On the Finite Caputo and Finite Riesz Derivatives

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Abstract: In this paper, we give some properties of the left and right finite Caputo derivatives. Such derivatives lead to finite Riesz type fractional derivative, which could be considered as the fractional power of the Laplacian operator modelling the dynamics of many anomalous phenomena in super-diffusive processes. Finally, the exact solutions of certain fractional diffusion partial differential equations are obtained by using the Adomain decomposition method and some new diffusion-wave equations are presented.

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1. Introduction

It is known that the classical calculus provides a power tool to explain and to model many important dynamically processes in most parts of applied areas of the sciences. But the experiments and reality teach us that there are many complex systems in nature with anomalous dynamics, including biology, chemistry, physics, geology, astrophysics and social sciences, and more in particular in transport of chemical contaminant through water around rocks, dynamics of viscoelastic materials as polymers, diffusion of pollution in the atmosphere, diffusion processes involving cells, signals theory, control theory, electromagnetic theory, and many more.

In most of the above-mentioned cases, this kind of anomalous processes have a macroscopic complex behavior, and their dynamics cannot be characterized by classical derivative models. It is also important to remark that the anomalous behavior of many complex

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processes include the multi-scaling in the time and space variables and so also the fractal characteristics of the media.

Stochastic tools have been used extensively during last 40 years to describe the dynamics of many anomalous processes, as such sub and super-diffusive processes. But the connections of these statistical models with some fractional differential equations, involving the fractional integral and derivative operators (Riemann-Liouville, Caputo, Liouville or Weyl and Riesz) have been formally established during the last 15 or 20 years and more intensively during the last 10 years by so many researches of a large list of different fields. It is possible to find so many reference of such fractional models and the applications of the fractional differential equations in the following monographic works by I. Podlubny (1999 [22]), B.J. West (1999 [26]), R. Metzler and J. Klafter (2000 [21]), E. Hilfer (Ed.) (2000 [13]), G.M. Zaslavsky (2005 [27]), and A.A. Kilbas, H.M. Sivastava and J.J. Trujillo (2006 [15]).

The Riemann-Liouville fractional integrals of order $\alpha > 0$, for suitable functions $\varphi(x)$ $(x \in \mathbb{R})$, in the interval $(a,b) \subseteq \mathbb{R}$, are well known (see for example [22, 24])

$$I_{a+}^{\alpha}\varphi(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-\zeta)^{\alpha-1} \varphi(\zeta)\ d\zeta, \tag{1.1}$$

$$I_{b-}^{\alpha}\varphi(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (\zeta-x)^{\alpha-1} \varphi(\zeta)\ d\zeta, \tag{1.2}$$

where $\Gamma(.)$ is the Gamma function. We refer to such fractional integrals I_{a+}^{α} and I_{b-}^{α} as left and right handed fractional integrals, respectively. Complementarily we define $I_{a+}^{0} = I_{b-}^{0} := I$ (Identity operator), as it is usual. Let us point out the namely the semigroup property for such fractional operators (see, for example, [24]), which can be written as follows: $\forall \alpha, \beta \geq 0$,

$$I_{a+}^{\alpha} I_{a+}^{\beta} = I_{a+}^{\beta} I_{a+}^{\alpha} = I_{a+}^{\alpha+\beta},$$
$$I_{b-}^{\alpha} I_{b-}^{\beta} = I_{b-}^{\beta} I_{b-}^{\alpha} = I_{b-}^{\alpha+\beta}.$$

In section 2. of this paper, the corresponding right Caputo fractional derivative is introduced, as the corresponding complementary operator to the well known left Caputo fractional derivative, according to the left and right handed fractional integrals given in (1.1) and (1.2). Some properties of such derivative are presented. In section 2.4, we obtain the numerical solutions of some simple examples of fractional differential equations, involving both mentioned Caputo derivatives, by using the Adomian decomposition method (ADM). As an application, in section **??**, we introduce a new fractional finite Riesz derivative. Such fractional derivative could be considerd as the fractional power of the known Laplacian operator. Finally, we establish the exact solutions of certain boundary problems for the fractional diffusion-wave partial differential equations by using the ADM. We remark that the finite Riesz derivative could play an important role in modelling the dynamics of some anomalous phenomena, for example to model the dynamics of super-diffusive processes, as a alternative to the recently models presented by several authors by using the Riesz derivative (see for example [19, 20]).

2. Left and Right Caputo Fractional Derivatives

In this section we present the fractional differentiation on finite interval in the framework of the Caputo fractional calculus. Particular attention is devoted to the ADM to solve Cauchy type problems of fractional differential equations involving the Caputo derivatives in a way accessible to applied scientists. By applying such analytic-numerical technique we shall derive the explicit solutions of some simple linear differential equations of fractional order.

