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On the Quantum Correction of Black Hole Thermodynamics

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Abstract: Bekenstein-Hawking Black hole thermodynamics should be corrected to incorporate quantum gravitational effects. Generalized Uncertainty Principle(GUP) provides a perturbational framework to perform such modifications. In this paper we consider the most general form of GUP to find black holes thermodynamics in microcanonical ensemble. Our calculation shows that there is no logarithmic pre-factor in perturbational expansion of entropy. This feature will solve part of controversies in literatures regarding existence or vanishing of this pre-factor.

Keywords: Quantum Gravity, Generalized Uncertainty Principle, Black Holes Thermodynamics
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1. Introduction

Quantum geometry, string theory and loop quantum gravity all indicate that measurements in quantum gravity should be governed by generalized uncertainty principle[1-5]. As a result, there is a minimal length scale of the order of Planck length which can not be probed. In the language of string theory, this is related to the fact that a string can not probe distances smaller than its length. Therefore, it seems that a reformulation of quantum theory to incorporate gravitational effects from very beginning is necessary to investigate Planck scale physics. Introduction of this idea, has drawn considerable attentions and many authors have considered various problems in the framework of generalized uncertainty principle[6-20]. Quantum gravitational induced corrections to black hole thermodynamics as a consequence of GUP are studied with details in liter-

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atures. Adler and his coworkers[21] have argued that contrary to standard viewpoint, GUP may prevent small black holes total evaporation in exactly the same manner that the ordinary uncertainty principle prevents the Hydrogen atom from total collapse. They have considered these black holes remnants as a possible source of dark matter. Medved and Vagenas[22], have recently formulated the quantum corrected entropy of black holes in terms of an expansion and have claimed that this expansion is consistent with all previous findings. Bolen and Cavaglia, have obtained thermodynamical properties of Schwarzschild anti-de Sitter black holes using GUP [23]. They have considered two limits of their equations, quantum gravity limit and usual quantum mechanical regime and in each circumstances they have interpreted their results. Action for the exact string black hole has been considered by Grumiller and he has found exact relation for entropy of a string black hole[24]. Existence or vanishing of logarithmic prefactor in the expansion of black hole entropy has been considered in details by Medved. He has argued in [25] that "the best guess for the prefactor might simply be zero" regarding to mutual cancelation of canonical and microcanonical contributions. But later, considering some general considerations of ensemble theory, he has argued that canonical and microcanonical corrections could not cancel each other to result in vanishing logarithmic pre-factor in entropy[26]. Meanwhile, Hod has employed statistical arguments that constrains this prefactor to be a non-negative integer[27]. There are other literatures considering logarithmic corrections to black hole entropy[28,29], but there is no explicit statement about the ultimate value of this prefactor.

Here, using generalized uncertainty principle in its most general form as our primary input, we find explicit perturbational expansion of black hole entropy in microcanonical ensemble. By computing the coefficients of this expansion, we will show that there is no logarithmic prefactor in expansion of microcanonical entropy.

2. Generalized Uncertainty Principle

Usual uncertainty principle of quantum mechanics, the so-called Heisenberg uncertainty principle, should be re-formulated regarding to non-commutative nature of spacetime. It has been indicated that in quantum gravity there exists a minimal observable distance on the order of the Planck length which in the context of string theories, this observable distance is referred to GUP[1-5],[30-33]. A generalized uncertainty principle can be formulated as

$$\delta x \geq \frac{\hbar}{2\delta p} + const. G\delta p, \tag{1}$$

which, using string theoretical arguments regarding the minimal nature of l_p[4], can be written as

$$\delta x \geq \frac{\hbar}{2\delta p} + \alpha^2 l_p^2 \frac{\delta p}{2\hbar} \tag{2}$$

The corresponding Heisenberg commutator now becomes,

$$[x, p] = i\hbar(1 + \alpha' p^2). \tag{3}$$

Note that commutator (3) is not the direct consequence of relation (2), but can be considered as one of its consequences[11]. α is positive and independent of δx and δp but may in general depend to the expectation values of x and p. In the same manner one can consider the following generalization,

$$\delta x\delta p \geq \frac{\hbar}{2}\Big(1+\frac{\beta^2}{l_p^2}(\delta x)^2\Big), \tag{4}$$

which indicates the existence of a minimal observable momentum. It is important to note that GUP itself can be considered as a perturbational expansion[11]. In this viewpoint, one can consider a more general statement of GUP as follows

