

A Generalized Option Pricing Model

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Abstract: The Black Scholes model of option pricing constitutes the cornerstone of contemporary valuation theory. However, the model presupposes the existence of several unrealistic assumptions including the lognormal distribution of stock market price processes. There, now, subsists abundant empirical evidence that this is not the case. Consequently, several generalisations of the basic model have been attempted with relaxation of some of the underlying assumptions. In this paper, we postulate a generalization that contemplates a statistical feedback process for the stochastic term in the Black Scholes partial differential equation. Several interesting implications of this modification emanate from the analysis and are explored.

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1. Introduction

With the rapid advancements in the evolution and study of disordered systems and the associated phenomena of nonlinearity, chaos, self organized criticality etc., the importance of generalizations of the extant mathematical apparatus to enhance its domain of applicability to such disordered systems is cardinal to the further development of science. A possible mechanism for achieving this objective is through deformation of standard mathematics.

A considerable amount of work has already been done and success achieved in the broad areas of q -deformed harmonic oscillators [1], representations of q -deformed rotation and Lorentz groups [2-3]. q -deformed quantum stochastic processes have also been studied

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with realization of q-white noise on bialgebras [4], deformations of the Fokker Planck's equation [5], Langevin equation [6] and Levy processes [7-8] have also been analysed and results reported.

Though at a nascent stage, the winds of convergence of physics and finance are unmistakably perceptible with several concepts of fundamental physics like quantum mechanics, field theory and related tools of non-commutative probability, gauge theory, path integral etc. being applied for pricing of contemporary financial products and for explaining various phenomena of financial markets like stock price patterns, critical crashes etc [8-19].

The celebrated Black Scholes formula [20,21] constitutes the cornerstone of contemporary valuation theory. However, the model, although very robust and of immense practical utility is based on several unrealistic and rigid assumptions. Several generalizations have been attempted through relaxation of one or other assumption, thereby enhancing its spectrum of applicability.

In this paper, we attempt one such generalization based on the deformation of the standard Brownian motion. Section 2, which forms the essence of this paper, attempts a deformation of the standard Black Scholes pricing formula. In Section 3 we illustrate the theory developed in the previous section with a concrete example. Section 4 looks at the interpretation of the deformation index. Section 5 addresses issues relating to empirical relevance of the model. Section 6 the conclusions.

2. The Generalized Black Scholes Model

The standard analysis of the Black Scholes formula for option pricing presupposes that the stock price follows the lognormal distribution. However, significant empirical evidence now subsists of the stock returns deviating from the lognormal distribution with “fat tails” and a “sharp peak” which better fit the truncated Levy flights or other power law distributions [9, 22, 23]. To broadbase the Black Scholes model, generalizations by way of “Levy noise” and “jump diffusions” [24] have already been studied. In this paper, we propose a model that incorporates a “weighted Brownian motion” as the stochastic (noise) term, where the weights themselves are a function of the “Brownian motion / noise” i.e.

$$dW_t^P \rightarrow dU_t^P = f(U_t^P, t) dW_t^P \quad (1)$$

W_t^P is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E^P(dW_t dW_{t'}) = dt dt' \delta(t - t')$ and on some filtered probability space $(\Omega, (F_t), P)$. We, further, mandate that the function $f(U_t^P, t)$ satisfies the Novikov condition and that the process $U_t^P = \int_0^t f(U_s^P, s) dW_s^P$ is a local P -martingale with a non normal distribution. This requirement is not as restrictive as it may seem at first sight in context of the applications envisaged. We shall address this issue again in the sequel.

This generalization contemplates a statistical feedback process. In this context, several studies on stock market data have shown the existence of nonlinear characteristics and chaotic behavior that lend credence to the existence of a statistical feedback mechanism of market players. Explanations for the existence of “fat tails” in stock market

data have been offered through this statistical feedback process e.g. “extremal events” cause “disproportionate reactions” among market players. This deformed noise may also capture the “herd behavior” of stock market investors. The model also encompasses time dependent return processes since f is a function of U_t^P and t so that the drift term varies with time.

We define the European call option as a financial contingent claim that entails a right (but not an obligation) to the holder of the option to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). The option contract, therefore, has a terminal payoff of $\max(S_T - E, 0) = (S_T - E)^+$ where S_T is the stock price on the exercise date and E is the exercise price.