Definition 1 *[10, 24] (left and right handed Riemman-Liouville fractional derivatives)*

For functions $f(x)$ given in the interval $[a,b]$, The left and the right handed Riemman-Liouville fractional derivatives are given by

$$\mathfrak{D}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha-m+1}}, \tag{2.1}$$

$$\mathfrak{D}_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_x^b \frac{f(t)\,dt}{(t-x)^{\alpha-m+1}}, \tag{2.2}$$

where $m = [\alpha]+1$, respectively. Fractional derivatives (2.1) *and* (2.2) *are usually named Riemman-Liouville fractional derivatives.*

Definition 2 *[10] (finite Weyl fractional derivative)*

The finite Weyl fractional derivative of order α, $m-1<\alpha\le m$ of the function $f(t)\in \mathfrak{C}^m$, $t\in(0,a)$ is defined by

$$W_a^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^a (\zeta - t)^{m-\alpha-1} f^m(\zeta)\ d\zeta,$$

where $\mathfrak{C}^m$ is the class of all functions $f:(0,a)\to\mathbb{X}$, with $f^{(k)}(a)=0$, $k=0,1,...,m-1$.

Let $m = 1,2,\cdots$ and $AC^m([a,b])$ the space of functions φ which have continuous derivatives up to order $m-1$ on $[a,b]$ with $\varphi^{(m-1)}\in AC([a,b])$. Assuming that for $m-1<\alpha<m$ and a suitable function $\varphi(x)$ (for example, $\varphi\in AC^m([a,b])$). The corresponding left and right Caputo fractional derivatives, of order $\alpha>0$, to the Riemann-Liouville fractional integrals given in (1.1) and (1.2) are given by

$$\mathbf{D}_{a+}^{\alpha}\varphi(x) = I_{a+}^{m-\alpha} D^m \varphi(x), \tag{2.3}$$

$$\mathbf{D}_{b-}^{\alpha}\varphi(x) = (-1)^m I_{b-}^{m-\alpha} D^m \varphi(x), \tag{2.4}$$

2.1 Some properties of the left and right Caputo fractional operators

First we introduce the following Proposition:

Proposition 1 *[12]*

For a well-behaved function $\varphi(x)$ (with $x\in(a,b)$) we have

$$\left\{\begin{array}{l} D^n I_{b-}^n \varphi(x) = (-1)^n \varphi(x), \\ I_{b-}^n D^n \varphi(x) = (-1)^n \left\{\varphi(x) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(b)}{k!}(b-x)^k\right\}, \end{array}\right. \quad a<x<b,$$

and

$$D^m I^m_{a+} = I \text{ and } D^m I^m_{b-} = (-1)^m I.$$

Assume that $\varphi(x)$ is a suitable function, for example $\varphi(x) \in AC^m([a,b])$. For $\alpha \in (m-1,m)$ and $D^m\varphi(x) \neq 0$, we have the following Theorem:

Theorem 1

(1) $\lim_{\alpha\to m} \mathbf{D}^\alpha_{b-}\varphi(x) = (-1)^m \varphi^{(m)}(x)$.

(2) If $\mu \geq 0$, *and* $\alpha \in (m-1,m]$, *we have*

$$\mathbf{D}^\alpha_{b-}x^\mu = \left[\prod_{k=0}^{m-1}(\mu-k)\right]\frac{(-1)^m(b-x)^{m-\alpha}b^{\mu-m}}{\Gamma(m-\alpha+1)}$$
$$\times F_{1,2}(m-\mu,1;m-\alpha+1;\frac{b-x}{b}),\ b\neq 0.$$

(3)

$$I^\alpha_{b-}\mathbf{D}^\alpha_{b-}\varphi(x) = (-1)^m\varphi(x) + \sum_{k=0}^{m-1}\frac{(-1)^{m-k-1}(b-x)^k}{k!}D^k\varphi(b).$$

(4) For $\beta > m \geq \alpha > m-1$,

$$I^\beta_{b-}\mathbf{D}^\alpha_{b-}\varphi(x) = (-1)^m I^{\beta-\alpha}_{b-}\varphi(x) + \sum_{k=0}^{m-1}\frac{(-1)^{m-k-1}(b-x)^{k+\beta-\alpha}}{\Gamma(k+\beta-\alpha+1)}D^k\varphi(b).$$

(5) For $m \geq \beta > \alpha > m-1$,

$$I^\alpha_{b-}\mathbf{D}^\beta_{b-}\varphi(x) = (-1)^m D^{\beta-\alpha}_{b-}\varphi(x) + \sum_{k=0}^{m-1}\frac{(-1)^{m-k-1}(b-x)^{k+\beta-\alpha}}{\Gamma(k+\beta-\alpha+1)}D^{k+1}\varphi(b).$$

(6) Assume that $\beta > m \geq \alpha > m-1$ *and* $\beta = m+\nu$. *If* $D^m I^\beta_{b\pm}\varphi(x) \neq 0$, *then*

$$\mathbf{D}^\alpha_{b\pm}I^\beta_{b\pm}\varphi(x) = (\pm 1)^m I^{\beta-\alpha}_{b\pm}\varphi(x).$$

(7) Assume that $m \geq \alpha > m-1 \geq n$ *and* $k+n=m$. *If* $D^m I^n_{b\pm}\varphi(x) \neq 0$, *then*

$$\mathbf{D}^\alpha_{b\pm}I^n_{b\pm}\varphi(x) = (\pm 1)^n\ \mathbf{D}^{\alpha-n}_{b\pm}\varphi(x).$$

Proof.

