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Derivation of the Radiative Transfer Equation Inside a Moving Semi-Transparent Medium of Non Unit Refractive Index

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Abstract: The derivation of the radiative transfer equation inside a moving semi-transparent medium of non unit constant refractive index has been completely achieved, leading to an exactly similar equation as in the case of a unit index, unless it is expressed in a particular frame with particular time and space co-ordinates; defining first the "equivalent vacuum" and the "matter" space associated to its "matter" co-ordinates with the help of the Gordon's metric, it is shown that an observer at rest in vacuum perceives the isotropic moving medium as an anisotropic uniaxial medium of given optical axis, for which it is possible to derive general transmission and reflection rules for electromagnetic fields; however the exhibited refractive index characterising the moving medium, relatively to the observer located in vacuum, is not an effective index but only an apparent one without any energetic significance, and the specific intensity must be obtained relatively to a given observer at rest located inside the moving medium; finally the general form of the radiative transfer equation is obtained in the moving medium.

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1. Introduction

Several years ago, Mihalas [1] proposed an elegant way to obtain the invariant radiative transfer equation in a moving semi-transparent medium, noting that in some cases it was judicious to perform energetic calculations either in a comobile frame bound to a moving particle or in the frame bound to a given observer; then, when a radiation participates to the energy transfer, it is necessary to be able to compute the radiative

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fluxes in the appropriate frame; the energetic radiative fluxes being strongly related to the radiative intensity whose evolution is governed by the radiative transfer equation, one may naturally conceive to give the appropriate form of this equation either in the comobile frame or in the observer frame. This fundamental work however is restricted to media for which the refractive index is a unit index; this approximation is of great interest for gases for which the refractive index is very close to 1, but dense isotropic media, liquids or solids, have generally a much higher refractive index, and we may imagine situations where some liquids of high index are moving with a large speed: what is then in this case the correct form of the radiative transfer equation in such media, and is it possible to exhibit an invariant form of this equation, valid in both the comobile frame or the observer frame?

If the optics of moving dielectric media has recently received a strong interest in the literature [2, 3], to our knowledge no study focused on what happens from a radiative energetic point of view in moving semi-transparent media; it is to suspect however that the effects of high speeds may be spectacular, since some spectacular effects may arise from an optical point of view, as described in [2, 3]; the main tool used to exhibit such optical effects is the Gordon's metric tensor; indeed, many decades ago, Gordon [4] had the intuition that light in a moving dielectric medium "could see matter as a metric" in the sense where a moving dielectric medium acts on light as an effective gravitational field, and this is this property which is enhanced to produce in some particular conditions special optical effects; following this basic idea, it may be interesting to see if the Gordon's metric is the appropriate tool to exhibit an invariant form of the radiative transfer equation.

The purpose of this paper is then the derivation of the radiative transfer equation inside a grey (i.e. its optical properties are non frequency depending) moving semi-transparent medium characterised by its constant refractive index different from one; to do so, we shall first develop in section II the optical problem, that is determine the angle and frequency transformation for a propagating radiation, between the comobile location reference system bound to a moving particle embedded in the medium of non unit refractive index and the location reference system relatively to a given observer inside the medium: from the Gordon's metric, we shall construct an "equivalent vacuum" and its related "mater-light space" perceived both by the moving particle and the observer, such that the "vacuum" bound to the particle and the "vacuum" bound to the observer are related by a Lorentz transformation thanks to a particular rapidity different from the usual one; this latter result will provide us in section IV, analogously to what happens in the real vacuum, the radiation angle and frequency transformation in the refractive medium, from which, following the work of Mihalas, one deduces the invariant form of the radiative transfer equation. In section III, a closely related problem will be examined, which allows to interpret an uniaxial crystal (here are only studied the negative crystals) as an isotropic moving medium, for which it is possible to derive (here only the parallel polarisation was examined) the reflection and transmission laws for an electromagnetic field through an interface separating the crystal from an isotropic medium of unit refractive index.

2. The Optical Problem: Construction of an Equivalent Vacuum and Determination of the Fundamental Mater-Light Space

It is well known that in a motionless medium embedded in a flat space, for which the refractive index equals the unity, the radiative transfer equation (RTE) can be written in Cartesian co-ordinates as

$$P^{\alpha}\,\partial_{\alpha} I = \frac{h}{c}\left(\frac{\eta}{\nu^2} - \kappa\,\nu\,I\right) = \frac{\kappa\,h}{c\,\nu^2}\left[L^0\,(T) - L\right], \tag{1}$$

where $I = \frac{L}{\nu^3}$ is the specific intensity, L being the classical intensity, and $\vec{\vec{P}} = \frac{h\,\nu}{c}\left(1,\,\vec{\Omega}\right)$ is the impulsion-energy 4-vector; κ is the absorption coefficient, $L^0\,(T)$ the black body intensity at local thermodynamic equilibrium for a given temperature T, h the Boltzmann constant, c the light speed in the vacuum and ν the radiation frequency inside the medium; in absence of any relativistic event, the radiation frequency remains constant, and the formal RTE can be rewritten under the standard form

$$\frac{1}{c}\frac{\partial L}{\partial t} + \frac{\partial L}{\partial s} = \kappa\left[L^0\,(T) - L\right], \tag{2}$$

where t is the time and s the curvilinear abscissa along a luminous trajectory, with $\frac{\partial L}{\partial s} = \vec{\Omega}\ \vec{grad}\ L$.

The Gordon effective gravitational field can be expressed as [2]

$$g^{\mu\nu} = g^{\mu\nu\,(0)} - (\varepsilon\,\mu - 1)\,u^{\mu}\,u^{\nu} \Leftrightarrow g_{\mu\nu} = g^{(0)}_{\mu\nu} + \left(1 - \frac{1}{\varepsilon\,\mu}\right)u_{\mu}\,u_{\nu}, \tag{3}$$

where u is the mean 4-speed vector of the medium (relatively to a given observer), $g^{(0)}$ the vacuum Minkowski tensor, g the effective gravitational tensor, and ε and μ the relative dielectric and magnetic permittivity and permeability of the medium assumed hereafter isotropic, related to its refractive index n by $n^2 = \varepsilon\,\mu$; we shall consider only non magnetic media, with $\mu = 1$, so that $n^2 = \varepsilon$. Let us now remind a more mechanical demonstration of this latter result: in a transparent medium where the refractive index, hereafter assumed constant, i.e. non depending on space and/or time co-ordinates, is not 1, the proper time interval (PTI) for a photon can be written in Cartesian co-ordinates as

$$d\tau^2 = c^2\,dt^2 - n^2\left(dx^2 + dy^2 + dz^2\right) = -\,g_{\mu\nu}\,dx^{\mu}\,dx^{\nu}, \tag{4}$$

where the contravariant co-ordinates x^{μ} are $x^{\mu} = (x^0,\,x^1,\,x^2,\,x^3) = (ct,\,x,\,y,\,z)$; this is the most general form of the photon PTI in a motionless medium, simple translation of the fact that light propagates at speed $\frac{c}{n}$ in a dielectric for which the refractive index is different from one, for the PTI is a light one for a photon and $d\tau = 0 \Rightarrow \frac{ds}{dt} = \frac{c}{n}$; hence one deduces the covariant components of the metric tensor in Cartesian co-ordinates

$$g_{00} = -1 \quad g_{xx} = n^2 \quad g_{yy} = n^2 \quad g_{zz} = n^2, \tag{5}$$

and the contravariant components, since the tensor is diagonal

$$g^{00} = -1 \quad g^{xx} = \frac{1}{n^2} \quad g^{yy} = \frac{1}{n^2} \quad g^{zz} = \frac{1}{n^2}, \tag{6}$$

It has to be noticed that the PTI can be rewritten as

$$d\tau^2 = \left(1 - n^2\right) dx^{0^2} + n^2 \left(dx^{0^2} - dx^2 - dy^2 - dz^2\right), \tag{7}$$

At a given event $\{x^\mu\}$ in space-time, can be defined a 4-speed vector $u^\mu = \frac{dx^\mu}{d\tau}$ representing the mean motion of the dielectric; in a given comobile location reference system (LRS) bound to a particle moving with a speed $\vec{\beta} = \frac{\vec{v}}{c}$ relatively to a "fix" (for observer) LRS, the covariant components of the 4-speed are $u_{\mu'} = -\delta^0_{\mu'}$ where the primes indicate the co-ordinates relatively to the considered comobile LRS ; hence in this LRS one has

$$d\tau^2 = \left(1 - n^2\right) \left(u_{\mu'}\, dx^{\mu'}\right)^2 - n^2 g^{(0)}_{\mu'\nu'}\, dx^{\mu'}\, dx^{\nu'} = -g_{\mu'\nu'}\, dx^{\mu'}\, dx^{\nu'}, \tag{8}$$

where $g^{(0)}_{\mu'\nu'}$ represents the vacuum metric tensor in the comobile LRS; the cinematic transformation between the vacuum metric tensor relatively to the comobile LRS and the vacuum metric tensor relatively to the observer LRS, and the metric tensors associated to the dielectric is

$$g^{\mu\nu(0)} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g^{\mu'\nu'(0)} \quad \text{and} \quad g^{\mu\nu} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g^{\mu'\nu'}, \tag{9}$$

Considering a motion along the $\vec{x}$ axis, the former relations are developed for the diagonal components

$$\begin{array}{ll} g^{00(0)} = -\left(\frac{\partial x^0}{\partial x^{0'}}\right)^2 + \left(\frac{\partial x^0}{\partial x'}\right)^2 & g^{xx(0)} = -\left(\frac{\partial x}{\partial x^{0'}}\right)^2 + \left(\frac{\partial x}{\partial x'}\right)^2 \\ g^{yy(0)} = 1 \quad g^{zz(0)} = 1 & \end{array}, \tag{10}$$

and for the non diagonal one

$$g^{0x(0)} = -\left(\frac{\partial x^0}{\partial x^{0'}}\right)\left(\frac{\partial x}{\partial x^{0'}}\right) + \left(\frac{\partial x^0}{\partial x'}\right)\left(\frac{\partial x}{\partial x'}\right), \tag{11}$$

since the variables x^0 and x do not depend on y and z which remain unchanged; the Lorentz transform in Cartesian co-ordinates is simply

$$\begin{array}{ll} x^{0'} = \gamma\,(x^0 - \beta\, x) & x^0 = \gamma\,\left(x^{0'} + \beta\, x'\right) \\ x' = \gamma\,(x - \beta\, x^0) & x = \gamma\,\left(x' + \beta\, x^{0'}\right), \\ y' = y \quad z' = z & y = y' \quad z = z' \end{array} \tag{12}$$

from which one obtains

$$dx^0\, dx = \left|\frac{\partial\,(x^0, x)}{\partial\,(x^{0'}, x')}\right| dx^{0'}\, dx' = \begin{vmatrix} \frac{\partial x^0}{\partial x^{0'}} & \frac{\partial x}{\partial x^{0'}} \\ \frac{\partial x^0}{\partial x'} & \frac{\partial x}{\partial x'} \end{vmatrix} dx^{0'}\, dx' = dx^{0'}\, dx', \tag{13}$$

due to the scalar density conservation, it comes that

$$\sqrt{-\,Det\,\overline{\overline{g}}}\,d\,x^0\ dx\,dy\,dz\ =\ \sqrt{-\,Det\,\overline{\overline{g'}}}\,d\,x^{0'}\ dx'\,dy'\,dz', \tag{14}$$

hence one has $Det\,\overline{\overline{g}}\ =\ -\ n^6$: in Cartesian co-ordinates, the metric tensor determinant remains unchanged; developing relations (10)-(11), one has with the help of (12) the contravariant co-ordinates of the vacuum metric tensor relatively to the observer LRS

$$g^{00\,(0)}\ =\ -\,1\quad g^{xx\,(0)}\ =\ 1\quad g^{yy\,(0)}\ =\ 1\quad g^{zz\,(0)}\ =\ 1, \tag{15}$$

for the non diagonal component $g^{0x\,(0)}\ =\ 0$; the tensor being diagonal, one deduces the covariant co-ordinates as

$$g_{00}^{(0)}\ =\ -\,1\quad g_{xx}^{(0)}\ =\ 1\quad g_{yy}^{(0)}\ =\ 1\quad g_{zz}^{(0)}\ =\ 1, \tag{16}$$

while the metric tensor associated to the dielectric in the observer LRS is $g^{\mu\nu}\ =\ \frac{\partial\,x^\mu}{\partial\,x^{\mu'}}\,\frac{\partial\,x^\nu}{\partial\,x^{\nu'}}\ g^{\mu'\nu'}$, leading to

$$g^{00}\ =\ -\,\frac{\gamma^2\,\left(n^2\,-\,\beta^2\right)}{n^2}\qquad g^{xx}\ =\ \frac{\gamma^2\,\left(1\,-\,\beta^2\,n^2\right)}{n^2}\qquad g^{yy}\ =\ \frac{1}{n^2}\qquad g^{zz}\ =\ \frac{1}{n^2}, \tag{17}$$

for the diagonal components and

$$g^{0x}\ =\ -\,\frac{\beta\,\gamma^2\,(n^2\,-1)}{n^2}, \tag{18}$$

for the non diagonal one. The 4-speed vector being defined in vacuum as $\vec{\vec{u}}\ =\ \gamma\begin{pmatrix}1\\ \vec{\beta}\end{pmatrix}$ with $\gamma\ =\ \frac{1}{\sqrt{1\,-\,\beta^2}}$, one has for the contravariant co-ordinates of the 4-speed $u^0\ =\ \gamma$ and $u^x\ =\ \beta\,\gamma$; hence the contravariant components of the metric tensor given by (17) and (18) can be rewritten as

$$\begin{array}{ll} g^{00}\ =\ \frac{1}{n^2}\,\left[g^{00\,(0)}\ -\ (n^2\,-1)\ u^0\ u^0\right] & g^{xx}\ =\ \frac{1}{n^2}\,\left[g^{xx\,(0)}\ -\ (n^2\,-1)\ u^x\ u^x\right] \\ g^{0x}\ =\ \frac{1}{n^2}\,\left[g^{0x\,(0)}\ -\ (n^2\,-1)\ u^0\ u^x\right] & \end{array}, \tag{19}$$

or under a more compact form $g^{\mu\nu}\ =\ \frac{1}{n^2}\,\left[g^{\mu\nu\,(0)}\ -\ (n^2\,-1)\ u^\mu\ u^\nu\right]$, which is the Gordon metric [4]; it is then possible to obtain the covariant components with a simple inversion of the contravariant matrix, or more simply use Eq. (8) since $u_{\mu'}\ d\,x^{\mu'}\ =\ u_{\mu'}\ \frac{\partial\,x^{\mu'}}{\partial\,x^\mu}\,d\,x^\mu\ =\ -\ \delta^0_{\mu'}\ \frac{\partial\,x^{\mu'}}{\partial\,x^\mu}\,d\,x^\mu\ =\ -\ \frac{\partial\,x^{0'}}{\partial\,x^\mu}\,d\,x^\mu$, from which $u_{\mu'}\ d\,x^{\mu'}\ =\text{-}\,\gamma\ (d\,x^0\ \text{-}\,\beta\,dx)$; hence one obtains for the covariant components, since $d\,x^0\ dx\ =\ d\,x^{0'}\ dx'$:

$$\begin{array}{l} g_{00}\ =\ -\ \gamma^2\,\left(1\,-\,\beta^2\,n^2\right)\quad g_{xx}\ =\ \gamma^2\,\left(n^2\,-\,\beta^2\right)\quad g_{yy}\ =\ n^2\quad g_{zz}\ =\ n^2 \\ g_{0x}\ =\ -\,\beta\ \gamma^2\ (n^2\,-1) \end{array}, \tag{20}$$

which may be rewritten under the compact Gordon metric as $g_{\mu\nu} = n^2 \, g^{(0)}_{\mu\nu} + (n^2 - 1) \, u_\mu \, u_\nu$.

