

Extended Non Symmetric Gravitation Theory with a Scalar Field in Non Commutative Geometry

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Abstract: An extended method to reformulate the non symmetric gravitation theory in the non commutative geometry formalism is presented where all the lagrangian terms, including the various interaction ones with scalar fields, emerge naturally.

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1. Introduction

Recently, a new geometrical picture describing the various fundamental interactions has been proposed by A.Connes [1], [2], [3], [4]. It consists in the generalization of the classical differential geometry using a more profound mathematical formalism based on the discrete spaces non commutative geometry (NCG).

The success of the latter comes from the fact that it gives a geometric interpretation of the Higgs fields origin used in the in the standard model.

In this context, Chamseddine and collaborators [5], [6], [7] have reformulated General Relativity by considering a composed space- time consisting of a tensor product of a 4-dimensional manifold and a two points discrete space.

On the other hand, there exist many others theories inspired by General Relativity, which are based on a general non-symmetric metric $g_{\mu\nu}$, and in particular the Non Symmetric Gravitation theory (NGT) [8], [9], [10].

Yet, NGT as it was initially formulated lacked self-consistency; in particular the non-physical modes in the skewon sector are coupled with the physical ones.

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Recently, a new more consistent version of NGT was proposed by Moffat and Legaré [11], where these problems have been circumvented by adding new terms by hand to the action, but without any geometrical motivation.

In [12], we have derived an NGT action where the new terms added by Moffat in his new version of NGT are now a consequence of the discrete structure of space-time, with an additional interaction term $\varepsilon g_{ab}^{\mu\nu} R_{\mu\nu}^{ba}$.

In this work, we propose a new action for NGT without this interaction term. This was possible by generalizing the trace and introducing an operator P which permutes the indices of the Dirac γ matrices.

Moreover, the construction of this action with the scalar field coupled to NGT was possible by taking a general form of the generators of the 1-forms space $\Omega_D^1(\mathcal{B})$.

2. Formalism

In [6], the Hilbert-Einstein action was reformulated from the following Dirac operator:

$$D = \begin{pmatrix} \gamma^a \otimes e_a^\mu \partial_\mu \otimes 1 & \gamma^5 \otimes M_{12} \otimes K_{12} \\ \gamma^5 \otimes M_{21} \otimes K_{21} & \gamma^a \otimes e_a^\mu \partial_\mu \otimes 1 \end{pmatrix} = \begin{pmatrix} \gamma^a e_a^\mu \partial_\mu & \gamma^5 M_{12} K_{12} \\ \gamma^5 M_{21} K_{21} & \gamma^a e_a^\mu \partial_\mu \end{pmatrix}$$

where e_a^μ are the General Relativity (GR) vierbeins.

NGT as an extension of GR is based on the non-symmetric metric:

$$g^{\mu\nu} = e_a^\mu \tilde{e}_b^\nu \eta^{ab}$$

where here e_a^μ is the NGT vierbein and \tilde{e}_a^μ it's hyperbolic complex conjugate.

To generate non-symmetric terms and get an NGT action, we generalize the Dirac operator in the following form:

$$D = \begin{pmatrix} \gamma^a \otimes E_a^\mu \partial_\mu \otimes 1 & \gamma^5 \otimes M_{12} \otimes K_{12} \\ \gamma^5 \otimes M_{21} \otimes K_{21} & \gamma^a \otimes E_a^\mu \partial_\mu \otimes 1 \end{pmatrix} = \begin{pmatrix} \gamma^a E_a^\mu \partial_\mu & \gamma^5 M_{12} K_{12} \\ \gamma^5 M_{21} K_{21} & \gamma^a E_a^\mu \partial_\mu \end{pmatrix}$$

where E_a^μ is a 2×2 matrix (the generalized vierbein) defined by:

$$E_a^\mu = (E_a^\mu)^* = \begin{pmatrix} 0 & e_a^\mu \\ \tilde{e}_a^\mu & 0 \end{pmatrix}$$

and M_{12} , M_{21} (resp. KK_{12} , K_{21}) are 2×2 (resp. $NN \times N$) matrices. Here the γ^a 's are the ordinary Dirac matrices in the flat four-dimensional space-time and redefined such that [6]:

$$\gamma^{a*} = -\gamma^a, \{ \gamma^a, \gamma^b \} = \gamma^a \gamma^b + \gamma^b \gamma^a = -2\delta^{ab} \quad (1)$$

$$\gamma^{ab} = \frac{1}{2} [\gamma^a, \gamma^b] \quad , \quad \gamma^{(ab)} = \frac{1}{2} \{ \gamma^a, \gamma^b \} = -\delta^{ab}$$

In order to have a self-adjoint Dirac operator as it is required, one has to set the conditions:

$$K_{21}^* = K_{12} = K \tag{2}$$

and

$$M_{12} = M_{21}^* = M$$

The basic algebra \mathcal{A} is defined as:

$$\mathcal{A} = C_{\mathbb{R}}^{\infty}(X) \otimes (M_2(\mathbb{K}) \oplus M_2(\mathbb{K})) = C_{\mathbb{R}}^{\infty}(X, M_2(\mathbb{K})) \oplus C_{\mathbb{R}}^{\infty}(X, M_2(\mathbb{K})) \tag{3}$$

$$\mathcal{A} = \left\{ \alpha^{(1)} + \alpha^{(2)}; \alpha^{(i)} = \begin{pmatrix} a^{(i)} & 0 \\ 0 & b^{(i)} \end{pmatrix}; a^{(i)}, b^{(i)} \in C_{\mathbb{R}}^{\infty}(X, \mathbb{K}), i = 1, 2 \right\} \tag{4}$$

with

$$C_{\mathbb{R}}^{\infty}(X, M_2(\mathbb{K})) = C_{\mathbb{R}}^{\infty}(X) \otimes M_2(\mathbb{K})$$

where $C_{\mathbb{R}}^{\infty}(X)$ denotes the space of an infinite differentiable real function on a manifold X , and $M_2(\mathbb{K})$ is the set of the 2×2 hyperbolic complex matrices.

In what follows, we restrict ourselves to a sub-algebra \mathcal{B} of \mathcal{A} such that:

A representation of this sub-algebra on a Hilbert space \mathcal{H} is given by:

$$\pi(\alpha) = \pi(\alpha^{(1)} + \alpha^{(2)}) = \begin{pmatrix} 1 \otimes \alpha^{(1)} \otimes 1 & 0 \\ 0 & 1 \otimes \alpha^{(2)} \otimes 1 \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} & 0 \\ 0 & \alpha^{(2)} \end{pmatrix} \tag{5}$$

where \mathcal{H} is defined as[6]:

$$\mathcal{H} = L^2(S_1, dv_1) \oplus L^2(S_2, dv_2) \tag{6}$$

with $L^2(S_i, dv_i)$ is the square integrable functions over S_i such that

$$S_i = S_0 \otimes \mathbb{K}, \quad i = 1, 2 \tag{7}$$

where S_0 is the spinors space, and dv_i the volume element on X .