(1)

$$\lim_{\alpha\to m}\mathbf{D}^\alpha_{b-}\varphi(x) = (-1)^m\lim_{\alpha\to m}\left(\frac{\varphi^{(m)}(b)(b-x)^{m-\alpha}}{\Gamma(m-\alpha+1)} - \frac{1}{\Gamma(m-\alpha+1)}\int_x^b (s-x)^{m-\alpha}\varphi^{(m+1)}(s)\,ds\right)$$
$$= (-1)^m\left(\varphi^{(m)}(b) - \int_x^b\varphi^{(m+1)}(s)\,ds\right) = (-1)^m\varphi^{(m)}(x).$$

So, in (2.3) and (2.4) we can take $\alpha \in (m-1, m]$.

(2) From the fact that

$$I^{\beta}_{b-} x^{\ell} = \frac{(b-x)^{-\beta} b^{\ell}}{\Gamma(1-\beta)} F_{1,2}(-\ell, 1; 1-\alpha; \frac{b-x}{b}),\ \beta > 0,$$

where $F_{1,2}(a,b;c;z)$ *is the hypergeometric distribution introduced as [3]*

$$F_{1,2}(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!},$$

$c \neq 0, -1, -2, \cdots,\ |z| \leq 1$, *and* $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$. *So, we have*

$$\mathbf{D}^{\alpha}_{b-} x^{\mu} = \left[\prod_{k=0}^{m-1} (\mu - k)\right] \frac{(-1)^m (b-x)^{m-\alpha} b^{\mu-m}}{\Gamma(m-\alpha+1)} \times F_{1,2}(m-\mu, 1; m-\alpha+1; \frac{b-x}{b}),\ b \neq 0,$$

for $\mu \geq 0$, *and* $\alpha \in (m-1, m]$.

(3)

$$I^{\alpha}_{b-} \mathbf{D}^{\alpha}_{b-} \varphi(x) = I^{\alpha}_{b-} I^{m-\alpha}_{b-} D^m \varphi(x) = I^m_{b-} D^m \varphi(x) = (-1)^m \varphi(x) + \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} (b-x)^k}{k!} D^k \varphi(b).$$

(4) For $\beta > m \geq \alpha > m-1$,

$$I^{\beta}_{b-} \mathbf{D}^{\alpha}_{b-} \varphi(x) = I^{\beta}_{b-} I^{m-\alpha}_{b-} D^m \varphi(x) = I^{m+\beta-\alpha}_{b-} D^m \varphi(x) = (-1)^m I^{\beta-\alpha}_{b-} \varphi(x) + \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} (b-x)^{k+\beta-\alpha}}{\Gamma(k+\beta-\alpha+1)} D^k \varphi(b).$$

(5) For $m \geq \beta > \alpha > m-1$,

$$I^{\alpha}_{b-} \mathbf{D}^{\beta}_{b-} \varphi(x) = I^{\alpha}_{b-} I^{m-\beta}_{b-} D^m \varphi(x) = I^{m+\alpha-\beta}_{b-} D^{m-1} (D\varphi(x)) = (-1)^m D^{\beta-\alpha}_{b-} \varphi(x) + \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} (b-x)^{k+\beta-\alpha}}{\Gamma(k+\beta-\alpha+1)} D^{k+1} \varphi(b).$$

(6) Assume that $\beta > m \geq \alpha > m-1$ *and* $\beta = m + \nu$. *If* $D^m I^{\beta}_{b\pm} \varphi(x) \neq 0$, *then*

$$\mathbf{D}^{\alpha}_{b\pm} I^{\beta}_{b\pm} \varphi(x) = I^{m-\alpha}_{b\pm} D^m I^m_{b\pm} I^{\nu}_{b\pm} \varphi(x) = (\pm 1)^m I^{m-\alpha+\nu}_{b\pm} \varphi(x) = (\pm 1)^m I^{\beta-\alpha}_{b\pm} \varphi(x).$$

(7) Assume that $m \geq \alpha > m-1 \geq n$ *and* $k + n = m$. *If* $D^m I^n_{b\pm} \varphi(x) \neq 0$, *then*

$$\mathbf{D}^{\alpha}_{b\pm} I^n_{b\pm} \varphi(x) = I^{m-\alpha}_{b\pm} D^m I^n_{b\pm} \varphi(x) = (\pm 1)^n\ \mathbf{D}^{\alpha-n}_{b\pm} \varphi(x).$$

■

It is easy to check that the following Lemma is true.