We consider a non-dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{(\mu t + \sigma U_t^P)}$ under the probability measure P with drift μ and volatility σ . The logarithm of the stock price $Y_t = \ln S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dU_t^P = \mu dt + [\sigma f(U_t^P, t)] dW_t^P \quad (2)$$

Application of Ito’s formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2} [\sigma f(U_t^P, t)]^2 \right) S_t dt + [\sigma f(U_t^P, t)] S_t dW_t^P \quad (3)$$

Let $C(S_t, t)$ denote the instantaneous price of a call option with exercise price E at any time t before maturity when the price per unit of the underlying is S_t . We assume that $C(S_t, t)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C(S_t, t)$ yields

$$dC_t = \left[\left(\mu + \frac{1}{2} [\sigma f(U_t^P, t)]^2 \right) S_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} [\sigma f(U_t^P, t)]^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right] dt + \frac{\partial C}{\partial S} [\sigma f(U_t^P, t)] S_t dW_t^P, \quad (4)$$

Applying Girsanov’s theorem to the price process (3), we perform a change of measure and define a probability measure Q such that the discounted stock price process $Z_t = S_t e^{-rt}$ or equivalently

$$dZ_t = \left(\mu - r + \frac{1}{2} [\sigma f(U_t^P, t)]^2 \right) Z_t dt + [\sigma f(U_t^P, t)] Z_t dW_t^P \quad (5)$$

behaves as a martingale with respect to Q . This is performed by eliminating the drift term through the transformation

$$\frac{\left(\mu - r + \frac{1}{2} [\sigma f(U_t^P, t)]^2 \right)}{\sigma f(U_t^P, t)} \rightarrow \gamma_t \quad (6)$$

whence $W_t^Q = W_t^P + \gamma_t t$ is a Brownian motion without drift with respect to the measure Q and $dZ_t = [\sigma f(U_t^Q, t)] Z_t dW_t^Q$ which is driftless under the measure Q and hence, Z_t is a Q -martingale.

The equivalence of $[\sigma f(U_t^P, t)] Z_t dW_t^P$ and $[\sigma f(U_t^Q, t)] Z_t dW_t^Q$ follows from the fact that both W_t^Q, W_t^P are zero mean Weiner processes and that $f(U_t^Q, t)$ can be expressed in terms of $f(U_t^P, t)$ through $dZ_t = [\sigma f(U_t^Q, t)] Z_t dW_t^Q$ alongwith eq. (5). The noise terms in $dZ_t = [\sigma f(U_t^Q, t)] Z_t dW_t^Q$ and eq. (5), will, therefore, be equivalent stochastically.

The two measures P & Q are related through the Radon Nikodym derivative which in the deformed case takes the form

$$\xi(t) = \frac{dQ}{dP} = \exp\left(-\int_0^t \gamma_t dW_t^P - \frac{1}{2} \int_0^t \gamma_t^2 dt\right) \quad (7)$$

and the expectation operators under the two measures are related as

$$E^Q(X_t | F_s) = \xi^{-1}(s) E^P(\xi(t) X_t | F_s) \quad (8)$$

Our next step in martingale based pricing is to constitute a Q martingale process that hits the discounted value of the contingent claim i.e. call option. This is formed by taking the conditional expectation of the discounted terminal payoff from the claim under the Q 'measure i.e.

$$E_t = E^Q[e^{-rT} (S_T - E)^+ | F_t]. \quad (9)$$

We now constitute a self-financing strategy that exactly replicates the claim and whose value is known with certainty. For this purpose, we introduce a 'bond' in our model that evolves according to the following price process

$$\frac{dB_t}{B_t} = r dt, B_0 = 1, \quad (10)$$

where r is the relevant risk free interest rate.

Making use of ϕ_t units of the underlying asset and ψ_t units of the bond, where $\phi_t = \frac{\partial C(S_t, t)}{\partial S}$, $B_t \psi_t = C(S_t, t) - \phi_t S_t$, we can now construct a trading strategy that has the following properties

(1) it exactly replicates the price process of the call option i.e.

$$\phi_t S_t + \psi_t B_t = C(S_t, t), \forall t \in [0, T]. \quad (11)$$

(2) it is self financing i.e.

$$\phi_t dS_t + \psi_t dB_t = dV_t, \forall t \in [0, T]. \quad (12)$$

Using eqs. (1), (3), (11) & (12) we have

$$dC = \left(\phi_t \mu S_t + \frac{1}{2} \phi_t [\sigma f(U_t^P, t)]^2 S_t + \psi_t r B_t\right) dt + \phi_t [\sigma f(U_t^P, t)] S_t dW_t^P. \quad (13)$$

Matching the diffusion terms of (3) & (13) and using (11), we get the aforesaid expressions for ϕ_t and ψ_t respectively. The value of this portfolio at any time t can be shown to be

equal to $V_t = e^{rt} E_t$ with E_t being given by eq.(9). It follows that the value of the replicating portfolio and hence of the call option at time t is given by

$$\begin{aligned} V_t &= e^{rt} E_t = e^{-r(T-t)} E^Q[(S_T - E)^+ | F_t] = e^{-r(T-t)} E^Q[(S_T - E)1_{(S_T \geq E)} | F_t] \\ &= e^{-r(T-t)} \int_{\{U_T^Q: S(U_T^Q, T) \geq E\}} (S(U_T^Q, T) - E) \bar{f}(U_T^Q, T | U_T^Q, t) dU_T^Q \end{aligned} \quad (14)$$

