics, and to a 1993 paper that continued the geometric theme: “On Bell nonlocality without probabilities: More curious geometry.”[5] A few years later, Massad and Aravind[6] presented an elementary account of this work, translating the Majorana approach into standard methods more familiar to physicists—a worthwhile program, but, as the authors pointed out, one that sacrificed some of the simplicity and insight of the Majorana methods. In the meantime, a parallel track of investigations into Bell’s theorem without probabilities and Kochen-Specker type noncontextuality proofs had begun in the 1980’s, with contributions by Greenberger, Horne, Zeilinger, Peres and others; for a quick survey of this thread, see the first section of the 2003 article by Peres,[7] and references therein. Some additional useful references are given in the endnotes.[9, 8]

It is a privilege to contribute to the present collection of articles in celebration of the centenary of the birth of Ettore Majorana, whose fascinating method for visualizing spinors has proved so fruitful in the past and will doubtless lead to more insights in the future. And in honor also of Roger Penrose’s role in repopularizing the Majorana representation, I shall take a little time in what follows to explore just a bit more of the curious geometry which the Majorana representation reveals.

2. “Anticoherent” Spin States

In the usual way of speaking, a coherent spin state $|\hat{\mathbf{n}}\rangle$ is a spin state that corresponds as nearly as possible to a classical spin vector pointing in a given direction $\hat{\mathbf{n}}$. Coherent spin states are a conceptual attractor for physicists in part because they mimic classical spin angular momentum as much as possible. As useful as this makes the coherent states, in theory and in practice, it is also fair to say that today we care just as much, if not more, about ways in which quantum phenomena do *not* mimic classical phenomena—the distinct capabilities of quantum computers being just one example. From this perspective, a question arises as to whether there might in fact be certain limits to the usefulness of coherent states as a resource. As a case in point, Markham and Vedral have recently noted that coherent spin states are in a sense resistant to entanglement formation.[10]

It might then be worthwhile to ask the following question: What kind of spin state might serve as the *opposite* of a coherent state? If states of this kind could be thought of as being as “non-classical” as possible, then this might eventually make them distinctly useful conceptually or even practically.

Probably there are a number of reasonable ways to formalize the idea of a state that is “the opposite” of a coherent state. Perhaps the most natural approach is to consider a state $|\psi\rangle$ whose polarization vector vanishes, $\mathbf{p} \equiv \langle\psi|\mathbf{S}|\psi\rangle = \mathbf{0}$. Such a state “points nowhere,” in the mean, and this is certainly one way to serve as the opposite of a state that points, as much as possible, somewhere. For example, the state $|s = 1, m_{\hat{z}} = 0\rangle \equiv |0\rangle$

has vanishing polarization. So by this criterion at least, $|0\rangle$ would qualify as the opposite of coherent.

But one might also wish to make a stricter demand. For let us imagine that one experimenter measures $\hat{\mathbf{n}}_1 \cdot \mathbf{S}$ many times on a system prepared in state $|0\rangle$, while another experimenter measures $\hat{\mathbf{n}}_2 \cdot \mathbf{S}$ many times on the same kind of system prepared in the same state $|0\rangle$. The two experimenters will obtain the same mean value for their results. But, in general, the *variances* of their results will differ. At the level of the variances in their data, the signature will remain of the direction along which the spin has been measured. To this extent, the state $|0\rangle$ itself still contains a certain amount of directional information.

Erasing this directional signature would require finding a state $|\psi\rangle$ for which the variance in measurements of spin in any one direction is the same as the variance in measurements of spin in any other direction—in other words, a state $|\psi\rangle$ for which the function

$$\Delta S_{\hat{\mathbf{n}}}^2 = \frac{\langle \psi | (\hat{\mathbf{n}} \cdot \mathbf{S})^2 | \psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle \psi | \hat{\mathbf{n}} \cdot \mathbf{S} | \psi \rangle^2}{\langle \psi | \psi \rangle^2} \quad (1)$$

is uniform over the unit sphere.

Generally, the variance function for a state is nonuniform on the unit sphere; Fig. 1 shows several examples. But if the variance function $\Delta S_{\hat{\mathbf{n}}}^2$ is uniform over the sphere in the state $|\psi\rangle$, then we shall call $|\psi\rangle$ a *uniform state*.

If $|\psi\rangle$ is a uniform state with vanishing polarization vector, then we shall call $|\psi\rangle$ an *anticoherent state*. Anticoherent states will serve as our beginning notion of “the opposite” of coherent states.

A. Criteria for Uniformity and Anticoherence

In this section, we make a few observations on anticoherent states and establish some necessary and sufficient conditions for anticoherence and uniformity.

