

Quantization of the Scalar Field Coupled Minimally to the Vector Potential

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Abstract: A system of the scalar field coupled minimally to the vector potential is quantized by using canonical path integral formulation based on Hamilton-Jacobi treatment. The equation of motions are obtained as total differential equation and the integrability conditions are examined. © Electronic Journal of Theoretical Physics. All rights reserved.

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1. Introduction

Dirac approach [1,2] is widely used for quantizing the constrained Hamilton systems. The path integral is another approach used for the quantization of constrained systems of classical singular theories which is initiated by Faddeev [3]. Faddeev has applied this approach when only first-class constraints in the canonical gauge are present. Senjanovic [4] generalized Faddeev's method to second-class constraints. Fadkin and Vilkovisky [5,6] rederived both results in a broader context, where they improved procedure to the Grassman variables. Gitman and Tyutin [7] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge.

The Hamilton-Jacobi approach [8-10] is most powerful approach for treating constrained systems. The equations of motion for singular system are obtained as total differential equations in many variables. The integrability conditions for the system lead us to obtain the canonical reduced phase-space coordinates without using any fixing con-

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ditions . Muslih and Güler's have constructed the desired path integral in the context of canonical formalism [11-14], which is based on the Hamilton-Jacobi approach.

In this paper, we shall treat the scalar field coupled minimally to the vector potential as constrained system. The path integral quantization is obtained using both Hamilton-Jacobi approach and Faddeev approach and the results are compared.

2. Path Integral Formulation

In this section, we briefly review the Faddeev method and the Hamilton-Jacobi method for studying the path integral for constrained systems.

2.1 Faddeev Pop Method

Consider a mechanical system with n degrees of freedom and having α first-class constraints ϕ_α , but no second-class constraints, Faddeev has formulated the transition amplitude as [3]

$$\langle Out | S | In \rangle = \int \exp \left[i \int_{-\infty}^{\infty} (p_i \dot{q}_i - H_0) dt \right] \prod_t d\mu(q_i(t), p_i(t)), \quad (1)$$

where H_0 is the Hamiltonian of the system. The measure of integration is defined by

$$d\mu(q, p) = \left(\prod_{a=1}^{\alpha} \delta(\chi_a) \delta(\phi_a) \right) \det || \{ \chi_a, \phi_a \} || \prod_{i=1}^n dp_i dq^i. \quad (2)$$

and $\chi_a(p_i, q_i)$ are the gauge-fixing condition with

1. $\{ \chi_a, \chi_{a'} \} = 0$,
2. $\det || \{ \chi_a, \phi_a \} || \neq 0$.

2.2 Hamilton-Jacobi Path Integral Quantization

One starts from singular Lagrangian $L \equiv L(q_i, \dot{q}_i, \tau)$, $i = 1, 2, \dots, n$, with the Hess matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i \partial \dot{q}_j} \quad i, j = 1, 2, \dots, n, \quad (3)$$

of rank $(n - r)$, $r < n$. Then r momenta are dependent. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \dots, n - r, \quad (4)$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \dots, n. \quad (5)$$

The singular value of the system enables us to solve Eq.(4) for \dot{q}_a as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, p_b; \tau) \equiv w_a. \quad (6)$$

Substituting Eq. (6), into Eq. (5), we get

$$p_\mu = \left. \frac{\partial L}{\partial \dot{q}_\mu} \right|_{\dot{q}_a \equiv \omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, p_a; \tau). \quad (7)$$

Relations (7) indicate the fact that the generalized momenta P_μ are independent of P_a which is a natural result of the singular nature of the Lagrangian.

The canonical Hamiltonian H_0 is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a \equiv w_a; \tau) + p_a \dot{q}_a + P_\mu \dot{q}_\mu \Big|_{p_\mu = -H_\mu}. \quad (8)$$

The set of Hamilton-Jacobi Partial Differential Equations (HJPDE) is expressed as

$$H'_\alpha \left(\tau, q_\mu, q_a, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial \tau} \right) = 0, \quad \alpha = 0, n - p + 1, \dots, n, \quad (9)$$

where

$$H'_\alpha = p_\alpha + H_\alpha, \quad (10)$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq_r = \frac{\partial H'_\alpha}{\partial p_r} dt_\alpha, \quad r = 0, 1, \dots, n, \quad (11)$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad a = 0, \dots, n - p, \quad (12)$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \quad \alpha = 0, n - p + 1, \dots, n, \quad (13)$$

$$dZ = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha, \quad (14)$$

where $Z = S(t_\alpha, q_a)$ being the action. The set of Eqs. (11-14) are integrable if

$$dH'_\alpha = 0, \quad \alpha = 0, n - p + 1, \dots, n. \quad (15)$$

If conditions (15) are not satisfied identically, one may consider them as new constraints and a gain test the integrability conditions, then repeating this procedure, a set of conditions may be obtained.

In this case the path integral representation may be written as [11-14].

$$\langle Out | S | In \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp \left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right], \quad (16)$$

One should notice that the integrate (16) is an integration over the canonical phase-space coordinates q_a, p_a .