For the space of 1-forms denoted by $\Omega_D^1(\mathcal{B})$, one has as a representation:

$$\Omega_D^1(\mathcal{B}) = \pi(\Omega^1(\mathcal{B})) = \left\{ \pi(\omega) = \pi\left(\sum_i \alpha_i \delta\beta_i\right) = \sum_i \pi(\alpha_i) [D, \pi(\beta_i)] \right\} \tag{8}$$

Straightforward calculations give:

$$\pi(\omega) = \begin{pmatrix} \gamma^a E_a^\mu \omega_\mu^{(1)} & \gamma^5 K_{12} \Phi_{12} \\ \gamma^5 K_{21} \Phi_{21} & \gamma^a E_a^\mu \omega_\mu^{(2)} \end{pmatrix} \tag{9}$$

where

$$\omega_\mu^{(m)} = \sum_i \alpha_i^{(m)} \partial_\mu \beta_i^{(m)} \quad m = 1, 2 \quad (10)$$

and

$$\Phi_{mn} = \phi_{mn} M_{mn} \quad (11)$$

with

$$\phi_{mn} = \left(\sum_i \left(\alpha_i^{(m)} \beta_i^{(n)} - 1 \right) \right) \quad m \neq n = 1, 2 \quad (12)$$

where we have used the normalization condition:

$$\sum_i \alpha_i^{(1)} \beta_i^{(1)} = \sum_i \alpha_i^{(2)} \beta_i^{(2)} = 1 \quad (13)$$

Now, in order to get a representation of the 2-forms space $\Omega^2(\mathcal{B})$ without the junk forms (auxiliary fields), one has to take:

$$\Omega_D^2(\mathcal{B}) = \pi(\Omega_D^2(\mathcal{B})) / Aux^2 \quad (14)$$

where Aux^2 is the space of the auxiliary fields defined as:

$$Aux^2 = \{ \pi(\delta\omega) \quad / \quad \pi(\omega) = 0 \} \quad , \quad \omega \in \Omega^1(\mathcal{B}) \quad (15)$$

with

$$\pi(\delta\omega) = \sum_i \pi(\delta\alpha_i \delta\beta_i) = \sum_i [D, \pi(\alpha_i)] [D, \pi(\beta_i)] \quad (16)$$

Direct but lengthy calculations lead to:

$$\begin{aligned} \pi(\delta\omega)_{11} &= \gamma^a \gamma^b E_a^\mu E_b^\nu \left(\partial_\mu \omega_\nu^{(1)} - X_{\mu\nu}^{(1)} \right) + K_{12} K_{21} M_{12} M_{21} (\phi_{12} + \phi_{21}) \\ \pi(\delta\omega)_{22} &= \gamma^a \gamma^b E_a^\mu E_b^\nu \left(\partial_\mu \omega_\nu^{(2)} - X_{\mu\nu}^{(2)} \right) + K_{21} K_{12} M_{21} M_{12} (\phi_{21} + \phi_{12}) \\ \pi(\delta\omega)_{12} &= K_{12} \gamma^a \gamma^5 \left(E_a^\mu M_{12} \left(\partial_\mu \phi_{12} + \omega_\mu^{(1)} \right) - M_{12} E_a^\mu \omega_\mu^{(2)} - [E_a^\mu, M_{12}] Y_\mu^{(12)} \right) \\ \pi(\delta\omega)_{21} &= K_{21} \gamma^a \gamma^5 \left(E_a^\mu M_{21} \left(\partial_\mu \phi_{21} + \omega_\mu^{(2)} \right) - M_{21} E_a^\mu \omega_\mu^{(1)} - [E_a^\mu, M_{21}] Y_\mu^{(21)} \right) \end{aligned}$$

where the hyperbolic complex functions $X_{\mu\nu}^{(m)}$ and $Y_\mu^{(mn)}$ are given by:

$$X_{\mu\nu}^{(m)} = \sum_i \alpha_i^{(m)} \partial_\mu \partial_\nu \beta_i^{(m)} \quad m = 1, 2$$

and

$$Y_\mu^{(mn)} = \sum_i \alpha_i^{(m)} \partial_\mu \beta_i^{(n)} \quad m \neq n = 1, 2 \quad (17)$$

After some simplifications, we obtain for Aux^2 the following expression:

$$Aux^2 = \left(\begin{array}{cc} \gamma^a \gamma^b E_a^\mu E_b^\nu X_{\mu\nu}^{(1)} & K \gamma^a \gamma^5 [E_a^\mu, M_{12}] Y_\mu \\ K^* \gamma^a \gamma^5 [E_a^\mu, M_{21}] Z_\mu & \gamma^a \gamma^b E_a^\mu E_b^\nu X_{\mu\nu}^{(2)} \end{array} \right) \quad (18)$$

where $X_{\mu\nu}^{(1)} = X_{\nu\mu}^{(1)}, X_{\mu\nu}^{(2)} = X_{\nu\mu}^{(2)}, Y_\mu, Z_\mu$ are arbitrary hyperbolic complex functions.

The curvature tensor R^{AB} ($A, B = \overline{1, 5}$) is given by the Cartan structure equations [6],[7]:

$$R^{AB} = d\Omega^{AB} + \sum_C \Omega^{AC} \Omega^{CB}$$

where $\Omega^{AB} \in \Omega_D^1(\mathcal{B})$ are the components of the connection such that:

$$\begin{aligned} (\Omega^{AB})_{mm} &= \gamma^a E_a^\mu \omega_\mu^{(m)AB} & m = 1, 2 \\ (\Omega^{AB})_{mn} &= \gamma^5 K_{mn} M_{mn} \phi_{mn}^{AB} & m \neq n = 1, 2. \end{aligned}$$

and are subject to the unitarity condition:

$$(\Omega^{AB})^* = \Omega^{BA}$$

This leads to the following constraints on the fields:

$$\begin{aligned} \tilde{\omega}_\mu^{(1)AB} &= -\omega_\mu^{(1)BA}, \quad \tilde{\omega}_\mu^{(2)AB} = -\omega_\mu^{(2)BA} \\ \tilde{\phi}_{12}^{AB} &= \phi_{21}^{BA}, \quad \tilde{\phi}_{21}^{AB} = \phi_{12}^{BA} \end{aligned}$$

In order to get an explicit expression for R^{AB} and Aux^2 , we make the following choice for the matrix M:

$$M = \mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{19}$$

We note that this choice is not arbitrary; indeed when we go from the space $\pi(\Omega_D^2(\mathcal{B}))$ to the quotient space $\Omega_D^2(\mathcal{B}) = \pi(\Omega_D^2(\mathcal{B}))/Aux^2$, the fields ϕ_{mn}^{AB} vanish unless the matrix M obeys equation (A1). For simplicity, μ is taken to be equal to one.