When the polarization vector vanishes, $\Delta S_{\hat{\mathbf{n}}}^2$ reduces to $\langle (\hat{\mathbf{n}} \cdot \mathbf{S})^2 \rangle$. Denoting the uniform value of $\Delta S_{\hat{\mathbf{n}}}^2$ for an anticoherent state by c , we have

$$c + c + c = \Delta S_x^2 + \Delta S_y^2 + \Delta S_z^2 \quad (2)$$

$$= \langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle \quad (3)$$

$$= s(s + 1) \quad (4)$$

so that any anticoherent state must have $\Delta S_{\hat{\mathbf{n}}}^2 = s(s + 1)/3$, independent of $\hat{\mathbf{n}}$.

In addition, if $|\psi\rangle$ is anticoherent, then we must have $\langle S_i S_j \rangle = 0$ for perpendicular axes i and j . This can be seen from the identity

$$S_i S_j + S_j S_i = \left(\frac{1}{\sqrt{2}} S_i + \frac{1}{\sqrt{2}} S_j \right)^2 - \left(\frac{1}{\sqrt{2}} S_i - \frac{1}{\sqrt{2}} S_j \right)^2. \quad (5)$$

The right-hand side is $S_u^2 - S_v^2$, where the axes u and v are 45-degree rotations of axes i and j . So if we take expectation values of both sides in an anticoherent state $|\psi\rangle$, then

the right-hand side vanishes. Hence $\langle S_i S_j \rangle + \langle S_j S_i \rangle = 0$. But also $\langle S_i S_j \rangle - \langle S_j S_i \rangle = i\hbar \langle S_k \rangle = 0$, since $|\psi\rangle$ is polarization-free. Hence $\langle S_i S_j \rangle = 0$.

Conversely, if a state satisfies $\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = \langle S_x S_y \rangle = \langle S_y S_z \rangle = \langle S_z S_x \rangle = 0$ and $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle$, then it must be anticonherent. This is easily seen by expanding $\Delta S_{\mathbf{n}}^2 = \langle (\sin \theta \cos \phi S_x + \sin \theta \sin \phi S_y + \cos \theta S_z)^2 \rangle$.

A distinct criterion for uniformity that is sometimes useful can be given by considering $|\psi\rangle$ fixed and interpreting $\Delta S_{\mathbf{n}}^2$ as a vector in the Hilbert space $L^2[\mathcal{S}^2]$ of square-integrable functions on the sphere \mathcal{S}^2 , with inner product denoted

$\langle\langle Q|R \rangle\rangle \equiv \int_0^{2\pi} \int_0^\pi Q^*(\theta, \phi) R(\theta, \phi) \sin \theta d\theta d\phi$. By the Cauchy-Schwarz inequality, uniformity is equivalent to

$|\langle\langle Y_0^0 | \Delta S_{\mathbf{n}}^2 \rangle\rangle|^2 = \langle\langle \Delta S_{\mathbf{n}}^2 | \Delta S_{\mathbf{n}}^2 \rangle\rangle \langle\langle Y_0^0 | Y_0^0 \rangle\rangle$, where $Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$ is the normalized $\ell = 0$, $m = 0$ spherical harmonic. Explicitly then, the necessary and sufficient uniformity condition is[11]

$$\left[\int_0^{2\pi} \int_0^\pi \frac{1}{\sqrt{4\pi}} \Delta S_{\mathbf{n}}^2 \sin \theta d\theta d\phi \right]^2 = \int_0^{2\pi} \int_0^\pi [\Delta S_{\mathbf{n}}^2]^2 \sin \theta d\theta d\phi. \quad (6)$$

B. The Majorana Representation as a Strategy for Finding Anticonherent States

Even for relatively low spin quantum numbers, the anticonherence conditions for a general state $|\psi\rangle$ with given symbolic components will involve several complex variables and numerous terms, making it taxing to generate examples of anticonherent states by attacking the algebraic conditions head-on. But with the Majorana representation at our disposal, examples of anticonherent states can easily be found.

The Majorana representation of a $2s + 1$ -component spinor of the form $c_0|+s\rangle + \dots + c_{2s}|-s\rangle$ as a $2s$ -tuple of points on the sphere \mathcal{S}^2 is summarized by the following chain of bijective correspondences:

$$c_0|+s\rangle + \dots + c_{2s}|-s\rangle \leftrightarrow \text{roots of } M(z) \leftrightarrow \text{points in } \mathcal{S}^2, \quad (7)$$

where the Majorana polynomial $M(z)$ is formed from the spinor components according to

$$M(z) = \sum_{k=0}^{2s} c_k \binom{2s}{k}^{\frac{1}{2}} z^{2s-k}, \quad (8)$$

and where the mapping between points (X, Y, Z) of \mathcal{S}^2 and the roots of $M(z)$ in $\mathbf{C} \cup \{\infty\}$ is by stereographic projection, $(X, Y, Z) \leftrightarrow (X + iY)/(1 - Z)$.