$$\delta x\delta p \geq \frac{\hbar}{2}\Bigg(1+\frac{\alpha^2 l_p^2}{\hbar^2}(\delta p)^2+\frac{\beta^2}{l_p^2}(\delta x)^2+\gamma\Bigg), \tag{5}$$

where α, β and γ are positive and independent of δx and δp but may in general depend to the expectation values of x and p. Here, Planck length is defined as $l_p=\sqrt{\frac{\hbar G}{c^3}}$. Note that (5) leads to nonzero minimal uncertainty in both position $(\delta x)_{min}$ and momentum $(\delta p)_{min}$. In which follows, we use this more general form of GUP as our primary input and construct a perturbational framework to find thermodynamical properties of black hole and their quantum gravitational corrections. It should be noted that since GUP is a model independent concept[6], the results which we obtain are consistent with any promising theory of quantum gravity.

3. Black Holes Thermodynamics

Consider the most general form of GUP as equation (5). A simple calculation gives,

$$\delta x \simeq \frac{l_p^2\delta p}{\beta^2\hbar}\left[1\pm\sqrt{1-\beta^2\Bigg(\alpha^2+\frac{\hbar^2(\gamma+1)}{l_p^2(\delta p)^2}\Bigg)}\,\right]. \tag{6}$$

Here, to achieve standard values (for example $\delta x\delta p \geq \hbar$) in the limit of $\alpha=\beta=\gamma=0$, we should consider the minus sign. One can minimize δx to find

$$(\delta x)_{min} \simeq \pm\alpha l_p\sqrt{\frac{(1+\gamma)}{1-\alpha^2\beta^2}}. \tag{7}$$

The minus sign, evidently has no physical meaning for minimum of position uncertainty. Therefore, we find

$$(\delta x)_{min} \simeq \alpha l_p\sqrt{\frac{(1+\gamma)}{1-\alpha^2\beta^2}}. \tag{8}$$

This equation gives the minimal observable length on the order of Planck length. Since in our definition, α and β are dimensionless positive constant always less than one(extreme quantum gravity limit), $(\delta x)_{min}$ is defined properly. Equation (5) gives also

$$\delta p \simeq \frac{\hbar\delta x}{\alpha^2 l_p^2}\left[1\pm\sqrt{1-\alpha^2\Bigg(\beta^2+\frac{l_p^2(\gamma+1)}{(\delta x)^2}\Bigg)}\right]. \tag{9}$$

Here to achieve correct limiting results we should consider the minus sign in round bracket. From a heuristic argument based on Heisenberg uncertainty relation, one deduces the following equation for Hawking temperature of black holes[21],

$$T_H \approx \frac{\delta pc}{2\pi} \tag{10}$$

Based on this viewpoint, but now using generalized uncertainty principle in its most general form, modified black hole temperature in GUP is,

$$T_H^{GUP} \approx \frac{\hbar c \delta x}{2\pi\alpha^2 l_p^2}\left[1-\sqrt{1-\alpha^2\left(\beta^2+\frac{l_p^2(\gamma+1)}{(\delta x)^2}\right)}\,\right]. \tag{11}$$

Now consider a quantum particle that starts out in the vicinity of an event horizon and then ultimately absorbed by black hole. For a black hole absorbing such a particle with energy E and size R, the minimal increase in the horizon area can be expressed as [34]

$$(\Delta A)_{min} \geq \frac{8\pi l_p^2 ER}{\hbar c}, \tag{12}$$

then one can write

$$(\Delta A)_{min} \geq \frac{8\pi l_p^2 \delta pc\delta x}{\hbar c}, \tag{13}$$

where $E \sim c\delta p$ and $R \sim \delta x$. Using equation (9)(with minus sign) for δp and defining $A = 4\pi(\frac{\delta x_{min}}{2})^2$, we find

$$(\Delta A)_{min} \simeq \frac{8A}{\alpha^2}\left[1-\sqrt{1-\alpha^2\left(\beta^2+\frac{\pi l_p^2(\gamma+1)}{A}\right)}\right]. \tag{14}$$

Now we should determine δx. Since our goal is to compute microcanonical entropy of a large black hole, near-horizon geometry considerations suggests the use of inverse surface gravity or simply twice the Schwarzschild radius for δx. Therefore, $\delta x \approx 2r_s$ and defining $4\pi r_s^2 = A$ and $(\Delta S)_{min} = b = constant$, then it is easy to show that,

$$\frac{dS}{dA} \simeq \frac{(\Delta S)_{min}}{(\Delta A)_{min}} \simeq \frac{b\alpha^2}{8A\left[1-\sqrt{1-\alpha^2\left(\beta^2+\frac{\pi l_p^2(\gamma+1)}{A}\right)}\right]}. \tag{15}$$