In such a metric the photon PTI is

$$d\tau^2 = - g_{\mu\nu} \, dx^\mu \, dx^\nu = - g_{00} \, dx^{0^2} - g_{xx} \, dx^2 - 2 \, g_{0x} \, dx^0 \, dx - n^2 \left(dy^2 + dz^2\right) = 0, \tag{21}$$

for a constant refractive index the luminous trajectories are straight lines, and if light propagates along the $\vec{x}$ axis, then $dy = dz = 0$, from which one deduces

$$g_{xx} \left(\frac{dx}{dx^0}\right)^2 + 2 \, g_{0x} \frac{dx}{dx^0} + g_{00} = 0, \tag{22}$$

the resolution of this equation easily leads to

$$\frac{dx}{dx^0} = \frac{1}{\hat{n}} = \frac{1 + \beta n}{n + \beta} = \frac{dx' + \beta \, dx^{0'}}{\beta \, dx' + dx^{0'}}, \tag{23}$$

and if $\beta << 1$ one has for the effective refractive index $\frac{1}{\hat{n}} = \frac{1}{n} + \beta \left(1 - \frac{1}{n^2}\right)$, which is the famous well-known Fresnel's drag additional formula [5].

However, the physical significance of the former metric is not that obvious, since the g_{00} component associated to a time may be either negative or positive (or even zero), depending on the value of β if the latter one is greater or lower than $\frac{1}{n}$; we also make the choice of a new metric, equivalent to the precedent one, for which the unique eigen-value associated to a time is always negative: the calculation of the covariant metric tensor eigen-values shows that n^2 is a double eigen-value associated to the eigen-vectors $\vec{e}_y$ and $\vec{e}_z$, the two other eigen-values being solution of the characteristic equation

$$g^2 - (g_{xx} + g_{00}) \, g - \left(g_{0x}^2 - g_{xx} \, g_{00}\right) = g^2 - (g_{xx} + g_{00}) \, g - n^2 = 0, \tag{24}$$

leading to

$$\begin{array}{l} g_1 = \frac{g_{xx} + g_{00} - \sqrt{\Delta}}{2} = \frac{1}{2} \left[\gamma^2 \left(\beta^2 + 1\right) (n^2 - 1) - \sqrt{\Delta}\right] \\ g_2 = \frac{g_{xx} + g_{00} + \sqrt{\Delta}}{2} = \frac{1}{2} \left[\gamma^2 \left(\beta^2 + 1\right) (n^2 - 1) + \sqrt{\Delta}\right] \end{array}, \tag{25}$$

with

$g_1 \, g_2 = - n^2$ and $\Delta = (g_{xx} + g_{00})^2 + 4 \, (g_{0x}^2 - g_{xx} \, g_{00}) = (n^2 + 1)^2 + 4 \, \beta^2 \, \gamma^4 \, (n^2 - 1)^2 > 0$

One deduces from the former result that the g_1 eigen-value is strictly negative whatever the refractive index n and the β medium rapidity are, and that it may be associated to a time, the g_2 eigen-value being always positive and associated to a space variable: in well chosen axis, the metric tensor relatively to the observer LRS is diagonal and can be represented as

$$\overline{\overline{g}} = \begin{pmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & n^2 & 0 \\ 0 & 0 & 0 & n^2 \end{pmatrix}, \tag{26}$$

The determination of the associated Eigen-vector leads to

$$\begin{aligned} \vec{E_1} &= \vec{e_0} + \frac{\beta\,\gamma^2\left(n^2-1\right)}{\gamma^2\left(n^2-\beta^2\right)-g_1}\,\vec{e_x} = \vec{e_0} - \frac{g_{0x}}{g_{xx}-g_1}\,\vec{e_x} \\ \vec{E_2} &= -\frac{\beta\,\gamma^2\left(n^2-1\right)}{\gamma^2\left(1-\beta^2\,n^2\right)+g_2}\,\vec{e_0} + \vec{e_x} = \frac{g_{0x}}{g_2-g_{00}}\,\vec{e_0} + \vec{e_x} \end{aligned}, \tag{27}$$

these two vector are orthogonal relatively to this metric and can be normed so that $\vec{e_1}^{\,2} = g_1$ and $\vec{e_2}^{\,2} = g_2$ since $(g_{xx} - g_1)\,(g_{00} - g_1) = g_{0x}^2$: hence $\vec{E_1}^{\,2} = g_{00} - \frac{2\,g_{0x}^2}{g_{xx}-g_1} + \frac{g_{0x}^2\,g_{xx}}{(g_{xx}-g_1)^2} = 2\,g_1 - g_{00} + \frac{g_{xx}\,(g_{00}-g_1)}{g_{xx}-g_1} = \frac{g_1\,\sqrt{\Delta}}{g_{xx}-g_1}$ from which $\vec{E_1}^{\,2} = \vec{e_1}^{\,2}\,\frac{g_2-g_1}{g_{xx}-g_1}$ and $\vec{e_1} = \sqrt{\frac{g_{xx}-g_1}{g_2-g_1}}\left(\vec{e_0} - \frac{g_{0x}}{g_{xx}-g_1}\,\vec{e_x}\right)$; performing the same calculation with the second eigen-vector finally leads to the expression of the two normed eigen-vectors $\vec{e_1}$ and $\vec{e_2}$ associated to the two eigen-values g_1 and g_2 as

$$\begin{aligned} \vec{e_1} &= \frac{1}{\sqrt{g_2-g_1}}\left(\sqrt{g_{xx}-g_1}\,\vec{e_0} + \sqrt{g_{00}-g_1}\,\vec{e_x}\right) \\ \vec{e_2} &= \frac{1}{\sqrt{g_2-g_1}}\left(-\sqrt{g_2-g_{xx}}\,\vec{e_0} + \sqrt{g_2-g_{00}}\,\vec{e_x}\right) \end{aligned}. \tag{28}$$

The 4-event vector is defined as $\left(d\,\vec{\vec{M}}\right)^2 = -\,d\,\tau^2 = g_{\mu\nu}\,d\,x^\mu\,d\,x^\nu \Rightarrow d\,\vec{\vec{M}} = d\,x^\mu\,\vec{e}_\mu \Rightarrow \vec{\vec{M}} = x^\mu\,\vec{e}_\mu$, hence for a given event one has

$$\begin{aligned} \vec{\vec{M}} &= x^0\,\vec{e}_0 + x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z = x^{0'}\,\vec{e}_{0'} + x'\,\vec{e}_{x'} + y'\,\vec{e}_{y'} + z'\,\vec{e}_{z'} \\ &= \gamma\left(x^{0'} + \beta\,x'\right)\vec{e}_0 + \gamma\left(x' + \beta\,x^{0'}\right)\vec{e}_x + y'\,\vec{e}_{y'} + z'\,\vec{e}_{z'} \\ &= \gamma\left(\vec{e}_0 + \beta\,\vec{e}_x\right)x^{0'} + \gamma\left(\beta\,\vec{e}_0 + \vec{e}_x\right)x' + y'\,\vec{e}_{y'} + z'\,\vec{e}_{z'} \Rightarrow \begin{array}{l} \gamma\left(\vec{e}_0 + \beta\,\vec{e}_x\right)x^{0'} = x^{0'}\,\vec{e}_{0'} \\ \gamma\left(\beta\,\vec{e}_0 + \vec{e}_x\right)x' = x'\,\vec{e}_{x'} \end{array}, \end{aligned} \tag{29}$$

the variables $x^{0'}$ and x' being independent; it comes then

$$\begin{array}{l} \vec{e}_0 + \beta\,\vec{e}_x = \frac{\vec{e}_{0'}}{\gamma} \\ \beta\,\vec{e}_0 + \vec{e}_x = \frac{\vec{e}_{x'}}{\gamma} \end{array} \Rightarrow \begin{array}{l} \vec{e}_0 = \gamma\left(\vec{e}_{0'} - \beta\,\vec{e}_{x'}\right) \\ \vec{e}_x = \gamma\left(\vec{e}_{x'} - \beta\,\vec{e}_{0'}\right) \end{array}, \tag{30}$$

for this event one also has $\vec{\vec{M}} = x^1\,\vec{e}_1 + x^2\,\vec{e}_2 + y\,\vec{e}_y + z\,\vec{e}_z = x^0\,\vec{e}_0 + x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$ and for the pseudo-norm $\vec{\vec{M}}^{\,2} = g_1\,{x^1}^2 + g_2\,{x^2}^2 + n^2\,(y^2 + z^2)$; but

$$\begin{array}{l} \vec{e_1} = \frac{1}{\sqrt{g_2-g_1}}\left(\sqrt{g_{xx}-g_1}\,\vec{e_0} + \sqrt{g_{00}-g_1}\,\vec{e_x}\right) \\ \vec{e_2} = \frac{1}{\sqrt{g_2-g_1}}\left(-\sqrt{g_2-g_{xx}}\,\vec{e_0} + \sqrt{g_2-g_{00}}\,\vec{e_x}\right) \end{array} \Rightarrow \begin{array}{l} \vec{e_0} = \frac{1}{\sqrt{g_2-g_1}}\left(\sqrt{g_{xx}-g_1}\,\vec{e_1} - \sqrt{g_{00}-g_1}\,\vec{e_2}\right) \\ \vec{e_x} = \frac{1}{\sqrt{g_2-g_1}}\left(\sqrt{g_2-g_{xx}}\,\vec{e_1} + \sqrt{g_2-g_{00}}\,\vec{e_2}\right) \end{array}, \tag{31}$$

from which

$$\begin{aligned} x^1 &= \frac{\gamma}{\sqrt{g_2-g_1}}\left[\left(\sqrt{g_{xx}-g_1} + \beta\,\sqrt{g_2-g_{xx}}\right)x^{0'} + \left(\sqrt{g_2-g_{xx}} + \beta\,\sqrt{g_{xx}-g_1}\right)x'\right] \\ x^2 &= \frac{\gamma}{\sqrt{g_2-g_1}}\left[\left(\beta\,\sqrt{g_2-g_{00}} - \sqrt{g_{00}-g_1}\right)x^{0'} + \left(\sqrt{g_2-g_{00}} - \beta\,\sqrt{g_{00}-g_1}\right)x'\right] \end{aligned}, \tag{32}$$

performing the calculation of $g_1\,{x^1}^2 + g_2\,{x^2}^2$ one obtains finally

$$g_1\,{x^1}^2 + g_2\,{x^2}^2 = \gamma^2 \left\{ \begin{array}{l} \left(g_{00} + \beta^2\,g_{xx} + 2\beta\,g_{0x}\right){x^{0'}}^2 + \left(g_{xx} + \beta^2\,g_{00} + 2\beta\,g_{0x}\right){x'}^2 \\ + 2\left[\left(1+\beta^2\right)g_{0x} + \beta\left(g_{00} + g_{xx}\right)\right] x^{0'}\,x' \end{array} \right\} = -\,{x^{0'}}^2 + n^2\,{x'}^2, \tag{33}$$

hence the 4-event pseudo-norm remains unchanged, and

$$d\,x^1\;d\,x^2 = \left|\frac{\partial\left(x^1,x^2\right)}{\partial\left(x^{0'},x'\right)}\right| d\,x^{0'}\,dx' = \frac{\gamma^2}{g_2 - g_1}\left|\begin{array}{cc} \sqrt{g_{xx}-g_1} + \beta\sqrt{g_{00}-g_1} & \beta\sqrt{g_2-g_{00}} - \sqrt{g_2-g_{xx}} \\ \sqrt{g_{00}-g_1} + \beta\sqrt{g_{xx}-g_1} & \sqrt{g_2-g_{00}} - \beta\sqrt{g_2-g_{xx}} \end{array}\right| d\,x^{0'}\,dx', \\ = \frac{\gamma^2}{g_2-g_1}\left(1-\beta^2\right)\left(g_2-g_1\right)d\,x^{0'}\,dx' = d\,x^{0'}\,dx' \tag{34}$$

from which one deduces that $\sqrt{-\,Det\,\overline{\overline{g}}}\,d\,x^1\;d\,x^2\;dy\,dz \;=\; \sqrt{-\,Det\,\overline{\overline{g'}}}\,d\,x^{0'}\;dx'\,dy'\,dz'$, that is the conservation of the scalar density ; noticing furthermore that

$$\begin{array}{l} \left(\sqrt{g_{xx}-g_1} - \beta\sqrt{g_{00}-g_1}\right)\left(\beta\sqrt{g_{xx}-g_1} - \sqrt{g_{00}-g_1}\right) = -\,2\,\beta\;g_1 \\ \left(\sqrt{g_2-g_{xx}} + \beta\sqrt{g_2-g_{00}}\right)\left(\sqrt{g_2-g_{00}} + \beta\sqrt{g_2-g_{xx}}\right) = 2\,\beta\;g_2 \\ \left(\sqrt{g_{xx}-g_1} - \beta\sqrt{g_{00}-g_1}\right)\left(\sqrt{g_2-g_{xx}} + \beta\sqrt{g_2-g_{00}}\right) = 2\,\beta\;n^2 \\ \left(\beta\sqrt{g_{xx}-g_1} - \sqrt{g_{00}-g_1}\right)\left(\sqrt{g_2-g_{00}} + \beta\sqrt{g_2-g_{xx}}\right) = 2\,\beta \end{array}, \tag{35}$$