3. The Scalar Field Coupled Minimally to the Vector Potential

Consider the action integral for the scalar field coupled minimally to the vector potential as

$$S = \int d_4x L, \quad (17)$$

where the Lagrangian L is given by

$$L = -\frac{1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x) + (D_\mu\varphi)^*(x)D^\mu\varphi(x) - m^2\varphi^*(x)\varphi(x), \quad (18)$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (19)$$

and

$$D_\mu\varphi(x) = \partial_\mu\varphi(x) - ieA_\mu(x)\varphi(x). \quad (20)$$

Let us first discuss the system using Hamilton-Jacobi approach. In this approach the canonical momenta (4) and (15) take the forms

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \quad (21)$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0, \quad (22)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = (D_0\varphi)^* = \dot{\varphi}^* + ieA_0\varphi^*, \quad (23)$$

$$p_{\varphi^*} = \frac{\partial L}{\partial \dot{\varphi}^*} = (D_0\varphi) = \dot{\varphi} - ieA_0\varphi, \quad (24)$$

From Eqs. (21), (23) and (24), the velocities \dot{A}_i , $\dot{\varphi}^*$ and $\dot{\varphi}$ can be expressed in terms of momenta π_i , p_φ and p_{φ^*} respectively as

$$\dot{A}_i = -\pi_i - \partial_i A_0, \quad (25)$$

$$\dot{\varphi}^* = p_\varphi - ieA_0\varphi^*, \quad (26)$$

$$\dot{\varphi} = p_{\varphi^*} + ieA_0\varphi. \quad (27)$$

The canonical Hamiltonian H_0 is obtained as

$$H_0 = \frac{1}{4} F^{ij}F_{ij} - \frac{1}{2} \pi_i\pi^i + \pi^i \partial_i A_0 + p_{\varphi^*}p_\varphi + ieA_0\varphi p_\varphi - ieA_0\varphi^* p_{\varphi^*} - (D_i\varphi)^*(D^i\varphi) + m^2\varphi^*\varphi. \quad (28)$$

Making use of (9) and (10), we find for the set of HJPDE

$$H'_0 = \pi_4 + H_0, \quad (29)$$

$$H' = \pi_0 + H = \pi_0 = 0, \quad (30)$$

Therefore, the total differential equations for the characteristic (11-13) are obtained as

$$\begin{aligned} dA^i &= \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'}{\partial \pi_i} dA^0, \\ &= -(\pi^i + \partial_i A_0) dt, \end{aligned} \quad (31)$$

$$dA^0 = \frac{\partial H'_0}{\partial \pi_0} dt + \frac{\partial H'}{\partial \pi_0} dA^0 = dA^0, \quad (32)$$

$$\begin{aligned} d\varphi &= \frac{\partial H'_0}{\partial p_\varphi} dt + \frac{\partial H'}{\partial p_\varphi} dA^0, \\ &= (p_{\varphi^*} + ieA_0\varphi) dt, \end{aligned} \quad (33)$$

$$\begin{aligned} d\varphi^* &= \frac{\partial H'_0}{\partial p_{\varphi^*}} dt + \frac{\partial H'}{\partial p_{\varphi^*}} dA^0, \\ &= (p_\varphi - ieA_0\varphi^*) dt, \end{aligned} \quad (34)$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'}{\partial A_i} dA^0, \\ &= [\partial_l F^{li} + ie(\varphi^* \partial^i \varphi + \varphi \partial_i \varphi^*) + 2e^2 A^i \varphi \varphi^*] dt, \end{aligned} \quad (35)$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'}{\partial A_0} dA^0, \\ &= [\partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_\varphi] dt, \end{aligned} \quad (36)$$

$$\begin{aligned} dp_\varphi &= -\frac{\partial H'_0}{\partial \varphi} dt - \frac{\partial H'}{\partial \varphi} dA^0, \\ &= [(\vec{D} \cdot \vec{D} \varphi)^* - m^2 \varphi^* - ieA_0 p_\varphi] dt, \end{aligned} \quad (37)$$

and

$$\begin{aligned} dp_{\varphi^*} &= -\frac{\partial H'_0}{\partial \varphi^*} dt - \frac{\partial H'}{\partial \varphi^*} dA^0, \\ &= [(\vec{D} \cdot \vec{D} \varphi) - m^2 \varphi + ieA_0 p_{\varphi^*}] dt. \end{aligned} \quad (38)$$

The integrability condition ($dH'_\alpha = 0$) implies that the variation of the constraint H' should be identically zero, that is

$$dH' = d\pi_0 = 0, \quad (39)$$

which lead to a new constraint

$$H'' = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_\varphi = 0. \quad (40)$$

Taking the total differential of H'' , we have

$$dH'' = \partial_i d\pi^i + iep_{\varphi^*} d\varphi^* + ie\varphi^* dp_{\varphi^*} - ie\varphi dp_{\varphi} - iep_{\varphi} d\varphi = 0. \quad (41)$$

Then the set of equations (31-38) is integrable. Therefore, the canonical phase space coordinates (φ, p_{φ}) and $(\varphi^*, p_{\varphi^*})$ are obtained in terms of parameters (t, A^0) .