The Majorana representation of a coherent state consists of a single point with multiplicity $2s$. At the opposite extreme (in some intuitive sense), we can imagine states whose Majorana representations are spread “nicely” over the sphere. Nicest of all are what we shall call *perfect states*: states whose Majorana representations comprise the vertices of one of the five perfect solids (tetrahedron, octahedron, cube, icosahedron, and

dodecahedron). These states exist for $s = 2, 3, 4, 6$, and 10 . Constructing these states via the Majorana representation offers the possibility of generating concrete examples of anticonherent states without having to solve the anticonherence conditions directly. To this construction we now turn.

3. Perfect States

A. Concrete Examples of the Perfect States

Given the vertices of a perfect solid, the Majorana correspondence determines a perfect state, of which there are five specimens, up to rotational equivalence. It is straightforward to construct concrete examples of perfect states, which we denote by $|\psi_{\text{tet}}\rangle$, $|\psi_{\text{oct}}\rangle$, $|\psi_{\text{cube}}\rangle$, $|\psi_{\text{icos}}\rangle$, and $|\psi_{\text{dodec}}\rangle$. The results of these calculations are:

$$|s = 2, \psi_{\text{tet}}\rangle = \frac{1}{\sqrt{3}} \left(\sqrt{2}|+1\rangle - |-2\rangle \right) \quad (9a)$$

$$|s = 3, \psi_{\text{oct}}\rangle = \frac{1}{\sqrt{2}} (|+2\rangle - |-2\rangle) \quad (9b)$$

$$|s = 4, \psi_{\text{cube}}\rangle = \frac{1}{2\sqrt{6}} \left(\sqrt{5}|+4\rangle + \sqrt{14}|0\rangle + \sqrt{5}|-4\rangle \right) \quad (9c)$$

$$|s = 6, \psi_{\text{icos}}\rangle = \frac{1}{5} \left(-\sqrt{7}|+5\rangle + \sqrt{11}|0\rangle + \sqrt{7}|-5\rangle \right) \quad (9d)$$

$$|s = 10, \psi_{\text{dodec}}\rangle = a|+10\rangle + b|+5\rangle + |0\rangle - b|-5\rangle + a|-10\rangle. \quad (9e)$$

In $|\psi_{\text{dodec}}\rangle$, if we take $a = \sqrt{11/38 + 7b^2/38}$ then we ensure anticonherence, and if we take $b \approx 1.593$ (hence $a \approx 0.870$), then we obtain a dodecahedral Majorana representation.

The vertices of the perfect solids corresponding to the above states are, respectively, $\{(0, 0, 1), (\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}), (-\frac{\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3}), (-\frac{\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3})\}$; $\{(0, 0, 1), (1, 0, 0), (0, 1, 0), (-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$; $\{(\pm 1, \pm 1, \pm 1)/\sqrt{3}\}$; and, in the last two cases, the vertices defined for the icosahedron and dodecahedron in the standard Mathematica package `Geometry`Polytopes``, normalized to the unit sphere.

To expound briefly on one of these examples, consider the state $|\psi_{\text{dodec}}\rangle$. The roots of the Majorana polynomial for this state are found from

$$az^{20} + 4b\sqrt{969}z^{15} + 2\sqrt{46189}z^{10} - 4b\sqrt{969}z^5 + a = 0. \quad (10)$$

Eq. (10) is quartic in the variable $z^5 \equiv w$. If we imagine placing a dodecahedron flat on a table, then the twenty vertices form four tiers with five vertices in each tier, the five vertices of a given tier all lying in the same horizontal plane. Tier k corresponds to the k^{th} root of the quartic, $w_k = |w_k|e^{i\phi_k}$; and the five vertices in tier k correspond to the five fifth-roots of w_k , $|w_k|^{1/5}e^{i(\phi_k+2\pi n)/5}$ ($n = 0, 1, 2, 3, 4$). Via stereographic projection, the modulus $|w_k|^{1/5}$ determines the height of the k^{th} tier, and the complex phases $(\phi_k+2\pi n)/5$ serve to arrange the five vertices of the tier with equal spacing around a circle of latitude. With only four tiers of vertices to map, only a quartic polynomial is required, which is why a “sparse” state in the pattern of $|\psi_{\text{dodec}}\rangle$ suffices.

B. A Tetrahedral Basis in \mathbf{C}^5

Five tetrahedra can be oriented in space in an interlocking fashion so that their twenty vertices together comprise the vertices of a dodecahedron. (See Fig. 2.) This configuration offers a highly symmetrical way of inscribing five tetrahedra in the sphere. Now, as the tetrahedral perfect states happen to “live” in a space of five dimensions, a question naturally presents itself: Is it possible that a dodecahedral collection of tetrahedra inscribed in the unit sphere generates an orthonormal basis in \mathbf{C}^5 ? The answer (surprisingly it seems) is yes.