With this choice, we get the following expressions for the elements of the curvature tensor R^{AB} :

$$\begin{aligned} R_{11}^{AB} &= \gamma^a \gamma^b (\eta_{ab}^{\mu\nu} \mathbf{1} - \varepsilon g_{ab}^{\mu\nu} \tau_3) \left(R_{\mu\nu}^{(1)AB} - X_{\mu\nu}^{(1)AB} \right) + K \cdot K^* H_{12}^{AB} \tau \\ R_{22}^{AB} &= \gamma^a \gamma^b (\eta_{ab}^{\mu\nu} \mathbf{1} - \varepsilon g_{ab}^{\mu\nu} \tau_3) \left(R_{\mu\nu}^{(2)AB} - X_{\mu\nu}^{(2)AB} \right) + K^* \cdot K H_{21}^{AB} \bar{\tau} \\ R_{12}^{AB} &= K \cdot \gamma^a \gamma^5 \tilde{e}_a^\mu \left(\nabla_\mu \varphi_{12}^{AB} \bar{\tau} - \left(\varphi_{12}^{AC} \omega_\mu^{(2)CB} + \omega_\mu^{(2)AB} \right) \tau + Y_\mu^{AB} \tau_3 \right) \\ R_{21}^{AB} &= K^* \cdot \gamma^a \gamma^5 e_a^\mu \left(\nabla_\mu \varphi_{21}^{AB} \cdot \tau - \left(\varphi_{21}^{AC} \omega_\mu^{(1)CB} + \omega_\mu^{(1)AB} \right) \bar{\tau} - Z_\mu^{AB} \tau_3 \right) \end{aligned}$$

where:

$$R_{\mu\nu}^{(m)AB} = \partial_\mu \omega_\nu^{(m)AB} + \sum_C \omega_\mu^{(m)AC} \omega_\nu^{(m)CB} \quad m = 1, 2$$

and:

$$\begin{aligned} H_{12}^{AB} &= \varphi_{12}^{AC} \varphi_{21}^{CB} + \varphi_{12}^{AB} + \varphi_{21}^{AB} \\ H_{21}^{AB} &= \varphi_{21}^{AC} \varphi_{12}^{CB} + \varphi_{21}^{AB} + \varphi_{12}^{AB} \\ \nabla_\mu \varphi_{12}^{AB} &= \partial_\mu \varphi_{12}^{AB} + \omega_\mu^{(1)AC} \varphi_{12}^{CB} + \omega_\mu^{(1)AB} \\ \nabla_\mu \varphi_{21}^{AB} &= \partial_\mu \varphi_{21}^{AB} + \omega_\mu^{(2)AC} \varphi_{21}^{CB} + \omega_\mu^{(2)AB} \end{aligned}$$

We obtain for Aux^2 the following expression:

$$Aux^2 = \begin{pmatrix} \gamma^a \gamma^b (\eta_{ab}^{\mu\nu} - \varepsilon g_{ab}^{\mu\nu} \tau_3) X_{\mu\nu}^{(1)} & -K \cdot \gamma^a \gamma^5 \tilde{e}_a^\mu \tau_3 Y_\mu \\ K^* \cdot \gamma^a \gamma^5 e_a^\mu \tau_3 Z_\mu & \gamma^a \gamma^b (\eta_{ab}^{\mu\nu} - \varepsilon g_{ab}^{\mu\nu} \tau_3) X_{\mu\nu}^{(2)} \end{pmatrix} \tag{20}$$

where we have used the fact that:

$$M_{12} = M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{21} = M_{12}^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\tau} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$E_a^\mu E_b^\nu = \eta_{ab}^{\mu\nu} \mathbf{1} - \varepsilon g_{ab}^{\mu\nu} \tau_3$$

with:

$$\eta_{ab}^{\mu\nu} = (\alpha_a^\mu \alpha_b^\nu - \beta_a^\mu \beta_b^\nu) = +\eta_{ba}^{\nu\mu}$$

$$g_{ab}^{\mu\nu} = \alpha_a^\mu \beta_b^\nu - \beta_a^\mu \alpha_b^\nu = -g_{ba}^{\nu\mu}$$

$$\tau - \bar{\tau} = \tau_3$$

and

$$\alpha_a^\mu = \frac{1}{2} (e_a^\mu + \tilde{e}_a^\mu)$$

$$\beta_a^\mu = \frac{1}{2} \varepsilon (e_a^\mu - \tilde{e}_a^\mu)$$

ε is the hyperbolic complex number satisfying:

$$\varepsilon^2 = 1, \text{ and, } \tilde{\varepsilon} = -\varepsilon$$

The expression of the curvature tensor R^{AB} orthogonal to the space of the auxiliary fields Aux^2 is given by:

$$R_{11}^{AB} = \frac{1}{2} (\gamma^{ab} \eta_{ab}^{\mu\nu} \mathbf{1} + \varepsilon g_{ab}^{\mu\nu} \tau_3) \left(R_{\mu\nu}^{(1)AB} \right) - \frac{1}{2} (Tr(K.K^*) - 2(K.K^*) \tau) H_{12}^{AB}$$

$$R_{22}^{AB} = \frac{1}{2} (\gamma^{ab} \eta_{ab}^{\mu\nu} \mathbf{1} + \varepsilon g_{ab}^{\mu\nu} \tau_3) R_{\mu\nu}^{(2)AB} - \frac{1}{2} (Tr(K.K^*) - 2(K^*.K) \bar{\tau}) H_{21}^{AB}$$

$$R_{12}^{AB} = \frac{1}{2} K \cdot \gamma^a \gamma^5 \tilde{e}_a^\mu \left(\nabla_\mu \varphi_{12}^{AB} - \left(\varphi_{12}^{AC} \omega_\mu^{(2)CB} + \omega^{(2)AB} \right) \right)$$

$$R_{21}^{AB} = \frac{1}{2} K^* \cdot \gamma^a \gamma^5 e_a^\mu \left(\nabla_\mu \varphi_{21}^{AB} - \left(\varphi_{21}^{AC} \omega_\mu^{(1)CB} + \omega^{(1)AB} \right) \right)$$

with $R_{\mu\nu}^{(m)AB} = \partial_\mu \omega_\nu^{(m)AB} + \sum_C \omega_\mu^{(m)AC} \omega_\nu^{(m)CB} - (\mu \leftrightarrow \nu)$

$$= -R_{\nu\mu}^{(m)AB} \quad \text{with } m = 1, 2$$

Concerning the torsion T^A ($A = \overline{1, 5}$), it is given by the Cartan structure equations [6],[7]:

$$T^A = d\xi^A + \sum_B \Omega^{AB} \cdot \xi^B$$

where ξ^A are the generators of the space of 1-forms $\Omega_D^1(\mathcal{B})$ and which have the following expressions:

$$\xi^a = \pi \left(\tilde{e}_\mu^a \oplus \tilde{e}_\mu^a \delta \cdot x^\mu \oplus x^\mu \right) = \begin{pmatrix} \gamma^b E_b^\mu \tilde{e}_\mu^a & 0 \\ 0 & \gamma^b E_b^\mu \tilde{e}_\mu^a \end{pmatrix}, \quad a = 1, 2, 3, 4$$

$$\xi^{a*} = -\gamma^b E_b^\mu e_\mu^a \otimes \mathbf{1} = -\tilde{\xi}$$

$$\xi^5 = \pi \left(\lambda \oplus \tilde{\lambda} \delta \cdot 0 \oplus 1 \right) = \begin{pmatrix} 0 & \gamma^5 K M_{21} \lambda \\ -\gamma^5 K^* M_{21} \tilde{\lambda} & 0 \end{pmatrix} = -\xi^{5*} = -\tilde{\xi}^5$$

with:

$$d\xi^a = \pi \left(\delta \tilde{e}_\mu^a \oplus \tilde{e}_\mu^a \cdot \delta x^\mu \oplus x^\mu \right) = \gamma^b \gamma^c E_b^\mu E_c^\nu \partial_\mu \tilde{e}_\nu^a \otimes \mathbf{1}$$

$$d\xi^5 = \pi \left(\delta \lambda \oplus \tilde{\lambda} \delta \cdot 0 \oplus 1 \right) = \begin{pmatrix} KK^* M_{12} M_{21} (\lambda - \tilde{\lambda}) & K \gamma^a \gamma^5 E_a^\mu M_{12} \partial_\mu \lambda \\ -K^* \gamma^a \gamma^5 E_a^\mu M_{21} \partial_\mu \tilde{\lambda} & K^* K M_{21} M_{12} (\lambda - \tilde{\lambda}) \end{pmatrix}.$$

where λ is an hyperbolic complex function.