The expectation value of the contingent claim $\max(S_T - E, 0) = (S_T - E)^+$ under the measure Q depends only on the marginal distribution of the stock price process S_t under the measure Q which is obtained by writing it in terms of Q Brownian motion W_t^Q . We have, from eq.(2), for the deformed stock price process under the measure Q as

$$d(\ln S_t) = \mu dt + [\sigma f(U_t^P, t)] dW_t^P = \left(r - \frac{1}{2} [\sigma f(U_t^Q, t)]^2 \right) dt + [\sigma f(U_t^Q, t)] dW_t^Q \quad (15)$$

which on integration yields

$$S_t = S_0 \exp \left[\int_0^t [\sigma f(U_s^Q, s)] dW_s^Q + \int_0^t \left(r - \frac{1}{2} [\sigma f(U_s^Q, s)]^2 \right) ds \right]. \quad (16)$$

The value of the call option can now be computed by using eq. (14). The existence or otherwise of a closed form solution would depend on the explicit representation of the function $f(U, t)$.

The following observations are cardinal to the above analysis.

(a) We have, implicitly, made the standard assumption of the market satisfying the “No Arbitrage” condition. It is well known that long-term market equilibrium cannot subsist in the presence of arbitrage opportunities. This “No Arbitrage” condition guarantees the existence and measurability of γ_t defined by eq. (6) as is proved below:

For this purpose, we assume that there exist values of U_t^P for which $f(U_t^P, t) = 0$ and hence, γ_t does not exist. Let $X_t = \{U_t^P : f(U_t^P, t) = 0\}$. We construct a portfolio (ϕ, ψ) of the normalized stock process (\bar{S}_t) and the bond process

(B_t)

where

$$\phi = \begin{cases} \theta & \text{for } U_t^P \in X_t \\ 0 & \text{for } U_t^P \notin X_t \end{cases}$$

and

$\psi_t = \psi_0 + \phi_0 S_0 + \int_0^t e^{-rs} \phi_s dS_s - \int_0^t r e^{-rs} \phi_s ds - e^{-rt} \phi_t S_t, B_0 = 1$ and the normalized stock process i.e. the stock process adapted to a market with zero interest rates being given by $\bar{S}_t = S_t e^{-rt}$ and $d\bar{S}_t = e^{-rt} dS_t - r e^{-rt} S_t dt$.

The portfolio is self financing since $V_t = \psi_t + \phi_t \bar{S}_t$ and hence, $dV_t = \psi_t + \phi_t d\bar{S}_t$. Further, $V_t - V_0 = \int_0^t \phi_s d\bar{S}_s = \int_0^t e^{-rs} \left(\mu + \frac{1}{2} [of(U_s^P, s)]^2 - r \right) \phi_s S_s ds + \int_0^t e^{-rs} [of(U_s^P, s)] \phi_s S_s dW_s^P$
 $= \int_0^t \mathbb{1}_{X_s} e^{-rs} \left(\mu + \frac{1}{2} [of(U_s^P, s)]^2 - r \right) \theta_s S_s ds + \int_0^t \mathbb{1}_{X_s} e^{-rs} [of(U_s^P, s)] \theta_s S_s dW_s^P = \int_0^t \mathbb{1}_{X_s} e^{-rs} (\mu - r) \theta_s S_s ds \geq 0$

where \mathfrak{N}_{X_t} is the characteristic function of the set $X_t \forall U, t$. But under the “No Arbitrage” condition $V_t - V_0 \leq 0$. It, therefore, follows that $\mathfrak{N}_{X_t} = 0 \forall U, t$ and hence, $X_t = \phi$.

(b) In the standard Black Scholes theory, the Novikov condition is automatically satisfied due to the constancy of $\gamma_t \equiv \gamma$. However, in the deformed version, this condition needs to be explicitly imposed to ensure the applicability of the Girsanov’s theorem and hence, the existence of the equivalent martingale measure Q . Hence, we require that the function $f(U, t)$ to be such that $E^P \left\{ \exp \left[\frac{1}{2} \int_0^T (\gamma_s)^2 ds \right] \right\} < \infty$. As mentioned above, this condition is not very restrictive insofar as the applications of this model are concerned, since $f(U, t)$ would normally take the form of probability distributions and hence, be non zero bounded functions, thereby, automatically satisfying the square integrability requirements.