Specifically, suppose we choose a particular tetrahedron from among the ten contained in a given dodecahedron inscribed in the unit sphere, and that from this first tetrahedron we generate a series of four more tetrahedra by rotating the original tetrahedron in successive angular steps of $2\pi/5$ about an axis passing perpendicularly through the centers of two opposite pentagonal faces of the dodecahedron. The five tetrahedra that result from this process will between them include all 20 vertices of the dodecahedron. Denoting the five tetrahedral states corresponding to these five tetrahedra by $\{|\tau_1\rangle, |\tau_2\rangle, |\tau_3\rangle, |\tau_4\rangle, |\tau_5\rangle\}$, we find that $\langle\tau_i|\tau_j\rangle = \delta_{ij}$; the successive rotations in 3-space act in \mathbf{C}^5 to rotate the original tetrahedral state $|\tau_1\rangle$ into a succession of mutually orthogonal states. (To make an imprecise analogy, it is somewhat as if the vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ were to be rotated in 3-space by an angle of $2\pi/3$ about an axis spanned by $(1, 1, 1)$, with resulting actions $\hat{\mathbf{x}} \rightarrow \hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}} \rightarrow \hat{\mathbf{x}}$.)

At this point, I do not have an insightful argument to explain why the specified rotations of the dodecahedron should act in \mathbf{C}^5 to rotate the original tetrahedral state into a succession of mutually orthogonal states—just a symbolic Mathematica calculation of the inner products $\langle\tau_i|\tau_j\rangle$. So at present, the existence of the “tetrahedral basis” is something of a Platonic mystery.

Given a basis $\{|\tau_k\rangle\}$ of tetrahedral states corresponding to a set of five interlocking tetrahedra, there also exists another basis $\{|\bar{\tau}_k\rangle\}$ that is related to the original one by reflecting each of the five given tetrahedra through the center of the sphere. (See Fig. 3.) The orthogonality relations between these two bases are summarized by $\langle\tau_i|\bar{\tau}_j\rangle = c_{ij}(1 - \delta_{ij})$, where the c_{ij} are nonzero complex numbers. In other words, each vector in one of the bases is orthogonal to its corresponding vector in the “inverted” basis, but to no others. (Two orthonormal bases in \mathbf{C}^n may have this sort of relationship to one another provided $n \geq 4$.)

C. Anticoherence of the Perfect States

With perfect states in hand, we can determine whether any or all of them are anticoherent. Starting with the tetrahedral state $|\psi_{\text{tet}}\rangle$, one easily finds $\mathbf{p} = \mathbf{0}$ and $\Delta S_{\mathbf{n}}^2 = 2$, independent of $\hat{\mathbf{n}}$; the tetrahedral state is indeed anticoherent. See Fig. 4. Likewise, all of the perfect states listed above can be shown to be anticoherent. While these particular states were only representatives of equivalence classes under rotations, the property of

anticoherence is itself invariant under rotations, so it suffices to make the calculations with representatives. Thus, all perfect states are anticoherent.[12]

However, all anticoherent states are not perfect. To see this, consider the following family of states:

$$|s \geq 3, \psi_{ac}\rangle = e^{i\alpha} \sqrt{\frac{s+1}{6s}} | +s \rangle + \sqrt{\frac{2s-1}{3s}} |0\rangle + e^{i\alpha} \sqrt{\frac{s+1}{6s}} | -s \rangle. \quad (11)$$

It is straightforward to show that for integral spin $s \geq 3$, $|\psi_{ac}\rangle$ is anticoherent for any real α , with $\mathbf{p} = \mathbf{0}$ and variance $\Delta S_{\hat{\mathbf{n}}}^2 = s(s+1)/3$ independent of $\hat{\mathbf{n}}$.

Up to an overall scalar multiple, the Majorana polynomial of $|\psi_{ac}\rangle$ has the form

$$z^{2s} + e^{-i\alpha} \binom{2s}{s}^{\frac{1}{2}} \sqrt{\frac{4s-2}{2s+1}} z^s + 1, \quad (12)$$

which is quadratic in z^s . Thus, the Majorana representation of $|\psi_{ac}\rangle$ consists of two tiers of points on the sphere, each tier consisting of s points spaced equally around a circle of latitude. The cubic perfect state $|\psi_{cube}\rangle$ given in Eq. (9c) is an example of this general family, with $s = 4$ and $\alpha = 0$. For $s > 4$, the Majorana representation of $|\psi_{ac}\rangle$ will not form a perfect solid. See Fig. 5.

4. Higher Orders of Anticoherence

In the foregoing sections, we exhibited anticoherent states for all integral spin quantum numbers $s \geq 2$. For $s = 1/2$, $s = 1$, and $s = 3/2$, it is possible to show that no anticoherent states exist.[13] In general, one expects anticoherent states to exist for all $s \geq 2$, based on the following count of degrees of freedom.