Then, the components of T^A orthogonal to the auxiliary fields space take the form:

$$(T^a)_{11} = (\gamma^{cd} \eta_{cd}^{\mu\nu} \mathbf{1} + \varepsilon g^{\mu\nu} \tau_3) \left(\partial_\mu \tilde{e}_\nu^a + \omega_\mu^{(1)ab} \tilde{e}_\nu^b \right) + \frac{1}{2} (Tr(K.K^*) - 2(K.K^*) \tau) \tilde{\lambda} \varphi_{12}^{a5}$$

$$(T^a)_{22} = (\gamma^{cd} \eta_{cd}^{\mu\nu} \mathbf{1} + \varepsilon g^{\mu\nu} \tau_3) \left(\partial_\mu \tilde{e}_\nu^a + \omega_\mu^{(2)ab} \tilde{e}_\nu^b \right) - \frac{1}{2} (Tr(K.K^*) - 2(K^*.K) \bar{\tau}) \lambda \varphi_{21}^{a5}$$

$$(T_{12}^a) = K \cdot \gamma^d \gamma^5 \tilde{e}_d^\mu \lambda \omega_\mu^{(1)a5}$$

$$(T^a)_{21} = -K^* \cdot \gamma^d \gamma^5 e_d^\mu \tilde{\lambda} \omega_\mu^{(2)a5}$$

and:

$$(T^5)_{11} = (\gamma^{cd} \eta_{cd}^{\mu\nu} \mathbf{1} + \varepsilon g^{\mu\nu} \tau_3) \omega_\mu^{(1)5b} \tilde{e}_\nu^b + \frac{1}{2} (Tr(K.K^*) - 2(K.K^*) \tau) \left(\varphi_{12}^{55} \tilde{\lambda} - \lambda + \tilde{\lambda} \right)$$

$$(T^5)_{22} = (\gamma^{cd} \eta_{cd}^{\mu\nu} \mathbf{1} + \varepsilon g^{\mu\nu} \tau_3) \omega_\mu^{(2)5b} \tilde{e}_\nu^b - \frac{1}{2} (Tr(K.K^*) - 2(K^*.K) \bar{\tau}) \left(\varphi_{21}^{55} \lambda + \lambda - \tilde{\lambda} \right)$$

$$(T^5)_{12} = K \cdot \gamma^d \gamma^5 \tilde{e}_d^\mu \left(\partial_\mu \lambda + \omega_\mu^{(1)55} \lambda \right)$$

$$(T^5)_{21} = -K^* \cdot \gamma^d \gamma^5 e_d^\mu \left(\partial_\mu \tilde{\lambda} + \omega_\mu^{(2)55} \tilde{\lambda} \right)$$

The orthogonality condition of the T^A to the space Aux^2 leads to the following constraints on the fields:

$$\omega_\mu^{(2)a5} \tilde{\lambda} - \tilde{e}_\mu^b \varphi_{21}^{ab} = 0 \quad (21)$$

$$\omega_\mu^{(1)a5} \lambda + \tilde{e}_\mu^b \varphi_{12}^{ab} = 0 \quad (22)$$

$$\partial_\mu \tilde{\lambda} + \omega_\mu^{(2)55} \tilde{\lambda} - \tilde{e}_\mu^b \varphi_{21}^{5b} = 0 \quad (23)$$

$$\partial_\mu \lambda + \omega_\mu^{(1)55} \lambda + \tilde{e}_\mu^b \varphi_{12}^{5b} = 0 \quad (24)$$

The action \mathfrak{J} has the same form as that given in [5] namely:

$$\mathfrak{J} = \frac{1}{2} (E^A E^{B*} - E^{B*} E^A, R^{BA}) \quad (25)$$

where here $(,)$ denotes the generalized scalar product defined as:

$$(E^A E^{B*} - E^{B*} E^A, R^{BA}) = \int d^4 x \sqrt{e} \tilde{e} Tr_K \cdot \mathfrak{Tr} \left((E^B E^{A*} - E^{A*} E^B) R^{BA} \right) \quad (26)$$

with Tr_K is the trace over the K matrices, while \mathfrak{Tr} is the generalized trace defined in the appendix. The E^A are the generators of a subspace of the 1-form space $\Omega_D^1(A)$ defined by:

$$\left\{ \begin{array}{l} \pi(\omega) = \pi \left(\sum_i \alpha_i \delta \beta_i \right) = \sum_i \pi(\alpha_i) [D, \pi(\beta_i)] \\ \alpha_i \in A, \quad \beta_i \in \mathcal{B}, \alpha_i = \alpha_i^{(1)} \oplus \alpha_i^{(2)}, \quad \beta_i = \beta_i^{(1)} \oplus \beta_i^{(2)} \end{array} \right\}$$

A representation of this space is given by:

$$\pi(\omega) = \begin{pmatrix} \gamma^a \omega_\mu^{(1)} E_a^\mu & \gamma^5 K \Phi_{12} M_{12} \\ \gamma^5 K^* \Phi_{21} M_{12} & \gamma^a \omega_\mu^{(2)} E_a^\mu \end{pmatrix}$$

and the generators are:

$$\begin{aligned} E^a &= \pi(e_\mu^a \tau \oplus e_\mu^a \tau \cdot \delta x^\mu \oplus x^\mu) = \gamma^b e_\mu^a \tau E_b^\mu \otimes \mathbf{1} = \gamma^a M_{12} \otimes \mathbf{1} \\ \tilde{E}^a &= \pi(\tilde{e}_\mu^a \bar{\tau} \oplus \tilde{e}_\mu^a \bar{\tau} \cdot \delta x^\mu \oplus x^\mu) = \gamma^b \tilde{e}_\mu^a \bar{\tau} E_b^\mu \otimes \mathbf{1} = \gamma^a M_{21} \otimes \mathbf{1} \\ E^5 &= \pi(\lambda \oplus \tilde{\lambda} \cdot \delta 0 \oplus 1) = \begin{pmatrix} 0 & \gamma^5 K M_{12} \lambda \\ -\gamma^5 K^* M_{21} \tilde{\lambda} & 0 \end{pmatrix} = \tilde{E}^5 \end{aligned}$$

In fact, if we use the generators ξ^A of the space $\Omega_D^1(A)$

$$\xi^a = \gamma^b E_b^\mu \tilde{e}_\mu^a \otimes \mathbf{1} = \begin{pmatrix} 0 & \gamma^b e_b^\mu \tilde{e}_\mu^a \\ \gamma^a & 0 \end{pmatrix} \otimes \mathbf{1}$$

to define the action, we get two contributions to the action: one coming from the term $\gamma^b \tilde{e}_b^\mu \tilde{e}_\mu^a = \gamma^a$ and which gives the NGT action, and the other one from the term $\gamma^b e_b^\mu \tilde{e}_\mu^a$ giving a meaningless contribution. This is however not the case for the generators E^A

where the matrix $\tau = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the expression of E^a insures the absence of such terms

in the action.