(c) Except for the Novikov condition, which needs to be explicitly imposed in the deformed model as mentioned in (b) above, our analysis is equivalent to the standard Black Scholes model since $f(U, t)$ can be expressed as a function of Y , the logarithm of the stock price S through eq. (2);

(d) The “No Arbitrage” condition together with the Novikov Condition guarantee the completeness of the market and hence, the availability of replicating portfolios for the valuation of any contingent claim. This is established by showing that there exists a self financing portfolio (ϕ, ψ) defined as in (a) above that exactly replicates the terminal payoff of any lower bounded contingent claim, say $C(S_T, T)$. Mathematically, this implies that there exists a real number ε such that $C(S_T, T) = V_T^\varepsilon = \varepsilon + \int_0^T (\phi_t dS_t + \psi_t dB_t)$ or equivalently

$$\begin{aligned} C(S_T, T) &= V_T^\varepsilon = \varepsilon + \int_0^T (\phi_t dS_t + \psi_t dB_t) = e^{rT} \left(\varepsilon + \int_0^T \phi_t d\bar{S}_t \right) \\ &= e^{rT} \left[\varepsilon + \int_0^T e^{-rt} \left(\mu + \frac{1}{2} [of(U_t^P, t)]^2 - r \right) \phi_t S_t dt + \int_0^T e^{-rt} [of(U_t^P, t)] \phi_t S_t dW_t^P \right] = e^{rT} \left[\varepsilon + \int_0^T e^{-rt} [of(U_t^Q, t)] \phi_t S_t dW_t^Q \right] \end{aligned}$$

By the Martingale Representation Theorem, there exists a function η_t such that

$C(S_T, T) = e^{rT} \left\{ E^Q [e^{-rT} C(S_T, T)] + \int_0^T \eta_t S_t dW_t^Q \right\}$. Hence, we can identify $\varepsilon = E^Q [e^{-rT} C(S_T, T)]$ and $\phi_t = e^{rt} [of(U_t^Q, t)]^{-1} \eta_t$. By selecting the bond component of the portfolio (ψ) according to $\psi_t = \psi_0 + \int_0^t e^{-rs} d\lambda_s$ where $\lambda_s = \int_0^s \phi_v dS_v - \phi_s S_s$, we can make our portfolio (ϕ, ψ) self financing. This is shown below. We have,

$$dV_t = d(\psi_t e^{rt} + \phi_t S_t) = r e^{rt} \psi_t dt + e^{rt} d(\psi_t) + d(\phi_t S_t) = r e^{rt} \psi_t dt + e^{rt} d(\lambda_t) + d(\phi_t S_t) = r e^{rt} \psi_t dt + \phi_t dS_t$$

as required. Furthermore,

$$V_t^\varepsilon = e^{rt} \left(\varepsilon + \int_0^t \phi_v d\bar{S}_v \right) = e^{rt} \left(\varepsilon + \int_0^t \eta_v S_v dW_v^Q \right) = e^{rt} E^Q (e^{-rT} V_T^\varepsilon | F_t) = e^{rt} E^Q (e^{-rT} C(S_T, T) | F_t)$$

showing that V_t^ε is lower bounded and hence, establishing the completeness of the market.

3. An Illustration of the Deformed Model

We now present a concrete example as an application of the aforesaid analysis. For the purpose, we consider a Brownian motion of the form

$$dW_t^P \rightarrow dU_t^P = f(U_t^P, t)^q dW_t^P \quad (17)$$

where $f(U_t^P, t)$ **is a probability density function.**

The incorporation of probability dependent term in the stochastic force enables us to describe nonlinear return processes where the randomness is not uniform across the entire return spectrum. In the standard theory, we envisage a random process that is independent of the level of returns and hence, if sufficient number of observations are accumulated, the entire spectrum of possible returns will be traversed. However, through this deformed noise function we can model return processes that change with the respective probability of such returns i.e. the degree of randomness changes across the return spectrum – highly frequented regions of the spectrum may have higher/lower returns depending on the nature of the deformation function. Hence, a biased yet random return process can be accommodated. Although, in theory, the entire return spectrum may still be traversed if sufficient number of observations are made, yet the dependence on probabilities enable the modeling of systems that require a cleavage of the return spectrum to create an effectively nonergodic space for the system. The model would also be versatile enough to encompass a return spectrum having the character of a multifractal which goes well with contemporary research findings in this area. Furthermore, unlike the standard case where $W_t^P = \int_0^t dW_s^P$ is normally distributed, $U_t^P = \int_0^t f(U_s^P, s) dW_s^P$ is no longer normally distributed but follows a skewed distribution depending on the explicit representation of the function $f(U_t^P, t)$ and parameter q .