A spin- s state is represented by $2s$ points on the unit sphere, one of which may be rotated to become the North Pole. For $s \geq 1$, this leaves $2s - 1$ other points on the unit sphere, and thus $4s - 2$ real degrees of freedom—except that one of the points may always be rotated to azimuthal position $\phi = 0$. Thus, whenever a problem is rotationally invariant, there are $4s - 3$ real degrees of freedom for a spin- s state with $s \geq 1$.

A constraint equation on the complex spinor components will reduce the number of degrees of freedom by two, except when the constraint equation fails to have a nontrivial imaginary part. Thus, the requirement that $\mathbf{p} = \mathbf{0}$ will in general reduce the number of real degrees of freedom by three, leaving us with a manifold of $4s - 6$ real degrees of freedom. And finally, the uniformity condition Eq. (6) reduces the number of degrees of freedom by one more, leaving $4s - 7$ degrees of freedom. For finitely many isolated solutions we should have $4s - 7 = 0$, or $s = 1.75$. So the count of degrees of freedom is consistent with the fact that no anticoherent states exist for $s \leq 1.5$, and it suggests that for $s \geq 2.0$, there will be enough freedom in the choice of a state to satisfy the requirements of anticoherence.[16]

In fact, by advancing to high enough spin quantum numbers, it should be possible to find states $|\psi\rangle$ in which ever-higher moments of the probability distribution $\{|\langle s, m_{\hat{n}}|\psi\rangle|^2\}$ are independent of \hat{n} . Thus, one expects that greater and greater antcoherence is possible for higher and higher values of spin. The salient problem would be in finding concrete examples of states with the desired order of antcoherence. See Table 1.

Order	Condition	Least s for existence	Example
0	none	$\frac{1}{2}$	any $s = \frac{1}{2}$
1	$\langle \hat{n} \cdot \mathbf{S} \rangle \neq f(\hat{n})$	1	$ s = 1, m_{\hat{z}} = 0\rangle$
2	$\langle (\hat{n} \cdot \mathbf{S})^2 \rangle \neq f(\hat{n})$	2	$ \psi_{\text{tet}}\rangle$
3	$\langle (\hat{n} \cdot \mathbf{S})^3 \rangle \neq f(\hat{n})$	2, $\frac{5}{2}$, or 3	$ \psi_{\text{oct}}\rangle$
4	$\langle (\hat{n} \cdot \mathbf{S})^4 \rangle \neq f(\hat{n})$?	?
\vdots	\vdots	\vdots	\vdots

Table 1 Ascending orders of antcoherence for pure states. Antcoherence of order q means that all of the conditions up to and including condition q apply. The condition for $q = 1$, $\langle \hat{n} \cdot \mathbf{S} \rangle \neq f(\hat{n})$, is equivalent to the vanishing of the polarization vector.

5. Antcoherent Mixtures

In this section we present just a few observations concerning antcoherent mixtures. We say that a mixed state ρ is antcoherent to order q if $\text{Tr}[\rho(\hat{n} \cdot \mathbf{S})^\ell]$ is independent of \hat{n} for $\ell \leq q$. Mixed states incorporate more degrees of freedom than do pure states, so one expects antcoherent mixtures to be easier to find in lower-dimensional spaces. Indeed, for any s , the reduced density operator for a fully entangled spin, $\rho = \frac{1}{2s+1}I$, is antcoherent to all orders $q > 0$. (But for $s = 1/2$, $\rho = \frac{1}{2}I$ is the only mixed state that is antcoherent to any order.)

If a mixed state is diagonal in an antcoherent basis, then the mixture itself is antcoherent. Specifically, if $\{|\alpha_k\rangle\}$ is an orthonormal basis of pure states, all of which are antcoherent to order q , then for any numbers r_1, \dots, r_{2s+1} with $0 \leq r_k \leq 1$ and $\sum_k r_k = 1$, it is easy to see that the mixed state $\rho = \sum_k r_k |\alpha_k\rangle\langle\alpha_k|$ is itself antcoherent to order q . Thus, for $s = 2$, the tetrahedral basis may be used to construct mixtures antcoherent to second order: $\rho = \sum_{k=1}^5 r_k |\tau_k\rangle\langle\tau_k|$.

The converse is not true, in the sense that we can have a nondegenerate mixed state that is antcoherent to some order, even when the basis diagonalizing the state is not antcoherent to any order. For example, with $s = 3/2$, consider the mixed state diagonal in the standard basis $\{|m_{\hat{z}}\rangle\}$ with eigenvalues $\frac{1}{18}, \frac{1}{2}, \frac{1}{3},$ and $\frac{1}{9}$. This mixture is easily shown to be antcoherent to order $q = 1$, even though none of its eigenvectors has this property.