A straightforward calculation gives the following expression:

$$\mathfrak{J} = \mathfrak{J}^{(1)} + \mathfrak{J}^{(2)} + \mathfrak{J}^{(3)} + \mathfrak{J}^{(4)}$$

with

$$\begin{aligned} \mathfrak{J}^{(1)} &= \frac{1}{2} (E^a E^{b*} - E^{b*} E^a, R^{ba}) = \frac{1}{2} \int d^4x \sqrt{e\tilde{e}} Tr \cdot \mathfrak{Tr} ((E^b E^{a*} - E^{a*} E^b) R^{ba}) \\ \mathfrak{J}^{(2)} &= \frac{1}{2} (E^5 E^{b*} - E^{b*} E^5, R^{b5}) = \frac{1}{2} \int d^4x \sqrt{e\tilde{e}} Tr \cdot \mathfrak{Tr} ((E^5 E^b - E^b E^5) R^{b5}) \\ \mathfrak{J}^{(3)} &= \frac{1}{2} (E^a E^{5*} - E^{5*} E^a, R^{5a}) = \frac{1}{2} \int d^4x \sqrt{e\tilde{e}} Tr \cdot \mathfrak{Tr} ((\tilde{E}^a E^5 - E^5 \tilde{E}^a) R^{5a}) \\ \mathfrak{J}^{(4)} &= \frac{1}{2} (E^5 E^{5*} - E^{5*} E^5, R^{55}) = \frac{1}{2} \int d^4x \sqrt{e\tilde{e}} Tr \cdot \mathfrak{Tr} ((E^5 E^5 - E^5 E^5) R^{55}) \end{aligned}$$

Now by using the properties of the generalized trace $\mathfrak{Tr} = \mathcal{P}tr$ (defined in the appendix), we obtain:

$$\mathfrak{J}^{(1)} = \int d^4x \sqrt{e\tilde{e}} \left\{ - (e_a^\mu \tilde{e}_\mu^a + \frac{1}{2} \delta_{ab} G^{\mu\nu}) (R_{\mu\nu}^{(1)ba} + R_{\mu\nu}^{(2)ba}) + \frac{1}{2} Tr(KK^*) (H_{12}^{aa} - H_{21}^{aa}) \right\}$$

and by using the compatibility conditions for \tilde{e}_μ^a :

$$\nabla_\mu \tilde{e}_\nu^a = \partial_\mu \tilde{e}_\nu^a + \omega_\mu^{ab} \tilde{e}_\nu^b - W^\lambda_{\mu\nu} \tilde{e}_\lambda^a = 0$$

we get:

$$(e_a^\mu \tilde{e}_b^\nu + \frac{1}{2} \delta_{ab} G^{\mu\nu}) R_{\mu\nu}^{ba} = -G^{\mu\nu} (R_{\sigma\lambda\nu}^\lambda - \frac{1}{2} \tilde{R}_{\lambda\mu\nu}^\lambda) = -G^{\mu\nu} R_{\mu\nu}.$$

Therefore $\mathfrak{J}^{(1)}$ becomes:

$$\mathfrak{J}^{(1)} = \int d^4x \sqrt{e\tilde{e}} \left\{ G^{\mu\nu} (R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)}) + \frac{1}{2} Tr(KK^*) (H_{12}^{aa} - H_{21}^{aa}) \right\} \quad (27)$$

Similarly, we get the following expressions for $\mathfrak{J}^{(2)}, \mathfrak{J}^{(3)}$ and $\mathfrak{J}^{(4)}$:

$$\begin{aligned} \mathfrak{J}^{(2)} &= \int d^4x \sqrt{e\tilde{e}} \left\{ \frac{1}{4} Tr (K.K^*) \tilde{\lambda} \tilde{e}_a^\mu \left(\nabla_\mu \varphi_{12}^{a5} - \varphi_{12}^{aB} \omega_\mu^{(2)B5} - \omega_\mu^{(2)a5} \right) \right\} \\ \mathfrak{J}^{(3)} &= \int d^4x \sqrt{e\tilde{e}} \frac{1}{4} Tr (K.K^*) \lambda e_a^\mu \left(\nabla_\mu \varphi_{21}^{5a} - \varphi_{21}^{5B} \omega_\mu^{(1)Ba} - \omega_\mu^{(1)5a} \right) \\ \mathfrak{J}^{(4)} &= 0 \end{aligned}$$

The action \mathfrak{J} now takes the form:

$$\begin{aligned} \mathfrak{J} &= \int d^4x \sqrt{e\tilde{e}} \left\{ G^{\mu\nu} \left(R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} \right) + \frac{1}{2} Tr (K.K^*) \left(\varphi_{12}^{aB} \varphi_{21}^{Ba} - \varphi_{21}^{aB} \varphi_{12}^{Ba} \right) \right. \\ &\quad \left. + \frac{1}{4} Tr (K.K^*) \tilde{\lambda} \tilde{e}_a^\mu \left(\nabla_\mu \varphi_{12}^{a5} - \varphi_{12}^{aB} \omega_\mu^{(2)B5} - \omega_\mu^{(2)a5} \right) \right. \\ &\quad \left. + \frac{1}{4} Tr (K.K^*) \lambda e_a^\mu \left(\nabla_\mu \varphi_{21}^{5a} - \varphi_{21}^{5B} \omega_\mu^{(1)Ba} - \omega_\mu^{(1)5a} \right) \right\} \end{aligned}$$

If we impose the strong condition $T^A = 0$, the fields ϕ_{mn}^{AB} vanish and the contribution of the discrete space is trivial. So, in order to get a dynamical contribution of the discrete space, we must impose the following weak conditions:

$$Tr_K (T^a) = 0 \quad \text{and} \quad T^5 \neq 0 \tag{28}$$

where Tr_K denotes the trace over the matrices K_{12}, K_{21} .