Eq. (17) is equivalent to the Langevin equation [25]

$$\frac{dU_t^P}{dt} = f(U_t^P, t)^q \frac{dW_t^P}{dt} = f(U_t^P, t)^q \eta(t) \quad (18)$$

$\eta(t)$ is a noise function that satisfies

$$\langle \eta(t) \rangle = 0 \quad (19)$$

$$\langle \eta(t') dt' \eta(t'') dt'' \rangle = \delta(t' - t'') dt' \quad (20)$$

The time evolution of the probability density $f(U_t^P, t)$ is given by the following equation [26] (The super(sub)scripts are suppressed for the sake of brevity)

$$f(U, t + \Delta t) = \int \bar{f}(U, t + \Delta t | U', t) \cdot f(U', t) dU' \quad (21)$$

\bar{f} is the transition probability between states. We now set $U' = U - \Delta U$ and expand the integrand as a Taylor's series around $\bar{f}(U + \Delta U, t + \Delta t | U, t) f(U, t)$ to obtain

$$\begin{aligned} \bar{f}(U, t + \Delta t | U', t) f(U', t) &= -\Delta U \frac{d}{dU} \bar{f}(U + \Delta U, t + \Delta t | U, t) f(U, t) + \\ &-\frac{\Delta U^2}{2} \frac{d^2}{dU^2} \bar{f}(U + \Delta U, t + \Delta t | U, t) f(U, t) + \dots \end{aligned} \quad (22)$$

Eq. (22) on integration gives

$$f(U, t + \Delta t) = -\frac{d}{dU} \left[\int \Delta U \bar{f}(U + \Delta U, t + \Delta t | U, t) d\Delta U \right] f(U, t) + \quad (23)$$

$$-\frac{1}{2} \frac{d^2}{dU^2} \left[\int \Delta U^2 \bar{f}(U + \Delta U, t + \Delta t | U, t) d\Delta U \right] f(U, t) + \dots$$

We can further simplify the above expression, noting that U is a martingale, as follows:-

$$\int \Delta U \bar{f}(U + \Delta U, t + \Delta t | U, t) d\Delta U = E_t[\Delta U] = E_t \left[\int_t^{t+\Delta t} f(U_s, s) dW_s \right] = 0 \quad (24)$$

and

$$\int \Delta U^2 \bar{f}(U + \Delta U, t + \Delta t | U, t) d\Delta U = E_t[\Delta U^2] = E_t \left[\int_t^{t+\Delta t} f(U_s, s) ds \right] = f(U_s, t) \Delta t + o(\Delta t) \quad (25)$$

where the last step follows from Ito isometry. We have ignored terms of second and higher orders in Δt . Using the results in eqs. (24) & (25) in eq. (23) and taking the limit as $\Delta t \rightarrow 0$ we obtain the Fokker Planck equation [26] for the time evolution of the deformed probability density (17) as

$$\frac{df}{dt} = \frac{1}{2} \frac{d^2 f^{2q+1}}{dU^2} \quad (26)$$

To obtain an explicit solution of eq. (26) for the probability density $f(U, t)$, we postulate a normalized scaled solution, which enables the separation of the U and t dependencies through the ansatz

$$f(U, t) = g(t) H(Ug(t)) = g(t) H(z) \quad (27)$$

Substitution from eq. (27) into eq. (26) and simplification yields

$$\frac{g(t)}{g(t)^{2q+3}} \frac{\partial}{\partial z} (zH(z)) = \frac{1}{2} \frac{\partial^2}{\partial z^2} H(z)^{2q+1} \quad (28)$$

Writing $\frac{2g(t)}{g(t)^{2q+3}} = -k$, we have

$$g(t) = [(q+1)k(t-t_0)]^{-\frac{1}{2(q+1)}} \quad (29)$$

which gives the solution of eq. (26) as

$$f(U, t) = A(t-t_0)^{-\frac{1}{2(q+1)}} \exp_{(1-2q)} \left\{ B \left[(U-U_0)(t-t_0)^{-\frac{1}{2(q+1)}} \right]^2 \right\} \quad (30)$$

where $A = [(q+1)k]^{-\frac{1}{2(q+1)}}$, $B = -\frac{kA^2}{4(2q+1)}$ and $\exp_q(x) = [1 + (1-q)x]^{\frac{1}{1-q}}$ is the q -exponential function. k can be determined from the normalization condition $\int_{-\infty}^{\infty} f(U, t) dU = 1$, $f(U, t)$ being a probability density function.

The transition probability density $\bar{f}(U, t|U_0, t_0)$, that is the key element in option pricing, is the probability density $f(U, t)$ with a special initial condition $f(U, t_0) = \delta(U - U_0)$ i.e. $\bar{f}(U, t|U_0, t_0)$ also obeys the Fokker Planck equation (26). Furthermore, it is seen that the solution for $f(U, t)$ given by eq. (30) meets the δ function initial condition in the limit $t \rightarrow t_0$, and is, therefore, also a solution for the transition probability density $\bar{f}(U, t|U_0, t_0)$.