Lemma 1

If $\varphi^{(p)}(b) = 0$, *for all* $p = 0, 1, 2, \cdots, [\alpha]$. *Then [10]*

(*a*) $\lim_{\alpha\to p} \mathbf{D}^{\alpha}_{b-}\varphi(x) = \mathbf{D}^{p}_{b-}\varphi(x) = \left(\mp\frac{d}{dx}\right)^{p}\varphi(x)$ *and*

(*b*) *The family* $\left\{\mathbf{D}^{\alpha}_{b-};\ \alpha \in \mathbb{R}\right\}$ *is a multiplicative group.*

(*c*) *If* $\mu > 0$, $\alpha \in (m-1, m]$ $(m \geq 1)$*and* $b > x$, *then*

$$\mathbf{D}^{\alpha}_{x,b-}(b-x)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(b-x)^{\mu-\alpha}.$$

2.2 The relationship between the operators $\mathbf{D}^{\alpha}_{b-}$ and $\mathfrak{D}^{\alpha}_{b-}$

The left and the right handed Riemman-Liouville fractional derivatives are given as [24],

$$\mathfrak{D}^{\alpha}_{a+}\varphi(x) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^{m}\int_{a}^{x}\frac{\varphi(t)\,dt}{(x-t)^{\alpha-m+1}}, \tag{2.5}$$

$$\mathfrak{D}^{\alpha}_{b-}\varphi(x) = \frac{(-1)^{m}}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^{m}\int_{x}^{b}\frac{\varphi(t)\,dt}{(t-x)^{\alpha-m+1}}, \tag{2.6}$$

where $m = [\alpha] + 1$, respectively.

When $\alpha \in (0, 1)$ we have the following relationship between the operators $\mathbf{D}^{\alpha}_{b-}$ and $\mathfrak{D}^{\alpha}_{b-}$ (see [24])

$$\mathfrak{D}^{\alpha}_{b-}\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\varphi(b)}{(b-x)^{\alpha}} - \mathbf{D}^{\alpha}_{b-}\varphi(x)\right].$$

Such relations can be extended easily to the case that $\alpha \in (m-1, m]$ as follows:

$$\mathfrak{D}^{\alpha}_{b-}\varphi(x) = \mathbf{D}^{\alpha}_{b-}\varphi(x) + \sum_{k=0}^{m-1}\frac{(-1)^{m-k}(b-x)^{k-\alpha}}{\Gamma(k-\alpha+1)}\left[D^{k}\varphi(x)\right]_{x=b}. \tag{2.7}$$

So, if $\alpha \in (m-1, m]$ and $\varphi^{(k)}(x) = 0$ for all $k = 0, 1, 2, \cdots, m-1$, then

$$\mathbf{D}^{\alpha}_{b-}\varphi(x) \equiv \mathfrak{D}^{\alpha}_{b-}\varphi(x).$$

Remark 1

(A) *From the relationship* (2.7) *between the operators* $\mathbf{D}^{\alpha}_{b-}$ *and* $\mathfrak{D}^{\alpha}_{b-}$, *we find that the finite Weyl derivative (Definition 2) coincide with the right handed Riemman-Liouville fractional derivative (Definition 1).*

(B) *By passing to the limit for* $\alpha \to m^{-}$ *and using* (2.7), *we have*

$$\lim_{\alpha\to m^{-}}\mathfrak{D}^{\alpha}_{b-}\varphi(x) = (-1)^{m}\varphi^{(m)}(x) + \sum_{k=0}^{m-1}(-1)^{m-k}\delta^{(m-k-1)}(b-x)\,\varphi^{(k)}(b)$$
$$\neq (-1)^{m}\varphi^{(m)}(x).$$

So, from Theorem 1 (1), *we see that the right Caputo fractional derivative* $\mathbf{D}^{\alpha}_{b-}$ *have the continuation property, in other hand this property is not verified in the case of the right handed Riemman-Liouville fractional derivative.*

The following results are well known (see, for example [15]). Here we will present a proof of it by using the ADM as an example. We must mention that the ADM to solve the ordinary fractional differential equations has been used by several authors with good results (see, for example, [23, 25]).