Making use of Eqs.(14) and (28-30), one gets the canonical action integral as

$$Z = \int d^4x \left(-\frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + p_{\varphi} p_{\varphi^*} + \vec{D}\varphi^* \cdot \vec{D}\varphi + m^2 |\varphi|^2 \right), \quad (42)$$

where

$$\vec{D} = \vec{\nabla} + ie\vec{A}. \quad (43)$$

Now the path integral representation (16) is given by

$$\langle out|S|In \rangle = \int \prod_i dA^i d\pi^i d\varphi dp_{\varphi} d\varphi^* dp_{\varphi^*} \exp \left[i \left\{ \int d^4x \left(-\frac{1}{2} \pi_i \pi^i - \frac{1}{4} F^{ij} F_{ij} + p_{\varphi} p_{\varphi^*} + (D_i \varphi)^* (D_i \varphi) - m^2 \varphi^* \varphi \right) \right\} \right]. \quad (44)$$

To apply the Faddeev method to the pervious system, we start with the total Hamiltonian

$$H_T = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + p_{\varphi^*} p_{\varphi} + ieA_0 \varphi p_{\varphi} - ieA_0 \varphi^* p_{\varphi^*} - (D_i \varphi)^* (D_i \varphi) + m^2 \varphi^* \varphi + \lambda \pi_0. \quad (45)$$

According to Dirac's method, the time derivative of the primary constraints should be zero, that is

$$\dot{H}' = \{H', H_T\} = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_{\varphi} \approx 0, \quad (46)$$

which leads to the secondary constraints

$$H'' = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_{\varphi} \approx 0. \quad (47)$$

There are no tertiary constraints, since

$$\dot{H}'' = \{H'', H_T\} = 0. \quad (48)$$

By taking suitable linear combinations of constraints, one has to find the first-class one, that is

$$\Phi = H' = \pi_0. \quad (49)$$

The equations of motion read as

$$\dot{A}^i = \{A_0^i, H_T\} = -(\pi^i + \partial_i A_0), \quad (50)$$

$$\dot{A}^0 = \{A^0, H_T\} = \lambda, \quad (51)$$

$$\dot{\varphi} = \{\varphi, H_T\} = (p_{\varphi^*} + ieA_0\varphi), \quad (52)$$

$$\dot{\varphi}^* = \{\varphi^*, H_T\} = (p_{\varphi} - ieA_0\varphi^*), \quad (53)$$

$$\dot{\pi}^i = \{\pi^i, H_T\} = \partial_l F^{li} + ie(\varphi^* \partial^i \varphi + \varphi \partial_i \varphi^*) + 2e^2 A^i \varphi \varphi^*, \quad (54)$$

$$\dot{\pi}^0 = \{\pi^0, H_T\} = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_{\varphi}, \quad (55)$$

$$\dot{p}_{\varphi} = \{p_{\varphi}, H_T\} = (\vec{D} \cdot \vec{D}\varphi)^* - m^2\varphi^* - ieA_0 p_{\varphi}, \quad (56)$$

$$\dot{p}_{\varphi^*} = \{p_{\varphi^*}, H_T\} = (\vec{D} \cdot \vec{D}\varphi) - m^2\varphi + ieA_0 p_{\varphi^*}. \quad (57)$$

We will contact ourselves with a partial gauge fixing by introducing gauge constraints for the first-class primary constraints only, just to fix the multiplier λ in Eq.(45). Since there are weakly vanishing, a gauge choice near at hand would be:

$$\phi' = A_0 = 0. \quad (58)$$

But for this forbids dynamic at all, since the requirement $\dot{A}_0 = 0$ implies $\lambda = 0$.

Making use of Eq.(1), we obtain the path integral quantization

$$\begin{aligned} \langle out|S|In\rangle = \int \exp \left[i \int_{-\infty}^{+\infty} \left(-\frac{1}{2} \right) \pi_i \pi^i - \frac{1}{4} F^{ij} F_{ij} + p_{\varphi} p_{\varphi^*} \right. \\ \left. + \vec{D}\varphi^* \cdot \vec{D}\varphi - m^2 |\varphi|^2 \right] d^4x dA^i d\pi^i d\varphi dp_{\varphi} d\varphi^* dp_{\varphi^*}. \quad (59) \end{aligned}$$

We showed that Eq.(44) and Eq.(59) are identical.

4. Conclusion

Path integral quantization of the scalar field coupled minimally to the vector potential is obtained by using the canonical path integral formulation [11-14]. The integrability conditions dH'_0 and dH' are satisfied, the system is integrable, hence the path integral is obtained directly as an integration over the canonical phase space coordinates $A^i, \pi^i, \varphi, P_{\varphi}, \varphi^*$ and p_{φ^*} without using any gauge fixing conditions.

The Hamilton-Jacobi quantization is simpler and more economical. Also there is no need to distinguish between first and second-class constraints, and there is no need to introduce Lagrange multipliers; all that is needed is the set of Hamilton-Jacobi partial differential equations and the equations of motion. If the system is integrable then one can construct the canonical phase space.

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