As an illustration, the conditional probability density of the logarithm of the stock prices would be

$$\bar{f}(Y_{t+\Delta t}|Y_t) = A(\Delta t)^{-\frac{1}{2(q+1)}} \exp_{(1-2q)} \left\{ B \left[\frac{(\ln \frac{S_{t+\Delta t}}{S_t} - \mu \Delta t)}{\sigma} (\Delta t)^{-\frac{1}{2(q+1)}} \right]^2 \right\}$$

under the probability measure P and

$$\bar{f}(Y_{t+\Delta t}|Y_t) = A(\Delta t)^{-\frac{1}{2(q+1)}} \exp_{(1-2q)} \left\{ B \left[\frac{1}{\sigma} \left(\ln \frac{S_{t+\Delta t}}{S_t} \right) (\Delta t)^{-\frac{1}{2(q+1)}} \right]^2 \right\}$$

under Q .

Using the expression (30) for $f(U, t)$ with $U_0 = 0, t_0 = 0$ (which does not result in any loss of generality) in eq. (16), we derive the expression for the stock price process under the martingale measure Q and, thereby, of the contingent claim using eq. (14). To approximate $\int_0^t f(U, s)^{2q} ds$ we note that for any arbitrary value of time s , the distribution of the random variable U_s can be mapped onto the distribution of a random variable ω at a fixed time T through the transformation $U_s = \left(\frac{T}{s}\right)^{-\frac{1}{2(1+q)}} U_T$. Hence, $\int_0^t f(U, s)^{2q} ds =$

$$\begin{aligned} & \int_0^t f \left(\left(\frac{T}{s}\right)^{-\frac{1}{2(1+q)}} U_T, s \right)^{2q} ds \\ &= A^{2q} \int_0^t s^{-\frac{q}{(q+1)}} \exp_{(1-2q)}^{2q} \left[B \left(U_T T^{-\frac{1}{2(q+1)}} \right)^2 \right] ds = C t^{\frac{1}{(q+1)}} \exp_{(1-2q)}^{2q} \left[B \left(U_T T^{-\frac{1}{2(q+1)}} \right)^2 \right] \end{aligned} \tag{31}$$

where $C = (q + 1) A^{2q}$.

Furthermore, $\int_0^t f(U, t)^q dW = U(t)$, in view of eq. (17). Substituting this result and that of eq. (31) in eq. (16), we get the following expression for the stock price process in the martingale measure Q

$$S_t = S_0 \exp \left\{ \sigma U_t + rt - \frac{1}{2} \sigma^2 C t^{\frac{1}{(q+1)}} \exp_{(1-2q)}^{2q} \left[B \left(U_T T^{-\frac{1}{2(q+1)}} \right)^2 \right] \right\} \tag{32}$$

from which the value of the call option can be recovered using (14). It may, however, be noted that in the standard case the exponential is linear in W and the stock price, therefore, is a monotonically increasing function of W . Hence, the condition $S_t - E > 0$ is satisfied for all values of W that exceed a threshold value. However, in this illustration, consequent to the noise induced drift, the exponential in the stock price process is now a quadratic function of the deformed Brownian motion U . We, therefore, have two roots of U that meet the condition $S_t - E = 0$. Accordingly, there will exist an interval (U_1, U_2)

within which the inequality $S_t - E > 0$ will hold. Furthermore, as $q \rightarrow 0$, $U_2 \rightarrow \infty$ thereby recovering the standard case. Hence, we have

$$V_t = e^{-r(T-t)} \int_{U_1}^{U_2} \left(S_0 e^{\left\{ \sigma U_T + rT - \frac{1}{2} \sigma^2 CT^{\frac{1}{q+1}} \exp^{2q}_{(1-2q)} \left[B \left(U_T T^{-\frac{1}{2(q+1)}} \right)^2 \right] \right\}} - E \right) f(U_T, T) dU \quad (33)$$

As in the standard case, in the martingale measure based risk neutral world, the stock price distribution under Q is dependent on the risk free interest rate r and not on the average return μ . We easily recover the standard results from the generalized model in the limit $q \rightarrow 0$.

4. Interpretation of the q Index

Towards examining the interpretation of the q index in the context of the application being envisaged, we study the impact of the deformation of the standard exponential distribution $g(U, \zeta) = C e^{BU^2 \zeta}$. For this purpose, we note that $f(U, t)$, with $U_0 = 0$, $t_0 = 0$, can be expanded in the form of a gamma distribution as $f(U, x) = A \zeta_0^{1/2} \frac{1}{\Gamma[(-2q)^{-1}]} \int_0^\infty x^{-(1+\frac{1}{2q})} e^{-x(1+2q\zeta_0 BU^2)} dx$ where $\zeta = t^{-(1+q)^{-1}}$. We assume that there exists a function $h(\zeta)$ that modifies the exponential distribution $g(U, \zeta)$ to $f(U, \zeta)$ i.e. that $f(U, \zeta) = A \int_0^\infty h(\zeta) e^{BU^2 \zeta} d\zeta$. Identifying $-2q\zeta_0 x$ with ζ and comparing the two expressions for f we obtain $h(\zeta) = \zeta_0^{1/2} \frac{1}{\Gamma[(-2q)^{-1}]} e^{(2q\zeta_0)^{-1} \zeta} (-2q\zeta_0)^{1/2q} \zeta^{-(1+\frac{1}{2q})}$. Using this expression for $h(\zeta)$ we obtain the expected values of ζ and ζ^2 as $\langle \zeta \rangle = \zeta_0^{3/2}$ and $\langle \zeta^2 \rangle = (1-2q) \zeta_0^{5/2}$ which gives the coefficient of variation as $(1-2q) \zeta_0^{-1/2} - 1$. Hence, it follows that if $f(U, t)$ is a probability distribution function that satisfies the nonlinear Fokker Planck eq. (26), then its explicit representation is given as in eq. (30) where the parameter q is linearly related to the relative variance of $\zeta = t^{-(1+q)^{-1}}$. Furthermore, since the relative variance depends on both q and $\zeta = t^{-(1+q)^{-1}}$, it follows that the function $f(U, t)$ generates an ensemble of returns corresponding to various values of q over a particular time scale and also that, for a given q the distributions of returns evolves anomalously across differing timescales.