the x^1 and x^2 co-ordinates can equivalently be rewritten as

$$\begin{array}{l} x^1 = \frac{\gamma}{\sqrt{g_2-g_1}}\left[\frac{2\,\beta\;g_2}{\sqrt{g_2-g_{xx}}+\beta\sqrt{g_2-g_{00}}}\,x^{0'} + \left(\sqrt{g_2-g_{xx}} + \beta\sqrt{g_2-g_{00}}\right)x'\right] \\ x^2 = \frac{\gamma}{g_2\,\sqrt{g_2-g_1}}\left[\left(\sqrt{g_2-g_{xx}} + \beta\sqrt{g_2-g_{00}}\right)x^{0'} + \frac{2\beta\,g_2\,n^2}{\sqrt{g_2-g_{xx}}+\beta\sqrt{g_2-g_{00}}}\,x'\right] \end{array}, \tag{36}$$

with the help of the auxiliary value $\sqrt{X} \;=\; \sqrt{g_2-g_{xx}} + \beta\sqrt{g_2-g_{00}}$, performing the calculation of $g_1\,{x^1}^2 + g_2\,{x^2}^2$, it comes for the equation verified by X that

$$X^2 + \frac{g_2\,\left(g_2-g_1\right)}{\gamma^2}\,X - 4\,\beta^2\,g_2^2\,n^2 = 0, \tag{37}$$

the discriminant of this equation is $\Delta \;=\; g_2^2\left[16\,\beta^2\,n^2 + \frac{\left(g_2-g_1\right)^2}{\gamma^4}\right] \;=\; g_2^2\,\left(1+\beta^2\right)^2\left(n^2+1\right)^2$, and since X is positive, one has

$$X = \frac{g_2}{2}\left[\left(1+\beta^2\right)\left(n^2+1\right) - \left(1-\beta^2\right)\left(g_2-g_1\right)\right] = g_2\left[g_1 - g_{00} + \beta^2\left(g_{xx}-g_1\right)\right] \\ = g_2\left(\beta^2\,g_{xx} - g_{00}\right) - n^2\left(1-\beta^2\right) = g_2\left(1+\beta^2\right) - n^2\left(1-\beta^2\right) = g_2\left[1 + g_1 + \left(1-g_1\right)\beta^2\right], \tag{38}$$

following the same steps, with the help of the auxiliary value $\sqrt{X} \;=\; \sqrt{g_2-g_{00}} + \beta\sqrt{g_2-g_{xx}}$, and performing once again the calculation of $g_1\,{x^1}^2 + g_2\,{x^2}^2$, it comes for the equation verified by X

$$X^2 - \frac{g_2\,\left(g_2-g_1\right)}{\gamma^2\,n^2}\,X - 4\,\beta^2\,\frac{g_2^2}{n^2} = 0, \tag{39}$$

the discriminant of this equation is $\Delta = \frac{g_2^2}{n^4}\left[16\,\beta^2\,n^2 + \frac{(g_2 - g_1)^2}{\gamma^4}\right] = \frac{g_2^2\left(1+\beta^2\right)^2\left(n^2+1\right)^2}{n^4}$, and since X is positive, one has

$$\begin{aligned} X &= \tfrac{g_2}{2\,n^2}\left[\left(1+\beta^2\right)\left(n^2+1\right) + \left(1-\beta^2\right)(g_2 - g_1)\right] = \tfrac{g_2}{n^2}\left[g_{xx} - g_1 + \beta^2\,(g_1 - g_{00})\right] \\ &= \tfrac{g_2}{n^2}\left(g_{xx} - \beta^2\,g_{00}\right) + 1 - \beta^2 = g_2\left(1+\beta^2\right) + 1 - \beta^2 = g_2 + 1 + (g_2 - 1)\,\beta^2 \end{aligned}, \tag{40}$$

Finally the system (35) can be rewritten as

$$\begin{aligned} \sqrt{g_2 - g_{xx}} + \beta\sqrt{g_2 - g_{00}} &= \sqrt{g_2\left[1 + g_1 + (1 - g_1)\,\beta^2\right]} \\ \beta\sqrt{g_{xx} - g_1} - \sqrt{g_{00} - g_1} &= \sqrt{\tfrac{1+g_1+(1-g_1)\,\beta^2}{g_2}} \\ \sqrt{g_{xx} - g_1} - \beta\sqrt{g_{00} - g_1} &= -\,g_1\sqrt{g_2 + 1 + (g_2 - 1)\,\beta^2} \\ \sqrt{g_2 - g_{00}} + \beta\sqrt{g_2 - g_{xx}} &= \sqrt{g_2 + 1 + (g_2 - 1)\,\beta^2} \end{aligned}, \tag{41}$$

hence for the x^1 and x^2 co-ordinates one has

$$\begin{aligned} x^1 &= \tfrac{\gamma}{\sqrt{-g_1}}\left[\sqrt{-g_1}\sqrt{\tfrac{g_2+1+(g_2-1)\,\beta^2}{g_2 - g_1}}\,x^{0'} + n\sqrt{\tfrac{1+g_1+(1-g_1)\,\beta^2}{g_2 - g_1}}\,x'\right] \\ x^2 &= \tfrac{\gamma}{\sqrt{g_2}}\left[\sqrt{\tfrac{1+g_1+(1-g_1)\,\beta^2}{g_2 - g_1}}\,x^{0'} + n\sqrt{-g_1}\sqrt{\tfrac{g_2+1+(g_2-1)\,\beta^2}{g_2 - g_1}}\,x'\right] \end{aligned}, \tag{42}$$

performing then the calculation of $g_1\,{x^1}^2 + g_2\,{x^2}^2$, it comes the useful following relation

$$n^2 - 1 - 2\,g_1 + \beta^2\left(n^2 - 1 + 2\,g_1\right) = \frac{g_2 - g_1}{\gamma^2} \Leftrightarrow g_1 + g_2 = \gamma^2\left(1+\beta^2\right)\left(n^2 - 1\right), \tag{43}$$

this leads for the 4-event that

$$\begin{aligned} \vec{\vec{M}} &= \tfrac{\gamma}{\sqrt{-g_1}}\left[\sqrt{-g_1}\sqrt{\tfrac{g_2+1+(g_2-1)\,\beta^2}{g_2 - g_1}}\,x^{0'} + n\sqrt{\tfrac{1+g_1+(1-g_1)\,\beta^2}{g_2 - g_1}}\,x'\right]\vec{e}_1 \\ &+ \tfrac{\gamma}{\sqrt{g_2}}\left[\sqrt{\tfrac{1+g_1+(1-g_1)\,\beta^2}{g_2 - g_1}}\,x^{0'} + n\sqrt{-g_1}\sqrt{\tfrac{g_2+1+(g_2-1)\,\beta^2}{g_2 - g_1}}\,x'\right]\vec{e}_2 + y\,\vec{e}_y + z\,\vec{e}_z, \\ &= x^{0'}\,\vec{e}_{0'} + x'\,\vec{e}_{x'} + y'\,\vec{e}_{y'} + z'\,\vec{e}_{z'} \end{aligned} \tag{44}$$

and since the variables $x^{0'}$ and x' are independent,

$$\begin{aligned} \vec{e}_{0'} &= \gamma\left[\sqrt{\tfrac{g_2+1+(g_2-1)\,\beta^2}{g_2 - g_1}}\,\vec{e}_1 + \tfrac{1}{\sqrt{g_2}}\sqrt{\tfrac{1+g_1+(1-g_1)\,\beta^2}{g_2 - g_1}}\,\vec{e}_2\right] \\ \vec{e}_{x'} &= \gamma\left[\sqrt{g_2}\sqrt{\tfrac{1+g_1+(1-g_1)\,\beta^2}{g_2 - g_1}}\,\vec{e}_1 - g_1\sqrt{\tfrac{g_2+1+(g_2-1)\,\beta^2}{g_2 - g_1}}\,\vec{e}_2\right] \end{aligned}, \tag{45}$$

It is now comfortable to introduce symbolic variables such as

$$\begin{aligned} \bar{x}^1 &= \tfrac{\gamma}{\sqrt{-g_1}}\left(x^{0'} + \beta\,n\,x'\right) \\ \bar{x}^2 &= \tfrac{\gamma}{\sqrt{g_2}}\left(\beta\,x^{0'} + n\,x'\right) \end{aligned}, \tag{46}$$

since they obviously verify the relation $g_1\,\bar{x}^{1^2} + g_2\,\bar{x}^{2^2} = -\,x^{0'^2} + n^2\,x'^2$; however those variables do not represent the $\overset{\Rightarrow}{M}$ 4-event, but it is convenient to notice the following substitution, very useful for latter calculations

$$\begin{array}{l} \sqrt{-\,g_1}\,\sqrt{\frac{g_2\,+1+(g_2\,-1)\,\beta^2}{g_2\,-\,g_1}} \leftrightarrow 1 \\ \sqrt{\frac{1+g_1\,+(1-g_1)\,\beta^2}{g_2\,-\,g_1}} \leftrightarrow \beta \end{array} , \tag{47}$$

note that this equivalence becomes a strict equality if and only if $n = 1$; hence for the symbolic 4-speed one has

$$\overset{\Rightarrow}{\bar{u}} = \gamma \begin{pmatrix} 1 \\ \vec{\beta} \end{pmatrix} = \gamma \left(\frac{\vec{e_1}}{\sqrt{-\,g_1}} + \frac{\beta\,\vec{e_2}}{\sqrt{g_2}} \right), \tag{48}$$

this is obviously an admissible 4-speed since its pseudo-norm is $\overset{\Rightarrow}{\bar{u}}^2 = \gamma^2\,\left(\beta^2 - 1\right) = -\,1$, but the symbolic 4-speed is not the real 4-speed, which is given, with the help of (47) in the $\left(\vec{e_1},\,\vec{e_2}\right)$ basis as

$$\overset{\Rightarrow}{u} = \gamma\,\left[\sqrt{\frac{g_2\,+1\,+\,(g_2\,-1)\,\beta^2}{g_2\,-\,g_1}}\,\vec{e}_1 + \frac{1}{\sqrt{g_2}}\,\sqrt{\frac{1\,+\,g_1\,+\,(1\,-\,g_1)\,\beta^2}{g_2\,-\,g_1}}\,\vec{e}_2\right] = \vec{e}_{0'}, \tag{49}$$

which is the case.

Let us now focus our attention on the 4-impulsion of a photon: in the former comobile LRS its contravariant components are symbolically written $\overset{\Rightarrow}{P} = \frac{h\,\nu'}{c}\begin{pmatrix} 1 \\ \vec{\Omega'} \end{pmatrix}$ where $\vec{\Omega'}$ is the photon propagation direction relatively to this LRS, from which

$$\begin{pmatrix} P^{0'} \\ P^{x'} \\ P^{y'} \\ P^{z'} \end{pmatrix} = n^2\,\frac{h\,\nu'}{c} \begin{pmatrix} 1 \\ \frac{\cos\,\Theta'}{n} \\ \frac{\sin\,\Theta'\,\cos\,\Phi'}{n} \\ \frac{\sin\,\Theta'\,\sin\,\Phi'}{n} \end{pmatrix}, \tag{50}$$

the covariant components being $P_{\mu'} = g_{\mu'\nu'}\,P^{\nu'}$, leading naturally to the fact that its pseudo-norm is $\overset{\Rightarrow}{P}^2 = P_{\mu'}\,P^{\mu'} = 0$, since $\overset{\Rightarrow}{P}$ is a light 4-event like; then the contravariant co-ordinates of the photon 4-impulsion vector relatively to the observer LRS are $P^{\mu} = P^{\mu'}\,\frac{\partial\,x^{\mu}}{\partial\,x^{\mu'}}$, leading to

$$\begin{array}{ll} P^0 = \frac{\gamma\,h\,\nu'\,n}{c}\,(n\,+\,\beta\,\cos\,\Theta') & P^x = \frac{\gamma\,h\,\nu'\,n}{c}\,(n\,\beta\,+\,\cos\,\Theta') \\ P^y = \frac{h\,\nu'\,n}{c}\,\sin\,\Theta'\,\cos\,\Phi' & P^z = \frac{h\,\nu'\,n}{c}\,\sin\,\Theta'\,\sin\,\Phi' \end{array} , \tag{51}$$

from which it is obvious that:

$P_0 = -\frac{\gamma h \nu' n^2}{c} (1 + n\beta \cos\Theta') \quad P_x = \frac{\gamma h \nu' n^2}{c} (\beta + n \cos\Theta')$