6. Conclusion

By way of conclusion, we point to some further questions worth investigating with regard to anticonherent states.

First of all, returning to the “non-classical” perspective that led us to consider anticonherent states in the first place, it would be worthwhile to examine the question of whether states of high anticonherence manage to avoid the resistance to entanglement formation noted in the case of coherent states by Markham and Vedral.[10]

Second, a question may be posed as to whether anticonherent states have any interesting dynamical features. For example, when a system in an initially anticonherent state $|\psi_0\rangle$ is placed in a uniform, static magnetic field $B_0\hat{\mathbf{n}}$, the observable degree of development away from the initial state as a function of time, $|\langle\psi_0|\psi_t\rangle|^2$, is independent of the field direction to lowest nonvanishing order in time. (This is easy to see by expanding the unitary evolution $e^{-iHt/\hbar}$ in power series.) And if the initial state has a greater degree of anticonherence, then the $\hat{\mathbf{n}}$ -independence of $|\langle\psi_0|\psi_t\rangle|^2$ extends to higher order in time. This dynamical feature of anticonherent states may make it worthwhile to examine whether anticonherent states may be exploited in some fashion—theoretically, computationally, or practically—in the context of slowly-varying or imperfectly-known external fields.

Acknowledgments

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- [10] D. Markham and V. Vedral, “Classicality of spin-coherent states via entanglement and distinguishability,” *Phys. Rev. A* **67**, 042113 (2003).
- [11] One can also define a uniformity function U on spin states via

$$U(\psi) = \frac{|\langle\langle Y_0^0 | \Delta S_{\mathbf{n}}^2 \rangle\rangle|}{\sqrt{\langle\langle \Delta S_{\mathbf{n}}^2 | \Delta S_{\mathbf{n}}^2 \rangle\rangle}}. \quad (13)$$

In this characterization, the uniformity of any state lies between 0 and 1, and the uniform states are those for which $U(\psi) = 1$.

- [12] If we place $r > 1$ repeated points at each vertex of a perfect solid, then we determine a “degenerate” perfect state. For example, a doubly degenerate octahedral state is given by

$$|s = 6, \text{ oct}, r = 2\rangle = \frac{1}{4\sqrt{2}} \left(\sqrt{14} | +4 \rangle - 2 | 0 \rangle + \sqrt{14} | -4 \rangle \right). \quad (14)$$

This state is anticoherent to order 3, not higher. A triply degenerate tetrahedral state is given by

$$|s = 6, \text{ tet}, r = 3\rangle = \frac{1}{3\sqrt{183}} \left(8\sqrt{14} | +3 \rangle - 4\sqrt{15} | 0 \rangle + 3\sqrt{14} | -4 \rangle - \sqrt{385} | -6 \rangle \right). \quad (15)$$

This state is anticoherent to order 2, not higher. The orthogonality relations in \mathbf{C}^{13} between $|s = 6, \text{tet}, r = 3\rangle$, $|s = 6, \text{oct}, r = 2\rangle$, and $|s = 6, \text{icos}, r = 1\rangle$ are an interesting question.

- [13] For $s = 3/2$ in particular, one can show that although there is, up to rotational equivalence, a single uniform state, nevertheless, this state is not anticoherent: the polarization vector in this state does not vanish. Alternatively, one can show that if a normalized spin-3/2 state is polarization-free, then the left-hand side of Eq. (6) is necessarily equal to $25\pi/4$, independent of the state, whereas the right-hand side is necessarily equal to $141\pi/20$, also independent of the state. Hence, uniformity is impossible for a polarization-free spin-3/2 state, and anticoherent spin-3/2 states do not exist.
- [14] C. Séquin, *Tangle of Five Tetrahedra*. Computer rendering.
<http://http.cs.berkeley.edu/~sequin/SFF/spec.tangle5tetra.html>.
- [15] M.C. Escher, *Double Planetoid*, 1949. Woodcut. In *M.C. Escher: 29 Master Prints* (Harry N. Abrams, 1983), p. 56.
- [16] The analysis of degrees of freedom suggests that in the case $s = 2$ we should have a continuous one-dimensional manifold of rotationally inequivalent anticoherent states, with the tetrahedral states representing a point on a continuum.