Lemma 2 *Let $x \in (a, b) \subseteq \mathbb{R}$, $\varphi_a \in \mathbb{R}$, and $\alpha \in (0, 1]$. Then the following Cauchy type problem*

$$ {}^{c}D^{\alpha}_{a+}\varphi(x) = \varphi(x) \tag{2.8}$$

$$ \varphi(a) = \varphi_a, \tag{2.9}$$

have the unique solution

$$\varphi(x) = \varphi_a E_\alpha\left((x-a)^\alpha\right). \tag{2.10}$$

Proof. First of all we can be sure that the problem (2.8)-(2.9) have a unique continuous solution by the application of the known existence theorem for Cauchy type problems involving the left Caputo derivative (see for example, [9] and [15]). Here, we apply the ADM to get the exact solution of the problem (2.8)-(2.9). In fact by the mentioned theorem we know that such problem is equivalent to the following Volterra integral equation

$$\varphi(x) = \varphi_a + I^{\alpha}_{a+}\varphi(x).$$

Now solving the last equality by using the ordinary ADM, with $\varphi_0(x) = \varphi_a$, we have

$$\begin{aligned}\varphi_1(x) &= I^{\alpha}_{a+}\varphi_0 = \varphi_a \frac{(x-a)^\alpha}{\Gamma(1+\alpha)},\\ &\vdots\\ \varphi_n(x) &= I^{\alpha}_{a+}\varphi_{n-1} = \varphi_a \frac{(x-a)^{n\alpha}}{\Gamma(1+n\alpha)},\ n \geq 1.\\ &\vdots\end{aligned}$$

Then, we can conclude that the unique solution of (2.8)-(2.9) is given by (2.10). ■

Lemma 3 *The unique continuous solution of the following Cauchy type problem*

$$ {}^{c}D^{\alpha}_{b-}\varphi(x) = \varphi(x),\ x \in (a, b) \subseteq \mathbb{R},\ \alpha \in (0, 1], \tag{2.11}$$

$$ \varphi(b) = \varphi_b \tag{2.12}$$

is given by

$$\varphi(x) = \varphi_b E_\alpha\left((b-x)^\alpha\right). \tag{2.13}$$

Proof. As in Lemma 2, we apply the corresponding existence theorem to Cauchy type problems involving the right Caputo derivative ${}^{c}D^{\alpha}_{b-}$, from which we know that the problem (2.11)-(2.12) has a unique continuous solution which is equivalent to the integral equation

$$\varphi(x) = \varphi_b - I^{\alpha}_{b-}\varphi(x).$$

Now we can apply the ADM, with $\varphi_0(x) = \varphi_b$, so we obtain that

$$\varphi_n(x) = I^{\alpha}_{b-}\varphi_{n-1} = \varphi_b \left(\frac{(b-x)^{n\alpha}}{\Gamma(1+n\alpha)}\right),$$

and we can conclude that (2.13) is the solution of (2.11)-(2.12). ■

2.3 A physical model

Consider a semi-infinite rod with a thermal source producing a temperature wave $f(t)$ localized at the front of the rod. The temperature wave intensity of the rod is registered by a detector at the times $t = 0$ and $t = b$.

An efficient way of obtaining some information about a layered medium structure and makeup is to measure its temperature wave expose reaction. The suggested technique allows us to restore the medium basic parameters with arbitrary accuracy.

Notice that in this case we study the primary phase of temperature oscillation in order to solve inverse problems. In the case of media exhibiting memory. So, we have the following boundary value problem in $\Omega = \{(x,t) : x \geq 0,\ t \in [0,b]\}$:

$$\begin{aligned} \mathbf{D}_{t,b-}^{\alpha}\varphi(x,t) &= \lambda^2 \frac{\partial^2 \varphi(x,t)}{\partial x^2},\ \alpha \in (0,2],\ x \geq 0,\ t \in [0,b], \\ \varphi(x,0) &= p(x),\ x \geq 0, \\ \varphi(x,b) &= q(x),\ x \geq 0,\ \alpha \in (1,2], \\ \varphi(0,t) &= f(t),\ t \in [0,b] \\ \varphi_x(0,t) &= g(t),\ t \in [0,b]. \end{aligned} \tag{2.14}$$

where λ is the thermal diffusivity and α is the anomalous diffusion index, (see for example [4]).

2.4 Applications of the Adomian method to solve fractional models

The ADM has been used to solve the problem (2.14) in the case $\alpha = 1, 2$ by Lesnic [16, 17, 18]. In this section we extend the ADM given in [16, 17, 18] to obtain the explicit solution of this boundary problem in the case that $\alpha \in (0,1)$ and $\alpha \in (1,2)$ for the following examples.

Example 1 *Consider the fractional boundary value problem*

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &= \mathbf{D}_{t,b-}^{2\alpha} u(x,t),\ \alpha \in (\frac{1}{2},1],\ x > 0,\ t \in (0,b), \\ u(0,t) &= \frac{2(b-t)^{2\alpha}}{\Gamma(2\alpha+1)},\ t \in (0,b), \\ u_x(0,t) &= 0,\ t \in (0,b), \\ u(x,0) &= x^2 + \frac{2b^{2\alpha}}{\Gamma(2\alpha+1)},\ u(x,b) = x^2. \end{aligned}$$