$P_y = \frac{h \nu' n^3}{c} \sin\Theta' \cos\Phi' \quad P_z = \frac{h \nu' n^3}{c} \sin\Theta' \sin\Phi'$

hence a simple calculation leads to $P_\mu P^\mu = 0$ which was expected; it is important however to notice here that these components suffer from a lack of physical clear significance so as for the $\overline{\overline{g}}$ metric tensor components expressed in the $\left(\vec{e_0}, \vec{e_x}, \vec{e_y}, \vec{e_z}\right)$ basis, since from the following tensorial relation

$$P^\mu = P^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} g^{\mu'\nu'} P_{\nu'} = P^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial x^\sigma} \frac{\partial x^{\nu'}}{\partial x^\tau} g^{\sigma\tau} g_{\mu'\nu'}, \tag{52}$$

one has

$$P^{0'} \frac{\partial x^\mu}{\partial x^{0'}} \left(1 - \frac{\partial x^{0'}}{\partial x^\sigma} \frac{\partial x^{\nu'}}{\partial x^\tau} g^{\sigma\tau} g_{0'\nu'}\right) = P^{x'} \frac{\partial x^\mu}{\partial x'} \left(\frac{\partial x'}{\partial x^\sigma} \frac{\partial x^{\nu'}}{\partial x^\tau} g^{\sigma\tau} g_{x'\nu'} - 1\right), \tag{53}$$

developing the former equality leads to

$$\begin{array}{l} 1 - \frac{\partial x^{0'}}{\partial x^\sigma} \frac{\partial x^{\nu'}}{\partial x^\tau} g^{\sigma\tau} g_{0'\nu'} = 1 - \frac{\gamma^4}{n^2} \left[n^2 \beta^4 - 2\beta^2 (2n^2 - 1) + n^2\right] \\ \frac{\partial x'}{\partial x^\sigma} \frac{\partial x^{\nu'}}{\partial x^\tau} g^{\sigma\tau} g_{x'\nu'} - 1 = \gamma^4 \left[\beta^4 - 2\beta^2 (2 - n^2) + 1\right] - 1 \end{array}, \tag{54}$$

but the $P^{0'}$ and $P^{x'}$ variables being independent and $\frac{\partial x^\mu}{\partial x^{0'}} \neq 0$ like $\frac{\partial x^\mu}{\partial x'} \neq 0$, Eq. (53) is verified if and only if $\begin{array}{l} \frac{\gamma^4}{n^2} \left[n^2 \beta^4 - 2\beta^2 (2n^2 - 1) + n^2\right] = 1 \\ \gamma^4 \left[\beta^4 - 2\beta^2 (2 - n^2) + 1\right] = 1 \end{array}$, that is if and only if $n = 1$; hence the $\vec{\vec{P}}$ representation in the $\left(\vec{e_0}, \vec{e_x}, \vec{e_y}, \vec{e_z}\right)$ basis is not a satisfactory one; this can be explained by the fact that there does not exist a canonical representation in this basis of the form $P^\mu = n^2 \frac{h\nu}{c} \frac{\Omega^\mu}{\sqrt{|g_{\mu\mu}|}}$ since the metric tensor $\overline{\overline{g}}$ is not diagonal in the former basis: the angle and frequency transformation must be then performed with the help of the diagonal metric; noticing that the 4-impulsion is then

$$\vec{\vec{P}} = \begin{pmatrix} P^1 \\ P^2 \\ P^y \\ P^z \end{pmatrix} = n^2 \frac{h\nu}{c} \begin{pmatrix} \frac{1}{\sqrt{-g_1}} \\ \frac{\cos\Theta}{\sqrt{g_2}} \\ \frac{\sin\Theta \cos\Phi}{n} \\ \frac{\sin\Theta \sin\Phi}{n} \end{pmatrix}, \tag{55}$$

for which it is easy to verify that $\vec{\vec{P}}^2 = 0$, moreover using relation (45), it comes from (50) that

$$\vec{\vec{P}} = n^2 \frac{h\nu'}{c} \left\{ \begin{array}{l} \gamma \left[\sqrt{\frac{g_2 + 1 + (g_2 - 1)\beta^2}{g_2 - g_1}} + \sqrt{g_2} \frac{\cos\Theta'}{n} \sqrt{\frac{1 + g_1 + (1 - g_1)\beta^2}{g_2 - g_1}}\right] \vec{e}_1 \\ + \gamma \left[\frac{1}{\sqrt{g_2}} \sqrt{\frac{1 + g_1 + (1 - g_1)\beta^2}{g_2 - g_1}} - g_1 \frac{\cos\Theta'}{n} \sqrt{\frac{g_2 + 1 + (g_2 - 1)\beta^2}{g_2 - g_1}}\right] \vec{e}_2 \\ + \frac{\sin\Theta'}{n} \left(\cos\Phi' \vec{e}_y + \sin\Phi' \vec{e}_z\right) \end{array} \right\}, \tag{56}$$

from which one obtains it is easy to obtain

$$\begin{aligned}
\nu &= \gamma\,\nu'\left[\sqrt{-\,g_1}\sqrt{\frac{g_2+1+(g_2-1)\,\beta^2}{g_2-g_1}}+\cos\Theta'\sqrt{\frac{1+g_1+(1-g_1)\,\beta^2}{g_2-g_1}}\right]\\
\nu\cos\Theta &= \gamma\,\nu'\left[\sqrt{\frac{1+g_1+(1-g_1)\,\beta^2}{g_2-g_1}}+\sqrt{-\,g_1}\,\cos\Theta'\sqrt{\frac{g_2+1+(g_2-1)\,\beta^2}{g_2-g_1}}\right],\\
\nu\sin\Theta\cos\Phi &= \nu'\sin\Theta'\cos\Phi'\\
\nu\sin\Theta\sin\Phi &= \nu'\sin\Theta'\sin\Phi'
\end{aligned} \tag{57}$$

one deduces from this result that angle Φ remains unchanged, while

$$\nu\sin\Theta = \nu'\sin\Theta', \tag{58}$$

noting $\mu = \cos\Theta$ and noticing that $\sqrt{\left[g_2+1+(g_2-1)\,\beta^2\right]\left[1+g_1+(1-g_1)\,\beta^2\right]} = 2\,\beta\sqrt{g_2}$, the two first equations of (58) can be rewritten as

$$\begin{aligned}
\nu &= \gamma\,\nu'\,n\sqrt{\frac{g_2+1+(g_2-1)\,\beta^2}{g_2\,(g_2-g_1)}}\left[1+2\,\beta\sqrt{-\frac{g_2}{g_1}}\,\frac{\mu'}{g_2+1+(g_2-1)\,\beta^2}\right]\\
\mu &= \frac{\left[g_2+1+(g_2-1)\,\beta^2\right]\mu'+2\,\beta\sqrt{-\frac{g_2}{g_1}}}{g_2+1+(g_2-1)\,\beta^2+2\,\beta\sqrt{-\frac{g_2}{g_1}}\,\mu'}
\end{aligned}, \tag{59}$$

from which it is easy to verify that for a unit refractive index one has

$$\nu = \gamma\,\nu'\,(1+\beta\,\mu') \quad \mu = \frac{\mu'+\beta}{1+\beta\,\mu'}, \tag{60}$$

which is the habitual and well-known angular aberration and frequency Doppler shift transformation in vacuum [2]; for an isotropic and grey dielectric, that is a refractive index independent on both the frequency and propagation direction, one has from (59)

$$\frac{\partial\mu}{\partial\mu'} = \gamma^2\left(\frac{\nu'}{\nu}\right)^2 n^2\,\frac{\left[g_2+1+(g_2-1)\,\beta^2\right]^2+4\,\beta^2\,\frac{g_2}{g_1}}{g_2\,(g_2-g_1)\left[g_2+1+(g_2-1)\,\beta^2\right]}\quad \frac{\partial\mu}{\partial\nu'} = 0 \quad \frac{\partial\nu}{\partial\nu'} = \frac{\nu}{\nu'}, \tag{61}$$

but

$$\begin{aligned}
\frac{\left[g_2+1+(g_2-1)\,\beta^2\right]^2+4\,\beta^2\,\frac{g_2}{g_1}}{\left[g_2+1+(g_2-1)\,\beta^2\right]^2} &= \frac{g_1\,g_2+2\,g_1+1+(g_1\,g_2-2\,g_1+1)\,\beta^2}{g_1\left[g_2+1+(g_2-1)\,\beta^2\right]}\\
= -\,\frac{n^2-1-2\,g_1+\left(n^2-1+2\,g_1\right)\beta^2}{g_1\left[g_2+1+(g_2-1)\,\beta^2\right]} &= \frac{g_2\,(g_2-g_1)}{\gamma^2\,n^2\left[g_2+1+(g_2-1)\,\beta^2\right]}
\end{aligned}, \tag{62}$$

so that

$$\frac{\partial\mu}{\partial\mu'} = \left(\frac{\nu'}{\nu}\right)^2, \tag{63}$$

hence one finally obtains

$$d\nu\,d\mu = \left|\frac{\partial\,(\nu,\,\mu)}{\partial\,(\nu'\,\mu')}\right| d\nu'\,d\mu' = \begin{vmatrix} \frac{\nu}{\nu'} & 0 \\ \frac{\partial\nu}{\partial\mu'} & \left(\frac{\nu'}{\nu}\right)^2 \end{vmatrix}\, d\nu'\,d\mu' = \frac{\nu'}{\nu}\,d\nu'\,d\mu', \tag{64}$$

the angle Φ remaining unchanged, noting $d\Omega = d\mu\, d\Phi$ the solid angle element, it comes from (64) the final important relation valid in the medium of index n as in vacuum

$$\nu\, d\nu\, d\Omega = \nu'\, d\nu'\, d\Omega'. \tag{65}$$

Let us then introduce the index $\overline{n}^2 = \frac{g_2\,(g_2 - g_1)}{g_2 + 1 + (g_2 - 1)\,\beta^2}$ such that the co-ordinates x^1 and x^2 can be rewritten:

$$x^1 = \frac{\gamma}{\sqrt{-\,g_1}}\left(\frac{n}{\bar{n}}\,x^{0'} + \frac{2\,\beta\, n\,\bar{n}}{g_2 - g_1}\,x'\right) \quad x^2 = \frac{\gamma}{\sqrt{g_2}}\left(\frac{2\,\beta\,\bar{n}}{g_2 - g_1}\,x^{0'} + \frac{n^2}{\bar{n}}\,x'\right), \tag{66}$$

note also that from the expression of $\overline{n}^2$, it is possible to obtain, after a rather difficult calculation the following relation

$$\frac{d\overline{n}}{d\beta} = \beta\,\gamma^2\,\overline{n}\left[1 - \frac{4\,\overline{n}^2\,(n^2+1)}{(g_2 - g_1)^3}\right] \Rightarrow \begin{array}{l} \frac{d\overline{n}}{d\beta} = 0 \;\; if\, n = 1 \\ \frac{d\overline{n}}{d\beta} = 0 \;\; if\, \beta = 0 \end{array}$$

The evolution of this index and its derivative with respect to β is plotted on the following figure, for a refractive index $n = 1.33$; below a rapidity $\beta = 0.6$, $\overline{n}$ remains practically constant so as its derivative; at $\beta = 0.75$, one has $\overline{n} = 1.50$ and the effects of the medium speed become appreciable; from $\beta = 0.75$ to 1, $\overline{n}$ and its derivative grow up very quickly, and for extremely high speeds, the refractive index effects cannot be longer ignored.

Then for the 4-speed one has

$$\vec{\vec{u}} = \gamma\,\frac{n}{\bar{n}}\left[\frac{\vec{e_1}}{\sqrt{-\,g_1}} + \frac{2\,\beta\,\bar{n}^2}{n\,(g_2 - g_1)}\,\frac{\vec{e_2}}{\sqrt{g_2}}\right], \tag{67}$$

since $\vec{\vec{u}}^2 = -1$ one obtains from (67) that $n^2\,(g_2 - g_1)^2 - 4\,\beta^2\,\bar{n}^4 = \frac{\bar{n}^2\,(g_2 - g_1)^2}{\gamma^2}$ and

$$\vec{\vec{u}} = \gamma\,\frac{n}{\bar{n}}\left[\frac{\vec{e_1}}{\sqrt{-\,g_1}} + \sqrt{1 - \left(\frac{\bar{n}}{\gamma\,n}\right)^2}\,\frac{\vec{e_2}}{\sqrt{g_2}}\right], \tag{68}$$

the previous expression of the 4-speed vector $\vec{\vec{u}}$ allows us to introduce a new rapidity $\overline{\beta} = \frac{2\,\beta\,\bar{n}^2}{n\,(g_2 - g_1)}$ such that $\bar{\beta} = \sqrt{1 - \left(\frac{\bar{n}}{\gamma\,n}\right)^2} = \sqrt{1 - \frac{1}{\bar{\gamma}^2}} \Leftrightarrow \bar{\gamma} = \gamma\,\frac{n}{\bar{n}} = \frac{1}{\sqrt{1 - \bar{\beta}^2}}$, with $\lim_{\beta \to 1} \bar{\beta} = 1$ whatever n is, so that the 4-speed vector can be written under the compact standard form

$$\vec{\vec{u}} = \bar{\gamma}\left(\frac{\vec{e_1}}{\sqrt{-\,g_1}} + \bar{\beta}\,\frac{\vec{e_2}}{\sqrt{g_2}}\right), \tag{69}$$

analogous to the symbolic 4-speed vector, replacing the rapidity β (expressed in vacuum) by the rapidity $\bar{\beta}$ expressed in the medium relatively to the observer LRS; introducing then the values of $\bar{n}$ and $\bar{\beta}$ in the angle/frequency transformations leads to

$$\nu = \bar{\gamma}\,\nu'\,\left(1 + \bar{\beta}\,\mu'\right) \quad \mu = \frac{\mu' + \bar{\beta}}{1 + \bar{\beta}\,\mu'} \quad \nu\, d\nu\, d\Omega = \nu'\, d\nu'\, d\Omega' \quad \nu\,\sin\Theta = \nu'\,\sin\Theta', \tag{70}$$

which has the remarkable form as the one obtained in vacuum; hence, one may expect to find a judicious set of co-ordinates $(\bar{x}^0,\, \bar{x})$ such that they verify the Lorentz transform, analogous to the habitual Lorentz transform replacing β by $\bar{\beta}$, that is

$$\begin{array}{lll} \bar{x}^0 = \bar{\gamma}\left(\bar{x}^{0'} + \bar{\beta}\,\bar{x}'\right) & & \bar{x}^{0'} = \bar{\gamma}\left(\bar{x}^0 - \bar{\beta}\,\bar{x}\right) \\ \bar{x} = \bar{\gamma}\left(\bar{x}' + \bar{\beta}\,\bar{x}^{0'}\right) & \Leftrightarrow & \bar{x}' = \bar{\gamma}\left(\bar{x} - \bar{\beta}\,\bar{x}^0\right), \\ \bar{y} = \bar{y}' \quad \bar{z} = \bar{z}' & & \bar{y}' = \bar{y} \quad \bar{z}' = \bar{z} \end{array} \tag{71}$$

from Eq. (66) one has

$$\begin{array}{l} x^{0'} = \frac{\gamma\, n}{\bar{n}}\left[x^1\,\sqrt{-\,g_1} - \frac{2\,\beta\,\bar{n}^2}{n\,(g_2\,-\,g_1)}\,x^2\,\sqrt{g_2}\right] = \bar{\gamma}\left(x^1\,\sqrt{-\,g_1} - \bar{\beta}\,x^2\,\sqrt{g_2}\right) \\ x' = \frac{1}{n}\,\frac{\gamma\, n}{\bar{n}}\left[x^2\,\sqrt{g_2} - \frac{2\,\beta\,\bar{n}^2}{n\,(g_2\,-\,g_1)}\,x^1\,\sqrt{-\,g_1}\right] = \frac{\bar{\gamma}}{n}\left(x^2\,\sqrt{g_2} - \bar{\beta}\,x^1\,\sqrt{-\,g_1}\right) \end{array}, \tag{72}$$

hence with the substitution

$$\bar{x}^{0'} = x^{0'} \quad \bar{x}' = n\,x' \quad \bar{x}^0 = x^1\,\sqrt{-\,g_1} \quad \bar{x} = x^2\,\sqrt{g_2}$$