Three point should be considered here. First note that b can be considered as one bit of information since entropy is an extensive quantity. Considering calibration factor of Bekenstein as $\frac{ln2}{2\pi}$, the minimum increase of entropy(i.e. b), should be $ln2$. Secondly, note that $\frac{dS}{dA} \simeq \frac{(\Delta S)_{min}}{(\Delta A)_{min}}$ holds since this is an approximate relation and give only relative changes of corresponding quantities. As the third remarks, our approach considers microcanonical ensemble since we are dealing with a Schwarzschild black hole of fixed mass. Now we should perform integration. There are two possible choices for lower limit of integration, $A = 0$ and $A = A_p$. Existence of a minimal observable length leads to existence of a minimum event horizon area, $A_p = 4\pi\left(\frac{(\delta x)_{min}}{2}\right)^2$. So it is physically reasonable to set A_p as lower limit of integration. This is in accordance with existing picture[21]. Based on these arguments, we can write

$$S \simeq \int_{A_p}^{A} \frac{b\alpha^2}{8A\left[1-\sqrt{1-\alpha^2\left(\beta^2+\frac{\pi l_p^2(\gamma+1)}{A}\right)}\right]}dA. \tag{16}$$

Integration gives,

$$S \simeq \mu\left[\ln\left|\frac{-2\sqrt{A(\zeta A+\eta)}+A(\zeta+1)+\eta}{-2\sqrt{A_p(\zeta A_p+\eta)}+A_p(\zeta+1)+\eta}\right| + \sqrt{\zeta}\ln\left|\frac{\eta+2\zeta A+2\sqrt{\zeta A(\zeta A+\eta)}}{\eta+2\zeta A_p+2\sqrt{\zeta A_p(\zeta A_p+\eta)}}\right|\right] \tag{17}$$

where,

$$\mu = \frac{b}{8\beta^2}, \qquad \eta = -\pi\alpha^2 l_p^2(\gamma+1), \qquad \zeta = 1-\alpha^2\beta^2,$$

$$A_p = \frac{\pi\alpha^2 l_p^2(1+\gamma)}{(1-\alpha^2\beta^2)} \tag{18}$$

This is the most general form of the black hole entropy which can be obtained from perturbational approach based on GUP.

Expansion of (17) gives

$$S \simeq \sum_{n=1}^{\infty} D_n (A-A_p)^n. \tag{19}$$

The coefficients of this expansion have very complicated form. The first coefficient is

$$D_1 = \mu\left(\frac{-\frac{2\zeta A_p+\eta}{\sqrt{A_p(\zeta A_p+\eta)}}+\zeta+1}{-2\sqrt{A_p(\zeta A_p+\eta)}+A_p(\zeta+1)+\eta} + \sqrt{\zeta}\frac{\frac{2\zeta^2 A_p+\zeta\eta}{\sqrt{\zeta A_p(\zeta A_p+\eta)}}+2\zeta}{2\sqrt{\zeta A_p(\zeta A_p+\eta)}+2\zeta A_p+\eta}\right). \tag{20}$$

The matter which is important in our calculations is the fact that expansion (19) has no logarithmic term. In other words, since expansion (19) contains only integer power of $A - A_p$, we conclude that in microcanonical ensemble, there is no logarithmic corrections due to quantum gravitational effects for thermodynamics of black holes. Adler *et al* have found vanishing entropy for remnant in their paper[21]. In other words, their result for entropy vanishes when one considers Planck mass limit. In our framework, when $A = A_p$, one finds $S = 0$ and therefore remnant has zero entropy. A result which physically can be acceptable since small classical fluctuations are not allowed at remnant scales because of the existence of the minimum length.

4. Summary

In this paper, using generalized uncertainty principle in its most general form as our primary input, we have calculated microcanonical entropy of a black hole. We have shown that in perturbational expansion there is no logarithmic pre-factor, which has been the source of controversies in literatures. Actually in calculation of entropy we should compute the number of possible microstates of the system and there are two possible choices for corresponding ensemble: canonical and microcanonical ensemble. We have shown that the contribution of microcanonical ensemble itself is vanishing. If there is any contribution related to canonical ensemble, it cannot cancel vanishing contribution of microcanonical ones. This argument resolves part of controversies regarding mutual cancelation of two contributions as have been indicated in introduction.

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