By using the constraints on the fields given by the condition (A2), and the equations (21), (22), (23), we obtain the following expression for the action:

$$\begin{aligned} \mathfrak{J} &= \int d^4x \sqrt{e\tilde{e}} \mathfrak{L} \quad \mathfrak{L} = \mathfrak{L}^{(1)} + \mathfrak{L}^{(2)} + \mathfrak{L}^{(3)} \\ \mathfrak{L}^{(1)} &= G^{\mu\nu} \left(R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} \right) = 2G^{\mu\nu} R_{\mu\nu} \\ \mathfrak{L}^{(2)} &= \frac{1}{2} Tr (K.K^*) \left(\varphi_{12}^{aB} \varphi_{21}^{Ba} - \varphi_{21}^{aB} \varphi_{12}^{Ba} \right) = 0 \\ \mathfrak{L}^{(3)} &= \frac{1}{4} Tr (K.K^*) \tilde{\lambda} \tilde{e}_a^\mu \left(\nabla_\mu \varphi_{12}^{a5} - \varphi_{12}^{aB} \omega_\mu^{(2)B5} - \omega_\mu^{(2)a5} \right) \\ &\quad + \frac{1}{4} Tr (K.K^*) \lambda e_a^\mu \left(\nabla_\mu \varphi_{21}^{5a} - \varphi_{21}^{5B} \omega_\mu^{(1)Ba} - \omega_\mu^{(1)5a} \right) \end{aligned}$$

If we put $\omega_\mu^{(2)55} = W_\mu$, $\tilde{W}_\mu = -W_\mu$, the expression for $\mathfrak{L}^{(3)}$ becomes:

$$\begin{aligned} \mathfrak{L}^{(3)} &= \frac{1}{4} Tr (K.K^*) \left(\tilde{\lambda} G^{\sigma\mu} \partial_\mu \left((\partial_\sigma - W_\sigma) \lambda \right) - \tilde{\lambda} G^{\nu\mu} \left((\partial_\sigma - W_\sigma) \lambda \right) \tilde{W}_{\mu\nu}^\sigma \right. \\ &\quad \left. - \tilde{\lambda} G^{\sigma\mu} \left((\partial_\sigma - W_\sigma) \lambda \right) W_\mu \right) + h.c.c. \end{aligned}$$

where we have used the compatibility conditions on e_a^σ .

In what follows, we consider several cases of special significance:

(i) The case $\lambda = \bar{\lambda} = 1$

Here, the expression for $\mathfrak{L}^{(3)}$ becomes:

$$\begin{aligned} \mathfrak{L}^{(3)} &= \frac{1}{4} Tr (K.K^*) \left\{ -G^{\sigma\mu} \partial_\mu W_\sigma + G^{\nu\mu} W_\sigma \tilde{W}_{\mu\nu}^\sigma + G^{\sigma\mu} W_\sigma W_\mu + h.c.c \right\} \\ &= \frac{1}{4} Tr (K.K^*) \left\{ 2G^{(\mu\nu)} W_\mu W_\nu - 2G^{[\mu\nu]} \partial_\nu W_\mu \right\} \end{aligned}$$

and therefore \mathfrak{L} takes the form:

$$\mathfrak{L} = G^{\mu\nu} R_{\mu\nu} - \frac{1}{4} x G^{(\mu\nu)} W_\mu W_\nu + \frac{1}{4} x G^{[\mu\nu]} \partial_\nu W_\mu \tag{29}$$

where $x = -Tr (K.K^*)$. We remind the reader that K being an hyperbolic complex matrix, x can be a positive number.

By using the following decomposition [10]:

$$\begin{aligned} W_{\mu\nu}^{\lambda} &= \Gamma_{\mu\nu}^{\lambda} - \frac{2}{3}\delta_{\mu}^{\lambda}W_{\nu} \\ R_{\mu\nu}(W) &= R_{\mu\nu}(\Gamma) + \frac{2}{3}W_{[\mu,\nu]} \end{aligned} \quad (30)$$

where

$$\begin{aligned} W_{\mu} &= \frac{1}{2}(W_{\mu\alpha}^{\alpha} - W_{\alpha\mu}^{\alpha}) \\ W_{[\mu,\nu]} &= \frac{1}{2}(W_{\mu,\nu} - W_{\nu,\mu}) \end{aligned}$$

and redefining W_{μ} such that:

$$\left(\frac{2}{3} + \frac{1}{4}x\right)W_{\mu} = \frac{2}{3}\bar{W}_{\mu}$$

together with:

$$\bar{W}_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \frac{2}{3}\delta_{\mu}^{\lambda}\bar{W}_{\nu}$$

we get:

$$G^{\mu\nu}R_{\mu\nu}(W) + \frac{1}{4}xG^{[\mu\nu]}\partial_{\nu}W_{\mu} = G^{\mu\nu}R_{\mu\nu}(\bar{W})$$

This leads to the following expression for the action:

$$\mathfrak{L} = G^{\mu\nu}R_{\mu\nu}(\bar{W}) - f(x)G_{\mu}^{(\mu\nu)}\bar{W}_{\mu}\bar{W}_{\nu}$$

where now the interaction constant $f(x)$ is given by:

$$f(x) = \frac{x}{\left(2 + \frac{3}{4}x\right)^2}$$

It is worth noting that this function has an extremum (maximum) at the point:

$$x = \frac{8}{3}, \quad \max f(x) = f\left(\frac{8}{3}\right) = \frac{1}{6}$$

so, the choice of this optimal value leads exactly to the modified Moffat's action of NGT [11]:

$$\mathfrak{L} = G^{\mu\nu}R_{\mu\nu}(\bar{W}) + \frac{1}{2}\sigma G_{\mu}^{(\mu\nu)}\bar{W}_{\mu}\bar{W}_{\nu}$$

where $\sigma = -\frac{1}{3}$.

Concerning the skew term added by hand by Moffat in his new action [11]:

$$\mathfrak{L}_{skew} = -\frac{1}{4}\mu^2(-g)^{\frac{1}{2}}G^{[\mu\nu]}G_{[\mu\nu]}$$

it can also be generated from the cosmological term:

$$\mathbf{J} = \frac{1}{2}(E^A E^{B*} - E^{B*} E^A, \xi^A \xi^{B*}) = \int \sqrt{e\tilde{e}} d^4x \mathfrak{L}_{cos}$$

which gives after some straightforward calculations (see the appendix):

$$\mathcal{L}_{\text{cos}} = -2G^{[\mu\nu]}G_{[\mu\nu]} - 4\text{Tr}(K.K^*)\lambda\tilde{\lambda} - 8 - G_{ba}^{\mu\nu}G_{\nu\mu}^{ab}$$

(ii) The case $\lambda = \exp(\varepsilon\Phi)$, $\tilde{\lambda} = \exp(-\varepsilon\Phi)$ (Φ is a real field).

By following the same steps as in case (i), we obtain:

$$\mathcal{L} = G^{\mu\nu}R_{\mu\nu}(\overline{W}) + \frac{1}{2}\sigma G_{\mu}^{(\mu\nu)}\overline{W}_{\mu}\overline{W}_{\nu} - \frac{2}{3}G^{(\mu\nu)}\partial_{\mu}\Phi\partial_{\nu}\Phi + \frac{2}{3}G^{(\mu\nu)}\varepsilon\overline{W}_{\mu}\partial_{\nu}\Phi$$

and

$$\mathcal{L}_{\text{cos}} = -2G^{[\mu\nu]}G_{[\mu\nu]} + \frac{8}{3} - G_{ba}^{\mu\nu}G_{\nu\mu}^{ab}$$

We thus have obtained an action in which NGT is coupled to a massless scalar field.