the Lorentz transform (72) is obtained: it has to be noticed that the fundamental variable $\bar{x}' = n\,x'$ induces a local dilatation along x', so that we choose for the other spatial variables the same dilatation, that is $\bar{y}' = n\,y'$ and $\bar{z}' = n\,z'$; from the 4-event vector, one constructs then an **"equivalent vacuum"** completely defined from the comobile LRS by its co-ordinates $\left(\bar{x}^{0'} = x^{0'},\, \bar{x}' = n\,x',\, \bar{y}' = n\,y',\, \bar{z}' = n\,z'\right)$ associated to the orthonormal basis $\left(\vec{\bar{e}}_{0'} = \vec{e}_{0'},\, \vec{\bar{e}}_{x'} = \frac{\vec{e}_{x'}}{n},\, \vec{\bar{e}}_{y'} = \frac{\vec{e}_{y'}}{n},\, \vec{\bar{e}}_{z'} = \frac{\vec{e}_{z'}}{n}\right)$, and from the observer LRS by its co-ordinates $(\bar{x}^0 = x^1\,\sqrt{-\,g_1},\, \bar{x} = x^2\,\sqrt{g_2},\, \bar{y} = n\,y,\, \bar{z} = n\,z)$ associated to the orthonormal basis $\left(\vec{\bar{e}}_0 = \frac{\vec{e}_1}{\sqrt{-\,g_1}},\, \vec{\bar{e}}_x = \frac{\vec{e}_2}{\sqrt{g_2}},\, \vec{\bar{e}}_y = \frac{\vec{e}_y}{n},\, \vec{\bar{e}}_z = \frac{\vec{e}_z}{n}\right)$: indeed, those light spaces can be easily explained, reminding that the photons PTI are expressed as

$$\begin{array}{l} d\,\tau^2 = -\,g_{\mu\nu}\,d\,x^\mu\,d\,x^\nu = c^2\,(dt')^2 - \left[(n\,dx')^2 + (n\,dy')^2 + (n\,dz')^2\right] \\ = d\,\bar{x}^{0'2} - \left(d\,\bar{x}'^2 + d\,\bar{y}'^2 + d\,\bar{z}'^2\right) \end{array}$$

and

$$\begin{array}{l} d\,\tau^2 = -\,g_{\mu\nu}\,d\,x^\mu\,d\,x^\nu = (\sqrt{-\,g_1}\,d\,x^1)^2 - \left[(\sqrt{g_2}\,d\,x^2)^2 + (n\,dy)^2 + (n\,dz)^2\right] \\ = d\,\bar{x}^{02} - (d\,\bar{x}^2 + d\,\bar{y}^2 + d\,\bar{z}^2) \end{array}$$

hence the co-ordinates sets $\left(\bar{x}^{0'},\, \bar{x}',\, \bar{y}',\, \bar{z}'\right)$ and $(\bar{x}^0,\, \bar{x},\, \bar{y},\, \bar{z})$ associated to the two basis $\left(\vec{\bar{e}}_{0'},\, \vec{\bar{e}}_{x'},\, \vec{\bar{e}}_{y'},\, \vec{\bar{e}}_{z'}\right)$ and $\left(\vec{\bar{e}}_0,\, \vec{\bar{e}}_x,\, \vec{\bar{e}}_y,\, \vec{\bar{e}}_z\right)$ represent vacuum light co-ordinates and basis, which are different from the mater co-ordinates and basis; from a comobile LRS point of view, the mater co-ordinates are $\left(x^{0'},\, x',\, y',\, z'\right)$ associated to the basis $\left(\vec{e}_{0'},\, \vec{e}_{x'},\, \vec{e}_{y'},\, \vec{e}_{z'}\right)$, so that it is convenient analogously to introduce, from the observer LRS point of view, the mater co-ordinates $\left(\tilde{x}^0 = x^1\,\sqrt{-\,g_1},\, \tilde{x} = \frac{x^2\,\sqrt{g_2}}{n},\, \tilde{y} = y,\, \tilde{z} = z\right)$ associated to the basis

$\left(\vec{\tilde{e}}_0 = \frac{\vec{e_1}}{\sqrt{-g_1}}, \vec{\tilde{e}}_x = n\frac{\vec{e_2}}{\sqrt{g_2}}, \vec{\tilde{e}}_y = \vec{e_y}, \vec{\tilde{e}}_z = \vec{e_z}\right)$, so that mater-light metric tensors can be expressed as

$$\overline{\overline{g'}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & n^2 & 0 & 0 \\ 0 & 0 & n^2 & 0 \\ 0 & 0 & 0 & n^2 \end{pmatrix} \text{ and } \overline{\overline{\tilde{g}}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & n^2 & 0 & 0 \\ 0 & 0 & n^2 & 0 \\ 0 & 0 & 0 & n^2 \end{pmatrix}, \quad (73)$$

hence the two metrics, relatively to the moving particle and the observer, are strictly equivalent, meaning that the natural mater curvilinear abscissa path is $d\,\tilde{s}^2 = d\,\tilde{x}^2 + d\,\tilde{y}^2 + d\,\tilde{z}^2$ relatively to the observer LRS, and that for a photon it is related to time by $d\tilde{t} = \frac{n}{c}\,d\tilde{s}$, the latter quantities being defined relatively to the observer LRS; then one has the relation between the two sets of co-ordinates

$$\begin{array}{l} \tilde{x}^0 = \gamma\,\bar{\gamma}\,\left[\left(1 - n\,\beta\,\bar{\beta}\right)\,x^0 + \left(n\,\bar{\beta} - \beta\right)\,x\right] \\ \tilde{x} = \frac{\gamma\,\bar{\gamma}}{n}\,\left[\left(\bar{\beta} - n\,\beta\right)\,x^0 + \left(n - \beta\,\bar{\beta}\right)\,x\right] \end{array} \Leftrightarrow \begin{array}{l} x^0 = \frac{\gamma\,\bar{\gamma}}{n}\,\left[\left(n - \beta\,\bar{\beta}\right)\,\tilde{x}^0 - n\,\left(n\,\bar{\beta} - \beta\right)\,\tilde{x}\right] \\ x = \frac{\gamma\,\bar{\gamma}}{n}\,\left[n\,\left(1 - n\,\beta\,\bar{\beta}\right)\,\tilde{x} - \left(\bar{\beta} - n\,\beta\right)\,\tilde{x}^0\right] \end{array}, \quad (73)$$

Furthermore, from the 4-event vector and the definition of the 4-speed vector, one has

$$\begin{array}{l} \vec{\vec{u}} = \frac{d\vec{\vec{M}}}{d\tau} = \frac{d\vec{\vec{M}}}{d\,\bar{x}^0}\,\frac{d\,\bar{x}^0}{d\tau} = \bar{\gamma}\,\left(\vec{\bar{e}}_0 + \bar{\beta}\,\vec{\bar{e}}_x\right) = \frac{d\,\bar{x}^0}{d\tau}\,\left(\vec{\bar{e}}_0 + \frac{d\bar{x}}{d\,\bar{x}^0}\,\vec{\bar{e}}_x + \frac{d\bar{y}}{d\,\bar{x}^0}\,\vec{\bar{e}}_y + \frac{d\bar{z}}{d\,\bar{x}^0}\,\vec{\bar{e}}_z\right) \\ \Rightarrow \frac{d\,\tilde{x}^0}{d\tau} = \bar{\gamma} \quad \vec{\bar{\beta}} = \frac{d\bar{x}}{d\,\bar{x}^0}\,\vec{\bar{e}}_x = \frac{d\tilde{x}}{d\,\tilde{x}^0}\,\vec{\tilde{e}}_x \quad \frac{d\bar{y}}{d\,\bar{x}^0} = \frac{d\bar{z}}{d\,\bar{x}^0} = 0 \end{array}, \quad (74)$$

but one also has $\vec{\vec{u}} = \frac{d\vec{\vec{M}}}{d\tau} = \frac{d\vec{\vec{M}}}{d\,x^{0'}}\,\frac{d\,x^{0'}}{d\tau} = \vec{e_{0'}} \Rightarrow \frac{d\,x^{0'}}{d\tau} = 1$, so that it comes the fundamental relation binding the proper times of a mass particle moving with speed βg?o(defined as if it were in vacuum)

$$d\,\tilde{x}^0 = \bar{\gamma}\,d\,x^{0'} \Leftrightarrow d\,\overline{x}^0 = \bar{\gamma}\,d\,\overline{x}^{0'} \quad (75)$$

3. Interpretation of the Fresnel'S Refractive Index as a Negative Uniaxial Anisotropy

From what precedes, it is obvious that the 4-speed is $\vec{\vec{u}} = \bar{\gamma}\,\left(\vec{\bar{e}}_0 + \bar{\beta}\,\vec{\bar{e}}_x\right)$ and is naturally defined in the vacuum bound to the observer located in the moving medium; it is important to remark here that the 4-speed perceived by an observer at rest <u>located in the medium</u> is the ***same*** 4-speed perceived by an observer at rest but <u>located in vacuum</u>; for the latter one, the 4-speed is:

$$\vec{\vec{u}}_v = \gamma\,\left(\vec{e_{\bar{0}}} + \beta\,\vec{e_{\bar{x}}}\right)$$

where $\left(\vec{e}_{\bar{0}}, \vec{e}_{\bar{x}}\right)$ is the equivalent "vacuum basis" of the moving medium perceived by the observer located in vacuum of unit refractive index, which is not the $\left(\vec{\bar{e}}_0, \vec{\bar{e}}_x\right)$ basis of the equivalent vacuum bound to the observer located inside the medium of refractive index n; it is thus defined as follows:

$$\vec{e}_{\bar{0}} = \gamma\left(\vec{\bar{e}}_{0'} - \beta\,\vec{\bar{e}}_{x'}\right) = \gamma\left(\vec{e}_{0'} - \frac{\beta}{n}\,\vec{e}_{x'}\right) = \frac{\gamma^2}{n}\left[\left(n - \beta^2\right)\vec{e}_0 + \beta\,(n-1)\,\vec{e}_x\right]$$
$$\vec{e}_{\bar{x}} = \gamma\left(\vec{\bar{e}}_{x'} - \beta\,\vec{\bar{e}}_{0'}\right) = \gamma\left(\frac{\vec{e}_{x'}}{n} - \beta\,\vec{e}_{0'}\right) = \frac{\gamma^2}{n}\left[\left(1 - n\,\beta^2\right)\vec{e}_x - \beta\,(n-1)\,\vec{e}_0\right]$$

with $\vec{e}_{\bar{0}}^{\,2} = -1 \quad \vec{e}_{\bar{x}}^{\,2} = 1 \quad \vec{e}_{\bar{0}}\,\vec{e}_{\bar{x}} = 0$

and equivalently:
$$\vec{e}_0 = \gamma^2\left[\left(1 - n\,\beta^2\right)\vec{e}_{\bar{0}} - \beta\,(n-1)\,\vec{e}_{\bar{x}}\right]$$
$$\vec{e}_x = \gamma^2\left[\beta\,(n-1)\,\vec{e}_{\bar{0}} + \left(n - \beta^2\right)\vec{e}_{\bar{x}}\right]$$

where β has to be understood as the speed of the mass particles evolving in the medium, but evaluated as if they were evolving in vacuum; hence if $n = 1$ one obviously has $\vec{e}_{\bar{0}} = \vec{e}_0$ and $\vec{e}_{\bar{x}} = \vec{e}_x$; replacing then $\vec{e}_{\bar{0}}$ and $\vec{e}_{\bar{x}}$ by their values in terms of $\vec{e}_0$ and $\vec{e}_x$ leads to $\vec{\vec{u}}_v = \gamma\left(\vec{e}_0 + \beta\,\vec{e}_x\right)$ which was expected ; furthermore, one has:

$$\vec{\bar{e}}_0 = \frac{\gamma\bar{\gamma}}{n}\left[\left(n - \beta\bar{\beta}\right)\vec{e}_0 + \left(n\beta - \bar{\beta}\right)\vec{e}_x\right]$$
$$\vec{\bar{e}}_x = \frac{\gamma\bar{\gamma}}{n}\left[\left(\beta - n\bar{\beta}\right)\vec{e}_0 + \left(1 - n\beta\bar{\beta}\right)\vec{e}_x\right]$$

so that after a simple calculation : $\vec{\vec{u}} = \gamma\left(\vec{e}_0 + \beta\,\vec{e}_x\right) = \vec{\vec{u}}_v$

which proves the result; one deduces then from what precedes that:

$$\begin{aligned}\vec{\bar{e}}_0 &= \gamma\bar{\gamma}\left[\left(1 - \beta\bar{\beta}\right)\vec{e}_{\bar{0}} + \left(\beta - \bar{\beta}\right)\vec{e}_{\bar{x}}\right]\\ \vec{\bar{e}}_x &= \gamma\bar{\gamma}\left[\left(\beta - \bar{\beta}\right)\vec{e}_{\bar{0}} + \left(1 - \beta\bar{\beta}\right)\vec{e}_{\bar{x}}\right]\end{aligned} \Leftrightarrow \begin{aligned}\vec{e}_{\bar{0}} &= \gamma\bar{\gamma}\left[\left(1 - \beta\bar{\beta}\right)\vec{\bar{e}}_0 - \left(\beta - \bar{\beta}\right)\vec{\bar{e}}_x\right]\\ \vec{e}_{\bar{x}} &= \gamma\bar{\gamma}\left[-\left(\beta - \bar{\beta}\right)\vec{\bar{e}}_0 + \left(1 - \beta\bar{\beta}\right)\vec{\bar{e}}_x\right]\end{aligned}$$

this result allows to define the vacuum-like co-ordinates inside the medium for the observer located in real vacuum: indeed, the existence of an observer located in vacuum induces the existence of an interface between the moving medium and the vacuum, that we shall assume orthogonal to the direction of motion of the moving medium; we shall this time define a fictive observer at rest in the vacuum such that at a given instant (hereafter designed as the initial vacuum instant) its position coincides with the position of the interface, and that at this time one has $d_V = \left\|\overrightarrow{O_V O_{V_f}}\right\|$ where d_V is the distance between the reference observer O_V in vacuum and the fictive observer O_{V_f} in vacuum, this distance being evaluated in vacuum; let us introduce $\left(x^{\bar{0}},\, \underline{x},\, \underline{y},\, \underline{z}\right)$ the vacuum-like co-ordinates of an event perceived by the reference observer at rest located in the real vacuum, $\left({x_f}^{\bar{0}},\, \underline{x_f},\, \underline{y_f},\, \underline{z_f}\right)$ the vacuum-like co-ordinates of the same event perceived by the fictive observer and $\left(\bar{x}^{0'},\, \bar{x}',\, \bar{y}',\, \bar{z}'\right)$ the vacuum-like co-ordinates of this event perceived by a moving particle in vacuum and bound to the moving interface; obviously, the interface moves with the β rapidity for the observers at rest in vacuum, while it moves with the $\bar{\beta}$ rapidity for observers at rest located in the moving medium, so that for the observers in vacuum one has:

$$\begin{aligned} {x_f}^{\bar 0} &= \gamma\left(\bar x^{0'} + \beta\,\bar x'\right) & x^{\bar 0} &= \gamma\left(\bar x^{0'} + \beta\,\bar x'\right) \\ \underline{x_f} &= \gamma\left(\beta\,\bar x^{0'} + \bar x'\right) \text{ from which: } & \underline{x} &= \gamma\left(\beta\,\bar x^{0'} + \bar x'\right) + d_V \\ \underline{y_f} &= \bar y' \quad \underline{z_f} = \bar z' & \underline{y} &= \bar y' \quad \underline{z} = \bar z' \end{aligned}$$

since the two observers in vacuum remain at rest and have the same time perception; similarly, one introduces a fictive observer in the medium such that at the initial vacuum instant, its position inside the moving medium coincides with the one of the moving interface and $d_M = \left\|\overrightarrow{O_M\,O_{M_f}}\right\|$ where d_M is the distance between the reference observer O_M in the moving medium and the fictive observer O_{M_f} in the moving medium, this distance being evaluated in the medium; note that the two fictive observers are created only for calculation rules: indeed, it $d_v\left(x^{\bar 0}\right) > d_V$ where $d_v\left(x^{\bar 0}\right)$ is the distance between the reference observer in vacuum and moving interface, then the moving medium leaves the reference observer in vacuum and goes towards the reference observer in the medium, so that the fictive observer which was in the medium at $x^{\bar 0} = \bar x^0 = 0$ is in vacuum for $\bar x^0 > 0$, while if $d_v\left(x^{\bar 0}\right) < d_V$, the moving medium goes towards the reference observer in vacuum and the fictive observer in vacuum at $\bar x^0 = x^{\bar 0} = 0$ is in the medium for $x^{\bar 0} > 0$; then for the moving medium:

$$\begin{aligned} {\bar x_f}^{0} &= \bar\gamma\left(\bar x^{0'} + \bar\beta\,\bar x'\right) & \bar x^{0} &= \bar\gamma\left(\bar x^{0'} + \beta\,\bar x'\right) \\ \bar x_f &= \bar\gamma\left(\bar\beta\,\bar x^{0'} + \bar x'\right) \text{ from which: } & \bar x &= \bar\gamma\left(\bar\beta\,\bar x^{0'} + \bar x'\right) - \overline{d_M} \text{ with } \overline{d_M} = n\, d_M \\ \bar y_f &= \bar y' \quad \bar z_f = \bar z' & \bar y &= \bar y' \quad \bar z = \bar z' \end{aligned}$$

note that no distinction has to be done for the basis vectors which are the same for the fictive and reference observers; hence from a same 4-event perceived both by an observer at rest located in the real vacuum and an observer at rest located in the moving medium, one has:

$$\vec{\vec{M}} = M^\mu\,\vec e_\mu = {\bar x_f}^0\,\vec e_0 + \bar x_f\,\vec e_x + \bar y_f\,\vec e_y + \bar z_f\,\vec e_z = {x_f}^{\bar 0}\,\vec e_{\bar 0} + \underline{x_f}\,\vec e_{\bar x} + \underline{y_f}\,\vec e_{\bar y} + \underline{z_f}\,\vec e_{\bar z}$$

hence from what precedes, it comes:

$$\begin{aligned} x^{\bar 0} &= \gamma\bar\gamma\left[\left(1-\beta\bar\beta\right)\bar x^0 + \left(\beta-\bar\beta\right)\left(\bar x + \overline{d_M}\right)\right] & \bar x^0 &= \gamma\,\bar\gamma\left[\left(1-\beta\bar\beta\right)x^{\bar 0} - \left(\beta-\bar\beta\right)\left(\underline{x} - d_V\right)\right] \\ \underline{x} - d_V &= \gamma\bar\gamma\left[\left(1-\beta\bar\beta\right)\left(\bar x + \overline{d_M}\right) + \left(\beta-\bar\beta\right)\bar x^0\right] \Leftrightarrow & \bar x + \overline{d_M} &= \gamma\bar\gamma\left[\left(1-\beta\bar\beta\right)\left(\underline{x} - d_V\right) - \left(\beta-\bar\beta\right)x^{\bar 0}\right] \\ \underline{y} &= \bar y \quad \underline{z} = \bar z & \bar y &= \underline{y} \quad \bar z = \underline{z} \end{aligned}$$

replacing then $\bar x^{0'}$ and $\bar x'$ by their values in terms of $x^{0'}$ and x', and relating $x^{0'}$ and x' to x^0 and x thanks to the Lorentz transform with the vacuum-like β rapidity finally leads to:

$$\begin{aligned} x^{\bar 0} &= \gamma^2\left[\left(1-n\,\beta^2\right)x^0 + \beta\,(n-1)\,(x-d_V)\right] & & x^0 = \tfrac{\gamma^2}{n}\left[\left(n-\beta^2\right)x^{\bar 0} - \beta\,(n-1)\,(\underline{x}-d_V)\right] \\ \underline{x} - d_V &= \gamma^2\left[\left(n-\beta^2\right)(x-d_V) - \beta\,(n-1)\,x^0\right] & \Leftrightarrow\; & x - d_V = \tfrac{\gamma^2}{n}\left[\left(1-n\,\beta^2\right)(\underline{x}-d_V) + \beta\,(n-1)\,x^{\bar 0}\right] \end{aligned}$$

so that $x^{\bar{0}} = x^0$ and $\underline{x} = x$ if $n = 1$. In these conditions, the 4-impulsion vector of a photon is, for the observer at rest located in the medium

$$\begin{aligned}\vec{\vec{P}} &= \frac{h\nu}{c}\left\{n^2\,\vec{\tilde{e}}_0 + n\left[\cos\Theta\,\vec{\tilde{e}}_x + \sin\Theta\left(\cos\Phi\,\vec{\tilde{e}}_y + \sin\Phi\,\vec{\tilde{e}}_z\right)\right]\right\} = P^{\tilde{\mu}}\,\vec{e_{\tilde{\mu}}} \\ &= \frac{h\nu\,n^2}{c}\left[\vec{\tilde{e}}_0 + \cos\Theta\,\vec{\tilde{e}}_x + \sin\Theta\left(\cos\Phi\,\vec{\tilde{e}}_y + \sin\Phi\,\vec{\tilde{e}}_z\right)\right] = \frac{E}{c}\begin{pmatrix}1\\ \vec{\Omega}\end{pmatrix}\end{aligned} \tag{76}$$

from the former definition of the 4-impulsion, one deduces that $\vec{\Omega} = \cos\Theta\,\vec{\tilde{e}}_x + \sin\Theta\left(\cos\Phi\,\vec{\tilde{e}}_y + \sin\Phi\,\vec{\tilde{e}}_z\right)$, and that $E = h\nu\,n^2 = n^2\,E^{(0)}$: it is a well known result that the energy of a photon in a medium of index n is n^2 times its energy in vacuum, but the important fact is that this result remains valid, even for a moving medium, in the point of view of the observer at rest located in the moving medium.

Replacing then $\vec{\tilde{e}}_0$ and $\vec{\tilde{e}}_x$ by their values in terms of $\vec{e}_0$ and $\vec{e}_x$, leads finally to

$$\vec{\vec{P}} = \frac{h\nu\,n}{c}\left\{\begin{array}{l}\gamma\bar{\gamma}\left[n - \beta\bar{\beta} + \left(\beta - n\bar{\beta}\right)\cos\Theta\right]\vec{e}_0 + \gamma\bar{\gamma}\left[n\beta - \bar{\beta} + \left(1 - n\beta\bar{\beta}\right)\cos\Theta\right]\vec{e}_x \\ +\sin\Theta\left(\cos\Phi\,\vec{e}_y + \sin\Phi\,\vec{e}_z\right)\end{array}\right\} = P^{\mu}\,\vec{e_{\mu}} \tag{77}$$

so that introducing the values of ν and $\cos\Theta$ in terms of ν' and $\cos\Theta'$ gives the values of P^{μ} obtained by Eq. (51). Replacing now $\vec{e}_0$ and $\vec{e}_x$ by their values in terms of $\vec{e_{\bar{0}}}$ and $\vec{e_{\bar{x}}}$ gives the important result

$$\vec{\vec{P}} = \frac{h\nu\,n^2}{c}\left\{\begin{array}{l}\gamma\bar{\gamma}\left[1 - \beta\bar{\beta} + \left(\beta - \bar{\beta}\right)\cos\Theta\right]\vec{e}_{\bar{0}} + \gamma\bar{\gamma}\left[\beta - \bar{\beta} + \left(1 - \beta\bar{\beta}\right)\cos\Theta\right]\vec{e}_{\bar{x}} \\ +\sin\Theta\left(\cos\Phi\,\vec{e}_{\bar{y}} + \sin\Phi\,\vec{e}_{\bar{z}}\right)\end{array}\right\} \tag{78}$$

this is the 4-impulsion energy of a photon evolving in the moving medium and expressed in the vacuum basis of an observer at rest located in the real vacuum, while this photon 4-impulsion energy perceived by an observer at rest located in the moving medium is simply $\vec{\vec{P}} = \frac{h\nu\,n^2}{c}\left[\vec{\tilde{e}}_0 + \cos\Theta\,\vec{\tilde{e}}_x + \sin\Theta\left(\cos\Phi\,\vec{\tilde{e}}_y + \sin\Phi\,\vec{\tilde{e}}_z\right)\right]$; under this form, Eq. (78) allows to define an apparent energy and direction $\hat{\Theta}$ of propagation of light in the moving medium for the observer located in vacuum, such that:

$$\frac{\hat{E}}{c} = \frac{h\nu\,n^2\gamma\bar{\gamma}}{c}\left[1 - \beta\bar{\beta} + \left(\beta - \bar{\beta}\right)\cos\Theta\right]$$

$$\cos\hat{\Theta} = \frac{\beta - \bar{\beta} + \left(1 - \beta\bar{\beta}\right)\cos\Theta}{1 - \beta\bar{\beta} + \left(\beta - \bar{\beta}\right)\cos\Theta}$$

$$\sin\hat{\Theta}\cos\hat{\Phi} = \frac{\sin\Theta\cos\Phi}{\gamma\bar{\gamma}\left[1 - \beta\bar{\beta} + \left(\beta - \bar{\beta}\right)\cos\Theta\right]}$$

$$\sin\hat{\Theta}\sin\hat{\Phi} = \frac{\sin\Theta\sin\Phi}{\gamma\bar{\gamma}\left[1 - \beta\bar{\beta} + \left(\beta - \bar{\beta}\right)\cos\Theta\right]}$$

from which $\hat{\Phi} = \Phi$ and

$$\cos\hat{\Theta} = \frac{\beta - \bar{\beta} + \left(1 - \beta\bar{\beta}\right)\cos\Theta}{1 - \beta\bar{\beta} + \left(\beta - \bar{\beta}\right)\cos\Theta}$$

$$\sin\hat{\Theta} = \frac{\sin\Theta}{\gamma\bar{\gamma}\left[1 - \beta\bar{\beta} + \left(\beta - \bar{\beta}\right)\cos\Theta\right]}$$

one can easily verify that $\cos^2\hat{\Theta} + \sin^2\hat{\Theta} = 1$, so that

$$\cos\Theta = \frac{\left(1 - \beta\bar{\beta}\right)\cos\hat{\Theta} - \left(\beta - \bar{\beta}\right)}{1 - \beta\bar{\beta} - \left(\beta - \bar{\beta}\right)\cos\hat{\Theta}}$$

$$\sin\Theta = \frac{\sin\hat{\Theta}}{\gamma\bar{\gamma}\left[1 - \beta\bar{\beta} - \left(\beta - \bar{\beta}\right)\cos\hat{\Theta}\right]}$$

hence $\overset{\Rightarrow}{P} = \frac{\hat{E}}{c}\begin{pmatrix}1\\ \overset{\rightarrow}{\hat{\Omega}}\end{pmatrix}$ with $\frac{\hat{E}}{c} = \frac{h\nu\, n^2}{c\gamma\bar{\gamma}\left[1-\beta\bar{\beta}+\left(\beta-\bar{\beta}\right)\cos\hat{\Theta}\right]}$

For a constant refractive index n, the light trajectories inside the medium are straight lines; indeed, these trajectories are the light geodesics determined by the geodesics equations

$\frac{d P^{\tilde{\alpha}}}{d\tilde{\sigma}} + \Gamma^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}}\, P^{\tilde{\beta}}\, P^{\tilde{\gamma}} = 0$ where $\tilde{\sigma}$ is a step parameter on the trajectory defined by $P^{\tilde{\alpha}} = \frac{d x^{\tilde{\alpha}}}{d\tilde{\sigma}}$ and $\Gamma^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}}$ are the Christoffel coefficients such that $\Gamma^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}} = \frac{\tilde{g}^{\tilde{\alpha}\tilde{m}}}{2}\left(\frac{\partial\,\tilde{g}_{\tilde{m}\tilde{\gamma}}}{\partial\, x^{\tilde{\beta}}} + \frac{\partial\,\tilde{g}_{\tilde{m}\tilde{\beta}}}{\partial\, x^{\tilde{\gamma}}} - \frac{\partial\,\tilde{g}_{\tilde{\beta}\tilde{\gamma}}}{\partial\, x^{\tilde{m}}}\right)$; obviously these coefficients are all 0 for a constant refractive index, so that the geodesics equations lead to $P^{\tilde{\alpha}} = cons\tan t$, or equivalently $\frac{d\,x^{\tilde{\alpha}}}{d\,x^{\tilde{\beta}}} = \frac{P^{\tilde{\alpha}}}{P^{\tilde{\beta}}} = cons\tan t$, and for light propagating in the $(\tilde{x},\, \tilde{y})$ plane, one has:

$d\tilde{x} = \frac{\cos\Theta}{n}\, d\,\tilde{x}^0 \quad d\tilde{y} = \frac{\sin\Theta}{n}\, d\,\tilde{x}^0$ hence $d\tilde{s}^2 = \left[1 + \left(\frac{d\tilde{y}}{d\tilde{x}}\right)^2\right] d\tilde{x}^2 = \frac{d\tilde{x}^2}{\cos^2\Theta} = \frac{d\,\tilde{x}^{0\,2}}{n^2}$

and one retrieves the obvious relation $\frac{d\tilde{s}}{d\,\tilde{x}^0} = \frac{1}{n}$, or equivalently $\frac{d\bar{s}}{d\,\bar{x}^0} = 1$; furthermore, from what precedes, it comes that:

$$\frac{d\underline{x}}{d\,x^{\bar{0}}} = \frac{\left(\beta-\bar{\beta}\right) d\,\bar{x}^0 + \left(1-\beta\bar{\beta}\right) d\bar{x}}{\left(1-\beta\bar{\beta}\right) d\,\bar{x}^0 + \left(\beta-\bar{\beta}\right) d\bar{x}} = \frac{\beta-\bar{\beta}+\left(1-\beta\bar{\beta}\right)\cos\Theta}{1-\beta\bar{\beta}+\left(\beta-\bar{\beta}\right)\cos\Theta} = \cos\hat{\Theta}$$

$$\frac{d\underline{y}}{d\,x^{\bar{0}}} = \frac{d\bar{y}}{\gamma\bar{\gamma}\left[\left(\beta-\bar{\beta}\right) d\,\bar{x}^0 + \left(1-\beta\bar{\beta}\right) d\bar{x}\right]} = \frac{\sin\Theta}{\gamma\bar{\gamma}\left[1-\beta\bar{\beta}+\left(\beta-\bar{\beta}\right)\cos\Theta\right]} = \sin\hat{\Theta}$$

hence $d\underline{s}^2 = \left[1 + \left(\frac{d\underline{y}}{d\underline{x}}\right)^2\right] d\underline{x}^2 = \frac{d\underline{x}^2}{\cos^2\hat{\Theta}} = d\,x^{\bar{0}2}$

Note that expressed in terms of $(x^0,\, x,\, y)$ co-ordinates set, the previous relations are equivalent to:

$$\frac{d\underline{x}}{d\,x^{\bar{0}}} = \frac{\left(n-\beta^2\right)\frac{dx}{d\,x^0} - \beta\,(n-1)}{\beta\,(n-1)\,\frac{dx}{d\,x^0} + 1 - n\,\beta^2} \quad \frac{d\underline{y}}{d\,x^{\bar{0}}} = \frac{n\,\frac{dy}{d\,x^0}}{\gamma^2\left[\beta\,(n-1)\,\frac{dx}{d\,x^0} + 1 - n\,\beta^2\right]}$$

so that when $\hat{\Theta} = 0$, one has $\frac{d\underline{x}}{d\,x^{\bar{0}}} = 1$ and $\frac{d\underline{y}}{d\,x^{\bar{0}}} = 0$, from which it comes:

$\frac{dy}{d\,x^0} = 0$ and $\frac{dx}{d\,x^0} = \frac{1+\beta n}{n+\beta} = \frac{1}{\hat{n}}$ which is the Fresnel's drag additional formula

for $\hat{\Theta} = \frac{\pi}{2}$, $\frac{d\underline{x}}{d\,x^{\bar{0}}} = 0$ and $\frac{d\underline{y}}{d\,x^{\bar{0}}} = 1$, from which one obtains:

$\frac{dx}{d\,x^0} = \frac{\beta\,(n-1)}{n-\beta^2}$ and $\frac{dy}{d\,x^0} = \frac{1}{\gamma^2\left(n-\beta^2\right)}$, hence $\left[\left(\frac{dx}{d\,x^0} = 0\right) and \left(\frac{dy}{d\,x^0} = 1\right)\right] \Leftrightarrow (n = 1)$

we retrieve the fact that it is impossible to define a physical direction for light in the moving medium when using the sets $\left(\overrightarrow{e}_0,\, \overrightarrow{e}_x,\, \overrightarrow{e}_y,\, \overrightarrow{e}_z\right)$ and $(x^0,\, x,\, y,\, z)$, since the two vectors $\overrightarrow{e}_0$ and $\overrightarrow{e}_x$ are not linearly independent; let us introduce now the habitual matter co-ordinates $\left(\hat{x}^0,\, \hat{x},\, \hat{y},\, \hat{z}\right)$ for the observer located in vacuum, such that for an event in vacuum the co-ordinates $\left(\hat{x}^0,\, \hat{x},\, \hat{y},\, \hat{z}\right)$ coincide with the vacuum co-ordinates $\left(x^{\bar{0}},\, \underline{x},\, \underline{y},\, \underline{z}\right)$, for an event in the medium perceived in the $\overrightarrow{e_{\bar{x}}}$ direction by the observer in vacuum they are:

$x^{\bar{0}} = \hat{x}^0 \quad \underline{x} = n_{||}\,\hat{x}$ then for a light propagating in the $\overrightarrow{e_{\bar{x}}}$ direction, one must have $\frac{d\hat{x}}{d\,\hat{x}^0} = \frac{1}{n_{||}}$, and for an event in the medium perceived in the $\overrightarrow{e_{\bar{y}}}$ or $\overrightarrow{e_{\bar{z}}}$ directions they are:

$x^{\bar{0}} = \hat{x}^0 \quad \underline{y} = n_{\perp}\,\hat{y} \quad \underline{z} = n_{\perp}\,\hat{z}$ where obviously $n_{\perp} = n$ and $\hat{y} = y$ as $\hat{z} = z$ if the observers in vacuum and in the medium perceive the same y and z co-ordinates; the observer at rest in vacuum perceives naturally the time in a given "direction" which is his fundamental time vector reference system, in the vacuum as well as for events located in the moving medium: this implies that the time direction $\vec{e_{\bar{0}}}$ perceived by the observer in vacuum for events located in the moving medium must be the time direction $\vec{e_0}$ perceived by this observer for events <u>located in the vacuum</u>, which is the case by construction of $\vec{e_{\bar{0}}}$; similarly, the observer at rest located in vacuum is unable to distinguish a light ray emerging from the moving medium in the perceived $\vec{e_{\bar{x}}}$ direction for a perceived frequency $\hat{\nu}$, governed by $\frac{d\hat{x}}{d\hat{x}^0} = \frac{1}{n_{||}}$, from any light travelling the vacuum for the same perceived direction and frequency and characterised by $\frac{dx}{dx^0}$, so that for the observer located in vacuum, the light emerging from the moving medium in the perceived direction $\vec{e_{\bar{x}}}$ will obey to:

$\frac{d\hat{x}}{d\hat{x}^0} = \frac{1}{n_{||}} = \frac{dx}{dx^0} = \frac{1}{\hat{n}}$ from which it comes $1 < n_{||} = \frac{n+\beta}{1+\beta n} < n = n_{\perp}$

Then, the Fresnel's refractive index can be interpreted as the apparent refractive index of the moving medium in the direction of motion of this medium effectively perceived by the observer at rest in vacuum; hence the two relations $\frac{dx}{dx^0} = \cos\hat{\Theta}$ and $\frac{dy}{dx^0} = \sin\hat{\Theta}$ can be rewritten as:

$$\begin{array}{l} \frac{d\hat{x}}{d\hat{x}^0} = \frac{\cos\hat{\Theta}}{n_{||}} \\ \frac{d\hat{y}}{d\hat{x}^0} = \frac{\sin\hat{\Theta}}{n_{\perp}} \end{array} \Rightarrow d\hat{s}^2 = \left[1 + \left(\frac{d\hat{y}}{d\hat{x}}\right)^2\right]\left(\frac{d\hat{x}}{d\hat{x}^0}\right)^2 d\hat{x}^{02} = \frac{n_{\perp}^2\cos^2\hat{\Theta} + n_{||}^2\sin^2\hat{\Theta}}{n_{\perp}^2\, n_{||}^2}\, d\hat{x}^{02}$$

from which one deduces the apparent refractive index: $n_e^2 = \frac{n_{\perp}^2\, n_{||}^2}{n_{\perp}^2\cos^2\hat{\Theta} + n_{||}^2\sin^2\hat{\Theta}}$

This apparent refractive index is the extraordinary wave refractive index for an uniaxial medium of optical axis $\vec{e_{||}} = \vec{e_{\bar{y}}}$ [6] : indeed, $n_e^2 = \frac{n_{\perp}^2\, n_{||}^2}{n_{\perp}^2\sin^2\left(\frac{\pi}{2}-\hat{\Theta}\right) + n_{||}^2\cos^2\left(\frac{\pi}{2}-\hat{\Theta}\right)}$ where $\frac{\pi}{2} - \hat{\Theta}$ is the angle between the unit wave vector $\vec{\hat{\Omega}}$ and the optical axis $\vec{e_{||}}$ of the medium, so that it comes $\vec{e_{||}} = \vec{e_{\bar{y}}}$: hence, if the observer at rest located in vacuum perceives the <u>*same y and z co-ordinates*</u> as the observer at rest located in the moving medium, the observer located in vacuum will perceive the isotropic moving medium (in the point of view of the observer located in the medium) as an uniaxial medium whose optical axis $\vec{e_{||}}$ is orthogonal to the perceived direction $\vec{e_{\bar{x}}}$ of motion of the medium and in the plane $\left(\vec{e_{\bar{x}}}, \vec{\hat{\Omega}}\right)$ where $\vec{\hat{\Omega}}$ is the perceived direction of propagation of light in the moving medium; in the fundamental orthonormal basis $\left(\vec{e_{\bar{x}}}, \vec{e_{\bar{y}}}, \vec{e_{\bar{z}}}\right)$, the dielectric tensor of the medium will be represented as $\overline{\overline{\varepsilon}} = \varepsilon_0 \begin{pmatrix} n_{\perp}^2 & 0 & 0 \\ 0 & n_{||}^2 & 0 \\ 0 & 0 & n_{\perp}^2 \end{pmatrix}$, with $1 < n_{||} < n_{\perp}$, so that the observer in vacuum will perceive the moving medium as a negative uniaxial medium, with reference matter co-ordinates:

$$x^{\bar{0}} = \hat{x}^0 \quad \underline{x} = n_e\left(\hat{\Theta}\right)\hat{x}\left(\hat{\Theta}\right) \quad \underline{y} = n_e\left(\hat{\Theta}\right)\hat{y}\left(\hat{\Theta}\right) \quad \underline{z} = n_e\left(\hat{\Theta}\right)\hat{z}\left(\hat{\Theta}\right)$$

hence in the uniaxial medium, the only possible reference co-ordinates are vacuum-like co-ordinates; since the two observers perceive the same y and z co-ordinates, one writes $\underline{y} = n\,\hat{y}$ and $\underline{z} = n\,\hat{z}$ with $\hat{y} = \tilde{y}$ and $\hat{z} = \tilde{z}$.

Let us now examine an electromagnetic field associated to the photon propagating in the moving medium: the co-ordinates $(\bar{x}^0, \bar{x}, \bar{y}, \bar{z})$ and $\left(\bar{x}^{0'}, \bar{x}', \bar{y}', \bar{z}'\right)$ associated to the basis $\left(\vec{\bar{e}}_0, \vec{\bar{e}}_x, \vec{\bar{e}}_y, \vec{\bar{e}}_z\right)$ and $\left(\vec{\bar{e}}_{0'}, \vec{\bar{e}}_{x'}, \vec{\bar{e}}_{y'}, \vec{\bar{e}}_{z'}\right)$ [respectively $\left(x^{\bar{0}}, \underline{x}, y, \underline{z}\right)$ and $\left(\bar{x}^{0'}, \bar{x}', \bar{y}', \bar{z}'\right)$ associated to $\left(\vec{e}_{\bar{0}}, \vec{e}_{\bar{x}}, \vec{e}_{\bar{y}}, \vec{e}_{\bar{z}}\right)$ and $\left(\vec{\bar{e}}_{0'}, \vec{\bar{e}}_{x'}, \vec{\bar{e}}_{y'}, \vec{\bar{e}}_{z'}\right)$] being related thanks to a vacuum Lorentz transform, the vacuum-like electromagnetic tensors are such that:

$F^{\underline{\mu}\underline{\nu}} = \frac{\partial\,\underline{x}^{\mu}}{\partial\,\bar{x}^{\mu'}}\,\frac{\partial\,\underline{x}^{\nu}}{\partial\,\bar{x}^{\nu'}}\;F^{\mu'\nu'}$ and $F^{\mu'\nu'} = \frac{\partial\,\bar{x}^{\mu'}}{\partial\,\bar{x}^{\mu}}\,\frac{\partial\,\bar{x}^{\nu'}}{\partial\,\bar{x}^{\nu}}\;F^{\bar{\mu}\bar{\nu}}$

from which one obtains after calculation:

$$\begin{aligned}
&E^{\underline{x}} = E^{\bar{x}'} = E^{\bar{x}} \quad E^{\underline{y}} = \gamma\,\left(E^{\bar{y}'} + c\,\beta\,B^{\bar{z}'}\right) = \gamma\,\bar{\gamma}\,\left[\left(1-\beta\,\bar{\beta}\right)\,E^{\bar{y}} + c\,\left(\beta-\bar{\beta}\right)\,B^{\bar{z}}\right]\\
&E^{\underline{z}} = \gamma\,\left(E^{\bar{z}'} - c\,\beta\,B^{\bar{y}'}\right) = \gamma\,\bar{\gamma}\,\left[\left(1-\beta\,\bar{\beta}\right)\,E^{\bar{z}} - c\,\left(\beta-\bar{\beta}\right)\,B^{\bar{y}}\right]\\
&B^{\underline{x}} = B^{\bar{x}'} = B^{\bar{x}} \quad B^{\underline{y}} = \gamma\,\left(B^{\bar{y}'} - \tfrac{\beta}{c}\,E^{\bar{z}'}\right) = \gamma\,\bar{\gamma}\,\left[\left(1-\beta\,\bar{\beta}\right)\,B^{\bar{y}} - \tfrac{1}{c}\,\left(\beta-\bar{\beta}\right)\,E^{\bar{z}}\right]\\
&B^{\underline{z}} = \gamma\,\left(B^{\bar{z}'} + \tfrac{\beta}{c}\,E^{\bar{y}'}\right) = \gamma\,\bar{\gamma}\,\left[\left(1-\beta\,\bar{\beta}\right)\,B^{\bar{z}} + \tfrac{1}{c}\,\left(\beta-\bar{\beta}\right)\,E^{\bar{y}}\right]
\end{aligned} \tag{79}$$