5. Empirical Evidence

The Black Scholes model assumes lognormal distributions of stock prices. However, deviations from such behaviour are, by now, well documented [28]. Empirical evidence testifies that probability distributions of stock returns are negatively skewed, have fat tails and show leptokurtosis [28]. Some of these features of empirical distributions are modeled through Levy distributions [29-32], stochastic volatility [33] or cumulant expansions [31] around the lognormal case. Each of these models, however, attempts to empirically attune the model parameters to fit observed data and hence, is equivalent

to interpolating or extrapolating observed data in one form or the other. In contrast, the deformed noise model preserves the analytical framework of the Black Scholes world by retaining only one source of stochasticity and hence remaining within the domain of complete markets. It also provides a complete form solution with enables the prediction of option prices ab initio in lieu of parameter fitting to match observed data.

In this context, the probability distribution function of eq. (30) generates power law distributions with consequential fat tails that are characteristic of stock price distributions. This fact is brought out explicitly by writing eq. (30), with $U_0 = 0$, $t_0 = 0$, in the form:-

$$f(U,t)=At^{-\frac{1}{2(q+1)}} \exp_{(1-2q)} \left[B \left(Ut^{-\frac{1}{2(q+1)}} \right)^2 \right] = At^{-\frac{1}{2(q+1)}} \left\{ 1+2q \left[B \left(Ut^{-\frac{1}{2(q+1)}} \right)^2 \right] \right\}^{\frac{1}{2q}} \sim (2qA^{2q}B)^{\frac{1}{2q}} U^{\frac{1}{q}} t^{-\frac{1}{2q}} \quad (34)$$

for sufficiently large values of t .

There is an intricate yet natural relationship between the power law tails observed in stock market data and probability distributions of the form (30) that emanates as the solution of the nonlinear Fokker Planck equation (26). The nonlinear Fokker Planck equation (26) is known to describe anomalous diffusion under time evolution. Empirical results [34-37] establish that temporal changes of several financial market indices have variances that are shown to undergo anomalous super diffusion under time evolution.

One of the most exhaustive set of studies on stock market data in varying dimensions has been reported in [38-42]. In [42], a phenomenological study was conducted of stock price fluctuations of individual companies using data from two different databases covering three major US stock markets. The probability distributions of returns over varying timescales ranging from 5 min. to 4 years were examined. It was observed that for timescales from 5 minutes upto 16 days the tails of the distributions were well described by a power law decay. For larger timescales results consistent with a gradual convergence to Gaussian behaviour was observed. In another study [38] the probability distributions of the returns on the S & P 500 were computed over varying timescales. It was, again, seen that the distributions were consistent with an asymptotic power law behaviour with a slow convergence to Gaussian behaviour. Similar findings were obtained on the analysis of the NIKKEI and the Hang –Sang indices [38].

A plausible explanation of the matching of empirical behaviour referred to in the preceding paragraphs and the probability distribution function (30) is based on the observation that if the stock prices show large deviations from the averages, then $f(U)$ would be small in line with the probabilities of extremal events being small. Since the exponent q is usually negative in the region of interest, the effective volatility would be accentuated. In terms of market behaviour, one could say that the traders would react extremally. On the other hand, mild deviations would cause moderate reactions from market players and hence, the effective volatility gets diminished.

6. Conclusions

Contemporary empirical research into the behavior of stock market price /return patterns has found significant evidence that financial markets exhibit the phenomenon of anomalous diffusion, primarily superdiffusion, wherein the variance evolves with time according to a power law t^α with $\alpha > 1.0$. The standard technique for the study of superdiffusive processes is through a stochastic process that evolves according to a Langevin equation and whose probability distribution function satisfies a nonlinear Fokker Planck equation of the form (26). The very fact that our deformed noise function satisfies the nonlinear Fokker Planck equation is motivation enough for an adoption of this deformed Brownian motion with statistical feedback for the modeling of financial processes.