(iii) The general case $\lambda = \Phi$, $\tilde{\lambda} = \tilde{\Phi}$:

We have:

$$\mathcal{L} = G^{\mu\nu}R_{\mu\nu}(W) - \frac{1}{8}x \left\{ G^{\mu\sigma} \left(\tilde{\Phi}D_{\sigma}D_{\mu}\Phi + \Phi D_{\mu}D_{\sigma}\tilde{\Phi} \right) - \frac{1}{2}G^{\mu\nu}W^{\sigma}{}_{\mu\nu} \left(\tilde{\Phi}D_{\sigma}\Phi - \Phi D_{\sigma}\tilde{\Phi} \right) \right\}$$

and:

$$\mathcal{L}_{\text{cos}} = -2G^{[\mu\nu]}G_{[\mu\nu]} + \frac{32}{3}\Phi\tilde{\Phi} - 8 - G_{ba}^{\mu\nu}G_{\nu\mu}^{ab}$$

where

$$D_{\sigma}\Phi = (\partial_{\sigma} - W_{\sigma})\Phi$$

$$D_{\sigma}\tilde{\Phi} = (\partial_{\sigma} + W_{\sigma})\tilde{\Phi}$$

$$D_{\mu}D_{\sigma}\Phi = (\partial_{\mu} - W_{\mu})(\partial_{\sigma} - W_{\sigma})\Phi$$

$$D_{\mu}D_{\sigma}\tilde{\Phi} = (\partial_{\mu} + W_{\mu})(\partial_{\sigma} + W_{\sigma})\tilde{\Phi}$$

We have thus obtained an action in which NGT is now coupled to a massive scalar field.

3. Conclusions

In this work, we have generalized the ordinary formalism of non commutative geometry to derive the various terms of the new Moffat's version of the NGT Lagrangian..

Moreover, and as a consequence of the discrete structure of space time, an additional unwanted interaction term has arised. In order to get rid of it, we were lead to generalize the notion of trace and introduce a γ matrices ordering operator P .

Furthermore, the construction of the action with the scalar field coupled to NGT was possible by taking a general form of the generators of the 1-forms space.

It is worth noting that in the term $\frac{1}{2}\sigma G_{\mu}^{(\mu\nu)}\overline{W}_{\mu}\overline{W}_{\nu}$, the value of the coupling constant σ comes naturally as the maximum of the function $f(x)$. This suggests that even the couplings may have a geometrical interpretation.

4. Appendix

4.1 The Cosmological Term

One can add to the action the following cosmological term:

$$\mathbf{J} = \frac{1}{2} (E^A E^{B*} - E^{B*} E^A, \xi^A \xi^{B*})$$

A straightforward calculation gives us the following expressions:

$$\begin{aligned} \frac{1}{2} Tr \mathfrak{T} \mathfrak{r} (E^b E^{a*} - E^{a*} E^b) (\xi^b \xi^{a*}) &= Tr \mathfrak{T} \mathfrak{r} (\gamma^b \gamma^a \tau - \gamma^a \gamma^b \bar{\tau}) (\gamma^c \gamma^d e_\nu^a \tilde{e}_\mu^b) (\eta_{cd}^{\mu\nu} - \varepsilon g_{cd}^{\mu\nu} \tau_3) \\ &= G^{\mu\nu} G_{\nu\mu} - G_{ab}^{\mu\nu} G_{\nu\mu}^{ba} - 12 \end{aligned}$$

and:

$$\begin{aligned} \frac{1}{2} (E^5 E^{b*} - E^{b*} E^5, \xi^b \xi^{5*}) &= \frac{1}{2} Tr \mathfrak{T} \mathfrak{r} (E^5 E^b - E^b E^5) \xi^b \xi^{5*} \\ &= \frac{1}{2} Tr \mathfrak{T} \mathfrak{r} (\gamma^a \gamma^5 K^* \tilde{\lambda}) (-K \gamma^a \gamma^5 \bar{\tau} \lambda) = -2 Tr (K.K^*) \lambda \tilde{\lambda} \\ \frac{1}{2} (E^a E^{5*} - E^{5*} E^a, R^{5a}) &= \frac{1}{2} Tr \mathfrak{T} \mathfrak{r} ((\tilde{E}^a E^5 - E^5 \tilde{E}^a) \xi^5 \xi^{a*}) \\ &= \frac{1}{2} Tr \mathfrak{T} \mathfrak{r} (\gamma^a \gamma^5 K \lambda) (-K^* \gamma^a \gamma^5 \bar{\tau} \tilde{\lambda}) = -2 Tr (K.K^*) \lambda \tilde{\lambda} \end{aligned}$$

The term $G^{\mu\nu} G_{\nu\mu}$ is written as:

$$G^{\mu\nu} G_{\nu\mu} = \eta^{\mu\nu} \eta_{\nu\mu} + g^{\mu\nu} g_{\nu\mu} = \eta^{\mu\nu} \eta_{\mu\nu} - g^{\mu\nu} g_{\mu\nu}$$

and

$$G^{\mu\nu} G_{\mu\nu} = e_a^\mu \tilde{e}_\mu^a e_b^\nu \tilde{e}_\nu^b = \eta^{\mu\nu} \eta_{\mu\nu} + g^{\mu\nu} g_{\mu\nu} = 4$$

so we get

$$\begin{aligned} G^{\mu\nu} G_{\nu\mu} &= 4 - 2g^{\mu\nu} g_{\mu\nu} = 4 - 2G^{[\mu\nu]} G_{[\mu\nu]} \\ \mathbf{J} &= \int \sqrt{e} \tilde{e} d^4 x \left(-2G^{[\mu\nu]} G_{[\mu\nu]} - 4Tr (K.K^*) \lambda \tilde{\lambda} - 8 - G_{ba}^{\mu\nu} G_{\nu\mu}^{ab} \right) \end{aligned}$$

4.2 The Condition $Tr_k T^a = 0$

If we impose the condition $Tr_k T^a = 0$ and $T^5 \neq 0$ we get:

$$Tr_k (T^a)_{11} = Tr_k (T^a)_{22} = 0 \Rightarrow Tr_\tau Tr_k (T^a)_{11} = Tr_\tau Tr_k (T^a)_{22} = 0$$

so:

$$\partial_\mu \tilde{e}_\nu^a + \omega_\mu^{(2)ab} \tilde{e}_\nu^b = 0 \quad (31)$$

$$\partial_\mu \tilde{e}_\nu^a + \omega_\mu^{(1)ab} \tilde{e}_\nu^b = 0 \quad (32)$$

One solution of (31) and (32) is given by :

$$\omega_\mu^{(1)ab} = \omega_\mu^{(2)ab}$$

Now the condition:

$$Tr_k (T_{12}^a) = Tr_k (T^a)_{21} = 0$$

implies:

$$\omega_\mu^{(2)a5} = \omega_\mu^{(2)5a} = \omega_\mu^{(1)a5} = \omega_\mu^{(1)5a} = 0$$