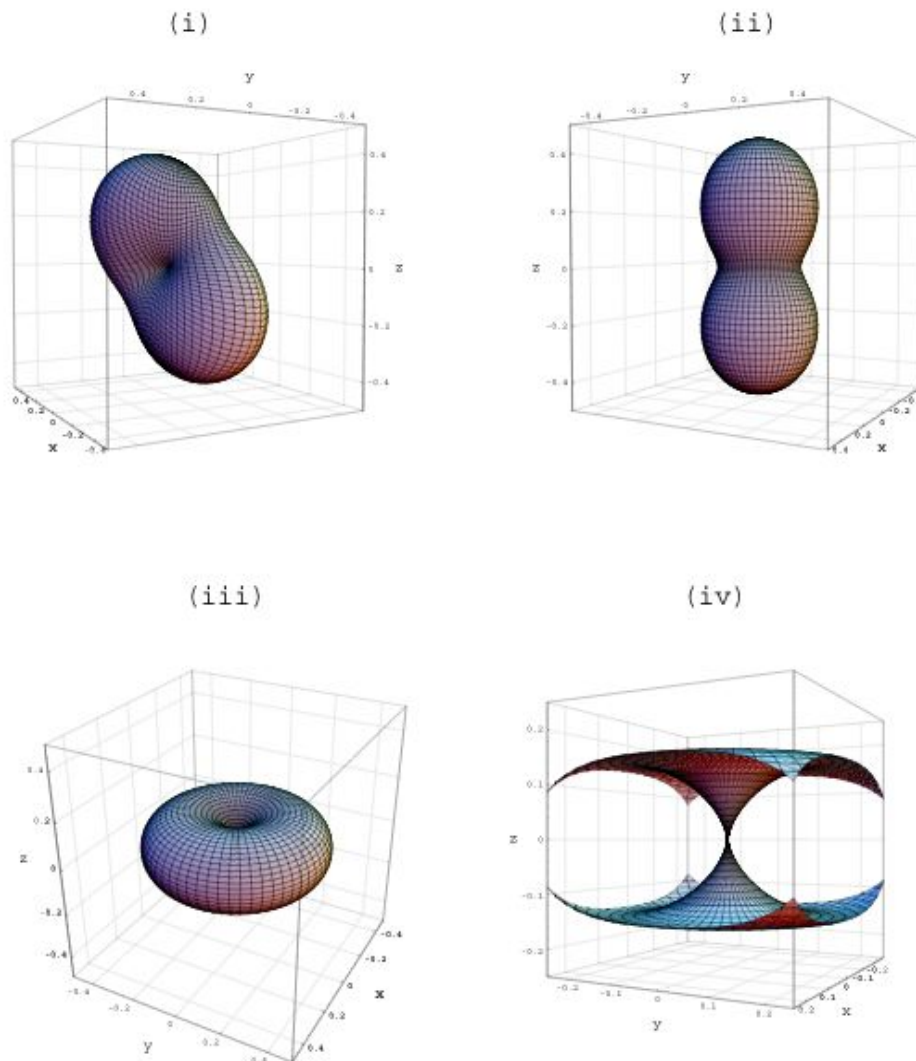


Fig. 1 Spherical parametric plots of the $L^2[\mathcal{S}^2]$ -normalized variance $\Delta S_{\mathbf{n}}^2 / \|\Delta S_{\mathbf{n}}^2\|$ for (i) a randomly chosen $s = 2$ state; (ii) the state $\sqrt{\frac{1}{2}}(|s = \frac{3}{2}, m_{\mathbf{z}} = +\frac{3}{2}\rangle + |s = \frac{3}{2}, m_{\mathbf{z}} = -\frac{3}{2}\rangle)$, with polarization vector $\mathbf{p} = \mathbf{0}$ and Majorana representation $\{(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0), (0, 0, -1)\}$; (iii) the coherent state $|s = 2, m_{\mathbf{z}} = +2\rangle$, with polarization vector $\mathbf{p} = 2\mathbf{z}$ and Majorana representation (SP, SP, SP, SP) (where “ SP ” = South Pole); and (iv) a cutaway view of plot (iii).

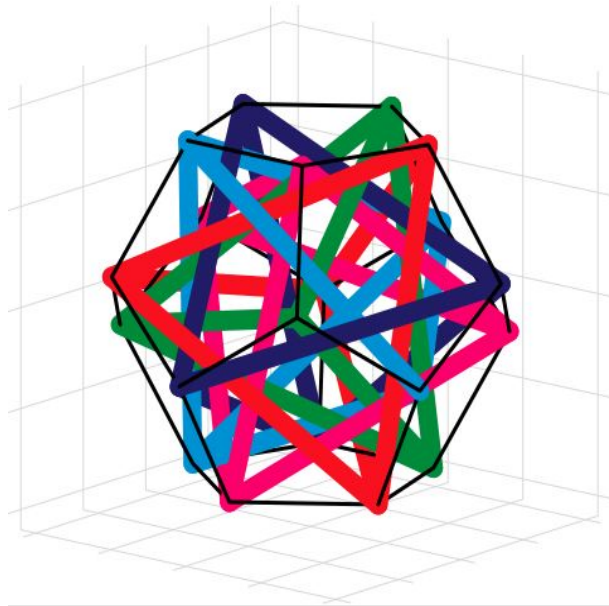
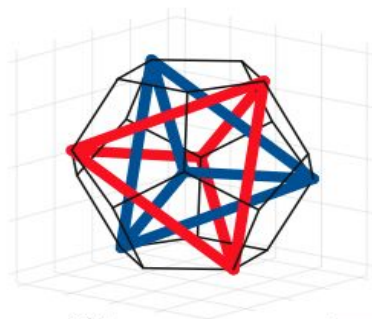
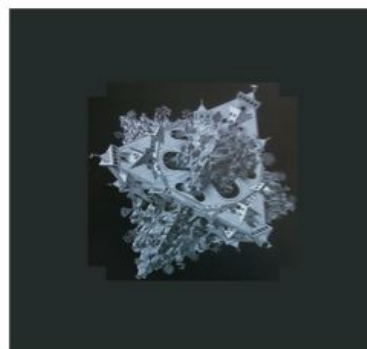