Until recently, stock market phenomena were assumed to result from complicated interactions among many degrees of freedom, and thus they were analyzed as random processes and one could go to the extent of saying that the Efficient Market Hypothesis [43-44] was formulated with one primary objective – to create a scenario which would justify the use of stochastic calculus [45] for the modeling of capital markets.

The Efficient Market Hypothesis contemplated a market where all assets were fairly priced according to the information available and neither buyers nor sellers enjoy any advantage. Market prices were believed to reflect all public information, both fundamental and price history and prices moved only as sequel to new information entering the market. Further, the presence of large number of investors was believed ensures that all prices are fair. Memory effects, if any at all, were assumed to be extremely short ranging and dissipated rapidly. Feedback effects on prices was, thus, assumed to be marginal. The investor community was assumed rational as benchmarked by the traditional concepts of risk and return.

An immediate corollary to the Efficient Market Hypothesis was the independence of single period returns, so that they could be modeled as a random walk and the defining probability distribution, in the limit of the number of observations being large, would be Gaussian.

Ever since the studies of Fama in 1964-65, evidence has been accumulating against the validity of the Efficient Market Hypothesis – the existence of negatively skewed observations and fat tails and distortion around the mean values are but a few [28, 31-35]. Most financial returns, including stock returns have shown deviation from Gaussian behaviour at short time scales with the variance not scaling with the sq. root of timescale, an attribute that is symptomatic of the possible existence of power law distributions like the one being envisaged in this study. A useful measure of quantifying deviations from the Gaussian distribution is the Hurst's exponent. If a population is Gaussian, a Hurst's exponent of 0.5 is mandated. Empirical evidence, however, shows that the Hurst's exponent for typical stock market data is around 0.6 for small timescales of about a day or less and tends to approach 0.5 asymptotically with the lengthening of the timescales. Empirical evidence also demonstrates the existence of memory effects, particularly in stock price volatilities that show long term memory effects with lag-s autocorrelations.

Further, these effects tend to fall off according to a power law rather than exponentially.

Furthermore, the access to enhanced computing power during the last decade has enabled analysts to try refined methods like the phase space reconstruction methods for determining the Lyapunov Exponents [46] of stock market price data, besides doing Rescaled Analysis [47] etc. A set of several studies has indicated the existence of strong evidence that the stock market shows chaotic behavior with fractal return structures and positive Lyapunov exponents. Results of these studies have unambiguously established the existence of significant nonlinearities and chaotic behavior in these time series [48-51].

As mentioned above, several studies [28,52-55] adopting largely diverse and independent approaches have established the existence of the following characteristics in the behavior of stock markets:-

- Long term correlation and memory effects
- Erratic markets under certain conditions and at certain times
- Fractal time series of returns
- Less reliable forecasts with increase in the horizon

thereby establishing strong evidence for the existence of chaotic behavior. In this context, the following are conventionally accepted as the inherent characteristics of a chaotic system [56-60]:-

- Exponential divergence of trajectories in phase space;
- Sensitive dependence on initial conditions;
- Fractal dimensions;
- Critical levels and bifurcations;
- Time dependent feedback systems;
- Far from equilibrium conditions.

This provides us with a second motivation for the adoption of this deformed Brownian motion structure as a model for the random kicks since our model is based on a statistical time dependent feedback into the system. This feedback may be modeled into the system macroscopically through the explicit representation of the probability distribution function $f(U, t)$ and microscopically through the stochastic process U .

It need be emphasized here that the above is purely a phenomenological model for modeling stock behavior. One could, for instance, postulate that the statistical feedback at the microscopic level represents the actions and interaction of the intra trader interactions among traders constituting the market. The statistical dependency in the noise could, further, be representing the aggregate behavior of these traders. Thus, we could model a market with non homogeneous reactions with consequent biased return structures

It is fair to say that the current stage of research in financial processes is dominated by the postulation of phenomenological models that attempt to explain a limited set of market behavior. There is a strong reason for this. A financial market consists of a huge number of market players. Each of them is endowed with his own set of beliefs about rational behavior and it is this set of beliefs that govern his actions. The market, therefore, invariably generates a heterogeneous response to any stimulus. Furthermore, “rational-

ity” mandates that every market player should have knowledge and understanding about the “rationality” of all other players and should take full cognizance in modeling his response to the market. This logic would extend to each and every market player so that we have a situation where every market player should have knowledge about the beliefs of every other player who should have knowledge of beliefs of every other player and so on. We, thus, end up with an infinitely complicated problem that would defy a solution even with the most sophisticated mathematical procedures. Additionally, unlike as there is in physics, financial economics does not possess a basic set of postulates like General Relativity and Quantum Mechanics that find homogeneous applicability to all systems in their domain of validity.

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