From the above results, we obtain:

$$\begin{aligned} R_{\mu\nu}^{(1)ab} &= R_{\mu\nu}^{(2)ab} = R_{\mu\nu}^{ab} \\ R_{\mu\nu} &= R_{\mu\nu} = R_{\mu\nu} \end{aligned}$$

while from (21) and (22) we get:

$$\varphi_{12}^{ab} = \varphi_{21}^{ab} = 0 \tag{33}$$

Similarly, the constraint:

$$Tr_k ((T^a)_{11} - (T^a)_{22}) = 0$$

leads to:

$$-\tau_3 \tilde{\lambda} \varphi_{12}^{a5} + \tau_3 \lambda \varphi_{21}^{a5} = 0$$

and therefore:

$$\tilde{\lambda} \varphi_{12}^{a5} = \lambda \varphi_{21}^{a5} \Leftrightarrow \tilde{\lambda} \varphi_{12}^{5a} = \lambda \varphi_{21}^{5a} \tag{34}$$

Consequently:

$$\varphi_{12}^{a5} \varphi_{21}^{5a} = \varphi_{21}^{a5} \varphi_{12}^{5a}$$

Taking now into account the equation (33) one has:

$$\varphi_{12}^{aB} \varphi_{21}^{Ba} = \varphi_{21}^{aB} \varphi_{12}^{Ba}$$

Finally from (23) and (24), we end up with:

$$\begin{aligned} \varphi_{21}^{5b} &= \tilde{e}_b^\mu \partial_\mu \tilde{\lambda} + \tilde{e}_b^\mu \omega_\mu^{(2)55} \tilde{\lambda} \\ \varphi_{12}^{b5} &= e_b^\mu \partial_\mu \lambda - e_b^\mu \omega_\mu^{(2)55} \lambda \end{aligned}$$

4.3 The Unitarity Condition

The unitarity condition $(\Omega^{AB})^* = (\Omega^{BA})$ which takes the form:

$$\begin{pmatrix} -\gamma^a E_a^\mu \tilde{\omega}_\mu^{(1)AB} & \gamma^5 K M_{12} \tilde{\phi}_{21}^{AB} \\ \gamma^5 K^* M_{21} \tilde{\phi}_{12}^{AB} & -\gamma^a E_a^\mu \tilde{\omega}_\mu^{(2)AB} \end{pmatrix} = \begin{pmatrix} \gamma^a E_a^\mu \omega_\mu^{(1)BA} & \gamma^5 K M_{12} \phi_{12}^{BA} \\ \gamma^5 K^* M_{21} \phi_{21}^{BA} & \gamma^a E_a^\mu \omega_\mu^{(2)BA} \end{pmatrix}$$

leads to the following constraints:

$$\begin{aligned} \tilde{\omega}_\mu^{(1)AB} &= -\omega_\mu^{(1)BA}, \quad \tilde{\omega}_\mu^{(2)AB} = -\omega_\mu^{(2)BA} \\ \tilde{\phi}_{12}^{AB} &= \phi_{21}^{BA}, \quad \tilde{\phi}_{21}^{AB} = \phi_{12}^{BA} \\ \tilde{R}_{\mu\nu}^{AB} &= -R_{\mu\nu}^{BA} \end{aligned}$$

4.4 The Generalized Trace

In the Euclidean case, the γ^a Dirac matrices satisfy:

$$\begin{aligned}\gamma^{a*} &= -\gamma^a \quad , \quad \{\gamma^a, \gamma^b\} = \gamma^a \gamma^b + \gamma^b \gamma^a = -2\delta^{ab} \\ \gamma^{ab} &= \frac{1}{2} [\gamma^a, \gamma^b] \quad , \quad \gamma^{(ab)} = \frac{1}{2} \{\gamma^a, \gamma^b\} = -\delta^{ab}\end{aligned}$$

Now, the generalized trace $\mathfrak{Tr} = P\text{tr}$ is defined as:

$$\begin{aligned}\mathfrak{Tr}\gamma^a\gamma^b &= P\text{tr}\gamma^a\gamma^b = \text{tr}\gamma^a\gamma^b = \text{tr}\gamma^{(ab)} = -\delta^{ab} \\ \mathfrak{Tr}\gamma^a\gamma^b\gamma^c\gamma^d &= P\text{tr}\gamma^a\gamma^b\gamma^c\gamma^d = \delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc} \\ \mathfrak{Tr}\gamma^a\gamma^b\gamma^{(cd)} &= P\text{tr}\gamma^a\gamma^b\gamma^{(cd)} = \delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc} \\ \mathfrak{Tr}\gamma^a\gamma^b\gamma^{[cd]} &= P\text{tr}\gamma^a\gamma^b\gamma^{cd} = -2\delta^{ac}\delta^{bd}\end{aligned}$$

where P is an operator which permutes the indices of the γ^a matrices, and "tr" holds for the trace over the Clifford algebra.

Moreover, one can consider \mathfrak{Tr} as an operator acting on the tensors \varkappa_{cd} such that:

$$\begin{aligned}\mathfrak{Tr}\gamma^a\gamma^b\varkappa_{cd} &= -\delta^{ab}\varkappa_{cd} \\ \mathfrak{Tr}\gamma^a\gamma^b\gamma^c\gamma^d\varkappa_{cd} &= \delta^{ab}\varkappa_{cc} - \varkappa_{ab} + \varkappa_{ba} \\ \mathfrak{Tr}\gamma^a\gamma^b\gamma^{(cd)}\varkappa_{cd} &= \delta^{ab}\varkappa_{cc} + \varkappa_{ab} + \varkappa_{ba} \\ \mathfrak{Tr}\gamma^a\gamma^b\gamma^{[cd]}\varkappa_{cd} &= -2\varkappa_{ab}\end{aligned}$$

References

- [1] T.Schucker, J.M.Zylinski, J.Geom.Phys. 16, (1995) 207.
- [2] A.Connes, in "Essay on Physics and Non-Commutative Geometry", THE INTERFACE OF MATHEMATICS AND PARTICLE PHYSICS, Clarendon Press, Oxford (1990).
- [3] D.Kastler, T.Schucker, J.Geom.Phys, 24 (1997)61.
- [4] A.H.Chamseddine, G.Felder, J.Frohlich, Nucl. Phys. B395 (1993) 672
- [5] A.H.Chamseddine, G.Felder, J.Frohlich, Commun.Math.Phys. 155 (1993) 205.
- [6] A.H.Chamseddine, J.Frohlich, O.Grandjean, J.Math. 36 (11) (1995).
- [7] J.Frohlich, O.Grandjean, A.Recknagel, J.Math.Phys. 36(1995)6255.
- [8] J.W.Moffat, Phys.Rev. D19, (1979)3554.
- [9] J.W.Moffat, J.Math.Phys. 29 (7) (1988)1655.
- [10] J.W.Moffat, J.Math.Phys. 21 (7) (1980)1798.
- [11] J.Legaré, J.W.Moffat,in "Field Equations and Conservation Laws in the Nonsymmetric. Gravitational Theory" arXiv:gr-qc/9412009
- [12] F.Khelili, J.Mimouni, N.Mebarki, J.Math.Phys. 42 (8) (2001)3615.