The solution of this problem is given by

$$u(x,t) = \frac{2(b-t)^{2\alpha}}{\Gamma(2\alpha+1)} + x^2.$$

Let us define the starting term

$$u_0(x,t) = \frac{1}{2}\left[p(t) + xq(t) + f_0(x) + \left(\frac{b-t}{b}\right)^{\alpha}(f_b(x) - f_0(x))\right],$$

and the recurrence relation as [16, 17, 18],

$$u_{n+1}(x,t) = \frac{1}{2}\left[L_t^{-1}L_{xx} + L_{xx}^{-1}L_t\right]u_n,\ n \geq 0,$$

where $L_t = {}^c D_{t,b-}^{2\alpha}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$,

$$L_t^{-1}w(t) = \int_t^b \frac{(s-t)^{2\alpha-1}}{\Gamma(2\alpha)}w(s)ds + \left(\frac{b-t}{b}\right)^\alpha \int_0^b \frac{(b-s)^{2\alpha-1}}{\Gamma(2\alpha)}w(s)ds,$$

$$L_{xx}^{-1} = \int_0^x dx' \int_0^{x'} dx''.$$

The ADM gives the solution in the following decomposed form

$$u(x,t) = \lim_{N\to\infty}\Phi_N(x,t),\quad \Phi_N(x,t) = \sum_{n=0}^{N} u_n(x,t),\ N \geq 0.$$

In this example, we use this starting term and recurrence relation in the decomposition method to avoid the error which may be appear in our solution (see [1, 2, 16, 17, 18]). Now, using Lemma 1 (*c*) *and Theorem 1* (3), *with* $m = 2$, *we can conclude that*

$$u(x,t) = -\frac{2tb^\alpha}{\Gamma(2\alpha+1)} + \left(x^2 + \frac{2(b-t)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2tb^\alpha}{\Gamma(2\alpha+1)}\right)\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}$$

$$= x^2 + \frac{2(b-t)^{2\alpha}}{\Gamma(2\alpha+1)}.$$

We check easily that when $\alpha \to 1$, *we have* $\varphi(x,t) = x^2 + (b-t)^2$, *which is the solution of the wave equation,*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2},\ 0 < t < b,\ x > 0,$$

with the conditions

$$u(0,t) = (b-t)^2,\ u_x(0,t) = 0,\ 0 \leq t \leq b,$$

$$u(x,0) = x^2 + b^2,\ u(x,b) = x^2,\ x > 0.$$

Example 2 *Consider the backward fractional heat equation given as [11]*

$$\frac{\partial u(x,t)}{\partial t} = k\mathbf{D}_{x,b-}^{2\alpha}u(x,t),\ \alpha \in (\frac{1}{2},1],\ x \in (0,b),\ t > 0,$$

subject to the following initial and boundary conditions

$$u(x,0) = f_0(x),\ x \in [0,b]$$

$$u(0,t) = p(t),\ u(b,t) = r(t),\ t > 0,$$

where $\mathbf{D}_{x,b-}^{2\alpha}u(x,t) = I_{b-}^{2-\alpha}\frac{\partial^2 u(x,t)}{\partial x^2}$. *We use the starting term*

$$u_0(x,t) = \frac{1}{2}\left[f_0(x) + p(t) + \left(\frac{b-x}{b}\right)^\alpha (r(t) - p(t))\right],$$

and the recurrence relation

$$u_{n+1}(x,t) = \frac{1}{2}\left[kL_t^{-1}L_{xx} + \frac{1}{k}L_{xx}^{-1}L_t\right]u_n,\ \ n \geq 0,$$

where $L_t = \frac{\partial}{\partial t}$, $L_{xx} = \mathbf{D}_{x,b-}^{2\alpha}$,

$$\begin{aligned} L_t^{-1}w(t) &= \int_0^t w(s)ds, \\ L_{xx}^{-1}v(x) &= \int_x^b \frac{(s-x)^{2\alpha-1}}{\Gamma(2\alpha)}v(s)ds + \left(\frac{b-x}{b}\right)^\alpha \int_0^b \frac{(b-s)^{2\alpha-1}}{\Gamma(2\alpha)}v(s)ds. \end{aligned}$$

To illustrate the decomposition method for solving this problem, consider $k = \pm 1$, $f_0(x) = \frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)}$, $p(t) = \frac{2b^{2\alpha}}{\Gamma(1+2\alpha)} + 2kt$ *and* $r(t) = 2kt$, *then the given recurrence relation and starting term gives*

$$\begin{aligned} u_0(x,t) &= \frac{1}{2}\left[\frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2(b-x)^\alpha b^\alpha}{\Gamma(1+2\alpha)} + 2kt\right], \\ u_n(x,t) &= \frac{1}{2^{n+1}}\left[\frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{2(b-x)^\alpha b^\alpha}{\Gamma(1+2\alpha)} + 2kt\right],\ \ n \geq 1, \end{aligned}$$

and

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\ &= \frac{2(b-x)^\alpha b^\alpha}{\Gamma(1+2\alpha)} + \left(\frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)} + 2kt - \frac{2(b-x)^\alpha b^\alpha}{\Gamma(1+2\alpha)}\right)\sum_{i=0}^{\infty}\frac{1}{2^{i+1}} \\ &= \frac{2(b-x)^{2\alpha}}{\Gamma(1+2\alpha)} + 2kt. \end{aligned}$$

As $\alpha \to 1$, *we have*

$$u(x,t) = (b-x)^2 + 2kt$$

which the solution of the forward and backward heat problem (see [17])

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2},\ \ 0 < x < b,\ t > 0,\ k = \pm 1,$$

with the initial and boundary conditions $u(x,0) = (b-x)^2$, $u(0,t) = 2kt$, $u(b,t) = b^2 + 2kt$.