Fig. 2 Five interlocking tetrahedra whose vertices collectively form a dodecahedron. Vertices of tetrahedra of the same color comprise the Majorana representation of a tetrahedral state $|\tau_k\rangle$. Tetrahedral states corresponding to tetrahedra of different colors are orthogonal. (For a more accomplished rendering, see the work by artist/computer scientist Carlos Séquin.[14])



(i)



(ii)

Fig. 3 (i) Reflections of the original five tetrahedra in the center of the sphere produce a new set of tetrahedra, corresponding to a new orthogonal basis $\{|\bar{\tau}_k\rangle\}$. (ii) M.C. Escher, *Double Planetoid*, 1949.[15]

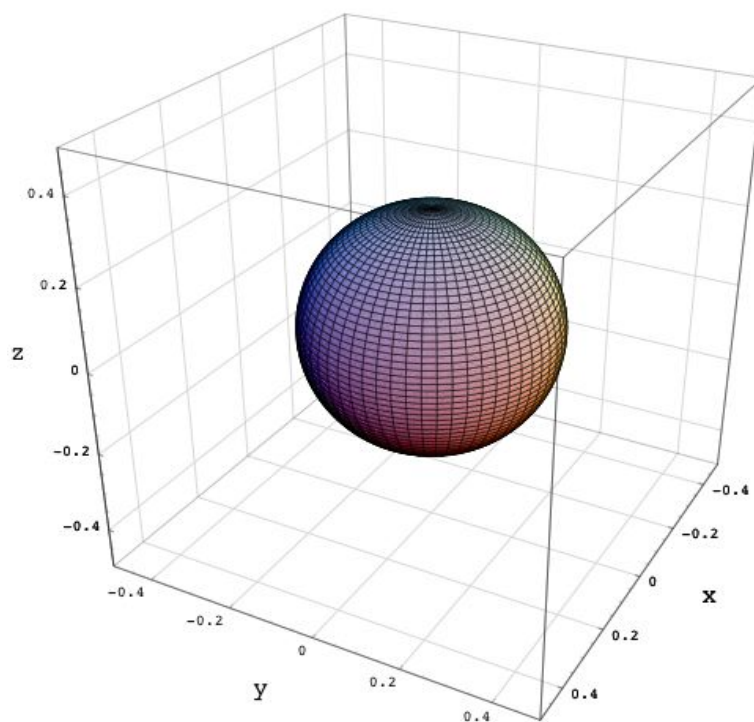


Fig. 4 A plot of the $L^2[\mathcal{S}^2]$ -normalized variance $\Delta S_{\mathbf{n}}^2 / \|\Delta S_{\mathbf{n}}^2\|$ for the tetrahedral state $|\psi_{\text{tet}}\rangle$ given in Eq. (9a). Compare Fig. 1.

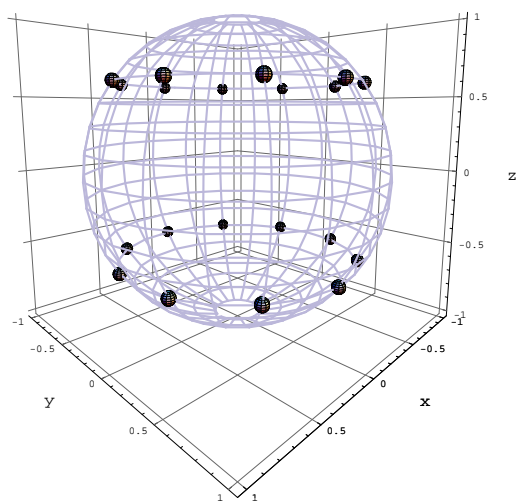


Fig. 5 The Majorana representation of the anticoherent state in Eq. (11), for $s = 10$. The unit sphere is shown in wire mesh. Points in the Majorana representation are shown as small spheres.