3. The finite fractional powers of the second order derivative

The finite Riesz potential, for $x \in (a,b) \subseteq \mathbb{R}$, was introduced in [24], as follows:

$$\begin{aligned} I_{(a,b)}^\alpha \varphi(x) &= \frac{I_{a+}^\alpha \varphi(x) + I_{b-}^\alpha \varphi(x)}{2\cos\left(\frac{\alpha\pi}{2}\right)} \\ &= \frac{1}{2\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right)}\int_a^b |x-\zeta|^{\alpha-1}\varphi(\zeta)\,d\zeta, \end{aligned}$$

for any $\alpha > 0$ with the exclusion of odd integer numbers for which $\cos\left(\frac{\alpha\pi}{2}\right)$ vanishes.

The finite Riesz potential given above has the semigroup only in restricted rang, e.g.

$$I^{\alpha}_{(a,b)} I^{\beta}_{(a,b)} = I^{\alpha+\beta}_{(a,b)} \text{ for } 0 < \alpha < 1,\ 0 < \beta < 1,\ \alpha + \beta < 1.$$

Now, we can introduce the finite Riesz derivative, in the framework of the Caputo derivative, as follows:

$$\begin{aligned} R^{\alpha}_{(a,b)}\varphi(x) &= I^{m-\alpha}_{(a,b)} D^m \varphi(x) = I^{m-\alpha}_{(a,b)}\left(D^m \varphi(x)\right) \\ &= -\frac{\mathbf{D}^{\alpha}_{a+}\varphi(x) + \mathbf{D}^{\alpha}_{b-}\varphi(x)}{2\cos\left(\frac{(m-\alpha)\pi}{2}\right)} \\ &= \frac{-1}{2\Gamma(\alpha)\cos\left(\frac{(m-\alpha)\pi}{2}\right)} \int_a^b |x-\zeta|^{m-\alpha-1} D^m \varphi(\zeta)\, d\zeta, \end{aligned}$$

for any positive $\alpha \in (m-1, m)$ and $x \in (a,b) \subseteq \mathbb{R}$.

In general, the Riesz fractional derivative $R^{\alpha}_{(a,b)}$ turns out to be related to the $\frac{\alpha}{2}$-power of the positive definite operator $-D^2 = -\frac{d^2}{dx^2}$ as follows:

$$R^{\alpha}_{(a,b)}\varphi(x) = \left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}} \varphi(x) = \begin{cases} -\frac{\mathbf{D}^{\alpha}_{a+}\varphi(x) + \mathbf{D}^{\alpha}_{b-}\varphi(x)}{2\cos\left(\frac{(1-\alpha)\pi}{2}\right)}, & \alpha \in (0,1), \\ -\frac{\mathbf{D}^{\alpha}_{a+}\varphi(x) + \mathbf{D}^{\alpha}_{b-}\varphi(x)}{2\cos\left(\frac{(2-\alpha)\pi}{2}\right)}, & \alpha \in (1,2), \\ -\frac{1}{\pi}\int_a^b \frac{D\varphi(\zeta)}{|t-\zeta|} d\zeta, & \alpha = 1, \end{cases} \quad (3.1)$$

where $x \in (a,b) \subseteq \mathbb{R}$.

From our definition of the finite Riesz derivative it is easy to prove the following Lemma.

Lemma 4

Let $\alpha \in (1,2)$, and $\varphi(x)$ be suitable function, for example $\varphi(x) \in AC^2$, then

(a) $\lim_{\alpha\to 2} R^{\alpha}_{(a,b)}\varphi(x) = -\frac{d^2}{dx^2}\varphi(x)$.

(b) $R^{\alpha}_{(a,b)}(kx + \ell) = 0$ *for all constants* k, ℓ.

(c) *For* $\alpha \in (m-1, m]$, *we have*

$$R^{\alpha}_{(a,b)} I^m_{a+}\varphi(x) = I^{m-\alpha}_{(a,b)}\varphi(x),$$

and

(d) $R^{\alpha}_{(a,b)} I^m_{b-}\varphi(x) = (-1)^m I^{m-\alpha}_{(a,b)}\varphi(x)$.

Proof.
(a)

$$\begin{aligned}\lim_{\alpha\to 2} R^{\alpha}_{(a,b)}\varphi(x) &= \lim_{\alpha\to 2} I^{2-\alpha}_{(a,b)} D^2\varphi(x)\\ &= \lim_{\alpha\to 2}\left[-\frac{\mathbf{D}^{\alpha}_{a+}\varphi(x)+\mathbf{D}^{\alpha}_{b-}}{2\cos\left(\frac{(2-\alpha)\pi}{2}\right)}\right]\varphi(x)\\ &= -\frac{d^2}{dx^2}\varphi(x).\end{aligned}$$

So, in (3.1) we can take $\alpha = 2$.
(b) $R^{\alpha}_{(a,b)}(kx+\ell) = I^{2-\alpha}_{(a,b)} D^2(kx+\ell) = 0$ *for all constants* k, ℓ.
(c) For $\alpha \in (m-1, m]$, *then by using Proposition 1, we have*

$$R^{\alpha}_{(a,b)} I^{m}_{a+}\varphi(x) = I^{m-\alpha}_{(a,b)} D^m\left(I^m_{a+}\varphi(x)\right) = I^{m-\alpha}_{(a,b)}\varphi(x),$$

(d) $R^{\alpha}_{(a,b)} I^{m}_{b-}\varphi(x) = I^{m-\alpha}_{(a,b)} D^m\left(I^m_{b-}\varphi(x)\right) = (-1)^m I^{m-\alpha}_{(a,b)}\varphi(x)$.
■

Remark 2

The finite Riesz derivative introduced above could be so convenient to be applied in model connected with physics, engineering, and applied science (see for example, [20-23]). Also we must point out that The finite Riesz derivative is a complementary operator to the well known Riesz derivative given explicitly by Samko et al. [24], in the d-dimensional case, as follows:

$$\begin{aligned}(-\Delta)^{\alpha/2}_d f(x) &= \frac{-\Gamma\left[(d-2+\alpha)/2\right]}{\pi^{(2-\alpha)/2}2^{2-\alpha}\Gamma\left[(2-\alpha)/2\right]}\int_{\Omega}\frac{\Delta f(\xi)}{\|x-\xi\|^{d-2+\alpha}}d\Omega(\xi)\\ &= -I^{2-\alpha}_d\left[\Delta f(x)\right].\end{aligned}$$

where $0<\alpha<2,\ x\in\Omega\subset\mathbb{R}^d$, *and*

$$I^{\beta}_d w(x) = \frac{\Gamma\left[(d+\beta)/2\right]}{\pi^{\beta/2}2^{\beta}\Gamma\left[\beta/2\right]}\int_{\Omega}\frac{w(\xi)}{\|x-\xi\|^{d+\beta}}d\Omega(\xi).$$

Example 3

We can consider the space-time fractional diffusion equation in a finite space domain, which is obtained from standard diffusion equation in a finite space domain, by replacing in the standard diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = C\frac{\partial^2}{\partial x^2}u(x,t), \quad x\in\mathbb{R},\ t\geq 0$$

the second-order space derivative by the finite Riesz derivative of order $\beta\in(1,2]$ *and the first-order time derivative by the Caputo derivative of order* $\alpha\in(0,1]$, *which can interpreted as a space and time derivative of fractional order, we obtain a sort of generalized diffusion equation.*

$$\mathbf{D}^{\alpha}_t u(x,t) = C R^{\beta}_x u(x,t),\ x\in(a,b),\ t\geq 0.$$

The infinite space domain was investigated with respect to its scaling and similarity properties in [14].

$$\mathbf{D}_t^{\alpha} u\left(x,t\right) = C R_x^{\beta} u\left(x,t\right), \ x \in \mathbb{R}, \ t \geq 0 \tag{3.2}$$

with to the initial condition

$$u\left(x,0\right) = \delta\left(x\right), \ x \in \mathbb{R}, \tag{3.3}$$

where $\mathbf{D}_t^{\alpha} \equiv \mathbf{D}_{t,0+}^{\alpha}$ *is the Caputo time fractional derivative of order* α*, with respect to* t*,* $R_x^{\beta} \equiv R_{(a,b),x}^{\beta}$ *is the space finite Riesz derivative of order* β *in* (a,b)*,* C *is the positive coefficient of diffusion,* $\alpha \in (0,1]$*,* $\beta \in (1,2]$*,* $u(x,t)$ *is a real function in the time-space variables, and* $\delta\left(x\right)$ *is the Dirac delta function. So the solution of* (3.2) *with the initial condition* (3.3) *is given as [14]*

$$u_{\alpha,\beta}\left(x,t\right) = \int_{-\infty}^{\infty} G_{\alpha,\beta}\left(x-y,t\right)\delta\left(y\right)\ dy,$$

where

$$G_{\alpha,\beta}\left(x,t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} E_{\alpha}\left[C(-ik)^{\beta} t^{\alpha}\right]\ dk.$$

and $E_{\alpha}\left(z\right)$ *is the Mittag-Leffler function defined as [22]*

$$E_{\alpha}\left(z\right) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(1+n\alpha\right)}.$$

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