choosing a monochromatic plane wave magnetic field perceived by the reference observer at rest located in the moving medium $\vec{B} = B_0\,\exp\left\{-\,\frac{2\,i\,\pi\,\nu}{c}\,\left[\bar{x}^0 - \left(\bar{x}\,\cos\Theta + \bar{y}\,\sin\Theta\right)\right]\right\}\,\vec{e}_z$, where $\vec{\Omega} = \cos\Theta\,\vec{e}_x + \sin\Theta\,\vec{e}_y$, leads to for the associated complex electric field (parallel polarisation):

$$\vec{E} = E_0\,\exp\left\{-\,\frac{2\,i\,\pi\,\nu}{c}\,\left[\bar{x}^0 - \left(\bar{x}\,\cos\Theta + \bar{y}\,\sin\Theta\right)\right]\right\}\,\left(-\sin\Theta\,\vec{e}_x + \cos\Theta\,\vec{e}_y\right)$$

here $\vec{E}$ and $\vec{B}$ are vacuum-like fields so that $B = \frac{E}{c}$, from which one deduces the components of the associated vacuum-like electromagnetic field relatively to the observer located in vacuum:

$$\begin{aligned}
&E^{\underline{x}} = -\,E\,\sin\Theta \quad E^{\underline{y}} = \gamma\,\bar{\gamma}\,\left[\left(1-\beta\,\bar{\beta}\right)\,\cos\Theta + \beta - \bar{\beta}\right]\,E \quad E^{\underline{z}} = 0\\
&B^{\underline{x}} = 0 \quad B^{\underline{y}} = 0 \quad B^{\underline{z}} = \gamma\,\bar{\gamma}\,\left[1-\beta\,\bar{\beta} + \left(\beta-\bar{\beta}\right)\,\cos\Theta\right]\,\frac{E}{c}
\end{aligned}$$

replacing Θ by its value in term of $\hat{\Theta}$ finally leads to:

$$\begin{aligned}
&E^{\underline{x}} = -\,E\,\frac{\sin\hat{\Theta}}{\gamma\,\bar{\gamma}\,\left[1-\beta\,\bar{\beta}-\left(\beta-\bar{\beta}\right)\,\cos\hat{\Theta}\right]} \quad E^{\underline{y}} = E\,\frac{\cos\hat{\Theta}}{\gamma\,\bar{\gamma}\,\left[1-\beta\,\bar{\beta}-\left(\beta-\bar{\beta}\right)\,\cos\hat{\Theta}\right]} \quad E^{\underline{z}} = 0\\
&B^{\underline{x}} = 0 \quad B^{\underline{y}} = 0 \quad B^{\underline{z}} = \frac{1}{\gamma\,\bar{\gamma}\,\left[1-\beta\,\bar{\beta}-\left(\beta-\bar{\beta}\right)\,\cos\hat{\Theta}\right]}\,\frac{E}{c}
\end{aligned}$$

from which one obtains: $\vec{D} = \varepsilon_0\,\underline{E}\,\left(-\sin\hat{\Theta}\,\vec{e}_{\bar{x}} + \cos\hat{\Theta}\,\vec{e}_{\bar{y}}\right)$ $\quad \vec{B} = \frac{\underline{E}}{c}\,\vec{e}_{\bar{z}}$

where $\underline{E} = \underline{E_0}\,e^{-i\,\bar{\Psi}}$, with $\underline{E_0} = \frac{E_0}{\gamma\,\bar{\gamma}\,\left[1-\beta\,\bar{\beta}-\left(\beta-\bar{\beta}\right)\,\cos\hat{\Theta}\right]}$ and $\bar{\Psi} = \frac{2\,\pi\,\nu}{c}\,\left[\bar{x}^0 - \left(\bar{x}\,\cos\Theta + \bar{y}\,\sin\Theta\right)\right]$

in the phase expression, the co-ordinates are the vacuum-like co-ordinates relatively to the reference observer inside the medium, $\overrightarrow{D}$ and $\overrightarrow{B}$ are the components of the vacuum-like electromagnetic field perceived by the observer at rest in vacuum, and $\overrightarrow{\hat{\Omega}} = \cos\hat{\Theta}\ \overrightarrow{e}_{\bar{x}} + \sin\hat{\Theta}\ \overrightarrow{e}_{\bar{y}}$ is the perceived unit wave vector associated to the electromagnetic field; hence, since the observer in vacuum perceives the moving medium as an anisotropic uniaxial medium, the apparent electromagnetic induction field inside the medium and perceived by this observer will obey to:

$\overrightarrow{k}\ \overrightarrow{D} = 0 \quad \overrightarrow{k}\ \overrightarrow{B} = 0$

$\overrightarrow{k} \wedge \overrightarrow{E} = \hat{\omega}\ \overrightarrow{B} \quad \overrightarrow{k} \wedge \frac{\overrightarrow{B}}{\mu_0} = -\hat{\omega}\ \overrightarrow{D}$ where $\overrightarrow{k}$ is the wave vector inside the medium:

$\overrightarrow{\hat{k}} = \frac{2\pi\hat{\nu}}{c}\ n_e\ \overrightarrow{\hat{\Omega}}$

hence $\overrightarrow{\hat{B}} = \hat{B}_0 \exp\left\{-\frac{2i\pi\hat{\nu}}{c}\left[\hat{x}^0 - n_e\left(\hat{x}\cos\hat{\Theta} + \hat{y}\sin\hat{\Theta}\right)\right]\right\}\overrightarrow{e_{\bar{z}}} = \hat{B}_0\ e^{-i\hat{\Psi}}\ \overrightarrow{e_{\bar{z}}}$ from which one deduces the electric induction:

$$\overrightarrow{\hat{D}} = \frac{n_e\ \hat{B}_0}{c\ \mu_0}\ e^{-i\hat{\Psi}}\left(-\sin\hat{\Theta}\ \overrightarrow{e_{\bar{x}}} + \cos\hat{\Theta}\ \overrightarrow{e_{\bar{y}}}\right) = \overline{\overline{\varepsilon}}\ \overrightarrow{\hat{E}}$$

then it easily comes for the electric field: $\overrightarrow{\hat{E}} = n_e\ c\ \hat{B}_0\ e^{-i\hat{\Psi}}\left(-\frac{\sin\hat{\Theta}}{n_\perp^2}\ \overrightarrow{e_{\bar{x}}} + \frac{\cos\hat{\Theta}}{n_{||}^2}\ \overrightarrow{e_{\bar{y}}}\right) = \hat{E}_0\ e^{-i\hat{\Psi}}\ \overrightarrow{e_E}$

one immediately verifies that $\overrightarrow{k} \wedge \overrightarrow{\hat{E}} = \hat{\omega}\ \overrightarrow{\hat{B}}$ and $\hat{B}_0^2 = \frac{n_\perp^2\ n_{||}^2\left(n_\perp^2\ \cos^2\hat{\Theta} + n_{||}^2\ \sin^2\hat{\Theta}\right)}{c^2\left(n_\perp^4\ \cos^2\hat{\Theta} + n_{||}^4\ \sin^2\hat{\Theta}\right)}\ \hat{E}_0^2 = \frac{N_e^2}{c^2}\ \hat{E}_0^2$, where N_e is the extraordinary ray refractive index for an uniaxial medium of optical axis $\overrightarrow{e_{||}} = \overrightarrow{e_{\bar{y}}}$ [6]. Then, the apparent electromagnetic field inside the medium is for the observer located in vacuum:

$\overrightarrow{\hat{E}} = n_e\ N_e\ \hat{E}_0\ e^{-i\hat{\Psi}}\left(-\frac{\sin\hat{\Theta}}{n_\perp^2}\ \overrightarrow{e_{\bar{x}}} + \frac{\cos\hat{\Theta}}{n_{||}^2}\ \overrightarrow{e_{\bar{y}}}\right)$

$\overrightarrow{\hat{D}} = \varepsilon_0\ n_e\ N_e\ \hat{E}_0\ e^{-i\hat{\Psi}}\left(-\sin\hat{\Theta}\ \overrightarrow{e_{\bar{x}}} + \cos\hat{\Theta}\ \overrightarrow{e_{\bar{y}}}\right)$ from which it comes $\hat{E}_0\ e^{-i\hat{\Psi}_V} =$

$\overrightarrow{\hat{B}} = \frac{N_e}{c}\ \hat{E}_0\ e^{-i\hat{\Psi}}\ \overrightarrow{e_{\bar{z}}}$

$\underline{E_0}\ e^{-i\bar{\Psi}_V}$,

where $\hat{\Psi}_V$ and $\bar{\Psi}_V$ are the vacuum-like phases, that is:

$$\hat{E}_0 \exp\left\{-\frac{2i\pi\hat{\nu}}{c}\left[x^{\bar{0}} - \left(\underline{x}\cos\hat{\Theta} + \underline{y}\sin\hat{\Theta}\right)\right]\right\} = \underline{E_0}\exp\left\{-\frac{2i\pi\nu}{c}\left[\bar{x}^0 - (\bar{x}\cos\Theta + \bar{y}\sin\Theta)\right]\right\}$$

the invariance for all $\bar{y} = \underline{y}$ implies $\hat{\nu}\sin\hat{\Theta} = \nu\sin\Theta$, and from the transformation formulas for the co-ordinates one has:

$$\nu\left[\bar{x}^0 - (\bar{x}\cos\Theta + \bar{y}\sin\Theta)\right] = \frac{\nu}{\gamma\bar{\gamma}\left[1-\beta\bar{\beta}+(\beta-\bar{\beta})\cos\hat{\Theta}\right]}\left\{x^{\bar{0}} - \left[(\underline{x}-d_V)\cos\hat{\Theta} + \underline{y}\sin\hat{\Theta}\right]\right\} + \nu\overline{d_M}\cos\Theta$$

hence $\hat{\nu} = \frac{\nu}{\gamma\bar{\gamma}\left[1-\beta\bar{\beta}-\left(\beta-\bar{\beta}\right)\cos\hat{\Theta}\right]}$ is the frequency perceived by the reference observer at rest in vacuum, depending on the propagation direction, ν being the frequency of the radiation for the reference observer located in the medium, and:

$$\nu\,\left(\bar{x}^0 - \bar{x}\,\cos\Theta\right) = \hat{\nu}\,\left(x^{\bar{0}} - \underline{x}\,\cos\hat{\Theta}\right) + \hat{\nu}\,d_V\,\cos\hat{\Theta} + n\,\nu\,d_M\,\cos\Theta$$

on the moving interface inside the medium, $\bar{x}' = 0$, from which $\underline{x} = \beta\,x^{\bar{0}} + d_V$ and $\bar{x} = \bar{\beta}\,\bar{x}^0 - n\,d_M$, so that:

$\nu\,\bar{x}^0\,\left(1 - \bar{\beta}\,\cos\Theta\right) = \hat{\nu}\,x^{\bar{0}}\,\left(1 - \beta\,\cos\hat{\Theta}\right)$ and since on the interface $\gamma\,\bar{x}^0 = \bar{\gamma}\,x^{\bar{0}}$, one has the fundamental relations:

$$\nu\,\bar{\gamma}\,\left(1 - \bar{\beta}\,\cos\Theta\right) = \hat{\nu}\,\gamma\,\left(1 - \beta\,\cos\hat{\Theta}\right)$$

$$\nu\,\sin\Theta = \hat{\nu}\,\sin\hat{\Theta}$$

with $\hat{E}_0\,\exp\left(\frac{2\,i\,\pi\,\hat{\nu}}{c}\,d_V\,\cos\hat{\Theta}\right) = \underline{E_0}\,\exp\left(-\frac{2\,i\,\pi\,\nu}{c}\,n\,d_M\,\cos\Theta\right)$

hence the internal apparent fields perceived by the observer at rest in vacuum are:

$$\vec{\hat{E}} = N_e\,n_e\,\underline{E_0}\,e^{-i\,\Gamma}\,\exp\left\{-\frac{2\,i\,\pi\,\hat{\nu}}{c}\,\left[\hat{x}^0 - n_e\,\left(\hat{x}\,\cos\hat{\Theta} + \hat{y}\,\sin\hat{\Theta}\right)\right]\right\}\,\left(-\frac{\sin\hat{\Theta}}{n^2}\,\vec{e}_{\bar{x}} + \frac{\cos\hat{\Theta}}{n_{||}^2}\,\vec{e}_{\bar{y}}\right)$$

$$\vec{\hat{B}} = \frac{N_e\,\underline{E_0}\,e^{-i\,\Gamma}}{c}\,\exp\left\{-\frac{2\,i\,\pi\,\hat{\nu}}{c}\,\left[\hat{x}^0 - n_e\,\left(\hat{x}\,\cos\hat{\Theta} + \hat{y}\,\sin\hat{\Theta}\right)\right]\right\}\,\vec{e}_{\bar{z}}$$

with $e^{-i\,\Gamma} = \exp\left[-\frac{2\,i\,\pi}{c}\,\left(\hat{\nu}\,d_V\,\cos\hat{\Theta} + n\,\nu\,d_M\,\cos\Theta\right)\right]$, while for the observer located in the medium, the true electromagnetic field is:

$$\vec{E} = E_0\,\exp\left\{-\frac{2\,i\,\pi\,\nu}{c}\,\left[\tilde{x}^0 - n\,\left(\tilde{x}\,\cos\Theta + \tilde{y}\,\sin\Theta\right)\right]\right\}\,\left(-\sin\Theta\,\vec{e}_x + \cos\Theta\,\vec{e}_y\right)$$

$$\vec{B} = \frac{n\,E_0}{c}\,\exp\left\{-\frac{2\,i\,\pi\,\nu}{c}\,\left[\tilde{x}^0 - n\,\left(\tilde{x}\,\cos\Theta + \tilde{y}\,\sin\Theta\right)\right]\right\}\,\vec{e}_z$$

The observer at rest in vacuum perceives emerging electromagnetic fields (parallel polarisation studied here) such that:

$$\vec{\hat{E}}_t = \hat{E}_{0t}\,e^{-i\,\hat{\Psi}_t}\,\left(-\sin\hat{\Theta}_t\,\vec{e}_{\bar{x}} + \cos\hat{\Theta}_t\,\vec{e}_{\bar{y}}\right) \qquad \vec{\hat{B}}_t = \frac{\hat{E}_{0t}}{c}\,e^{-i\,\hat{\Psi}_t}\,\vec{e}_{\bar{z}} \qquad \text{with } \hat{\Psi}_t = \frac{2\,\pi\,\hat{\nu}_t}{c}\,\left[\hat{x}^0 - \left(\hat{x}\,\cos\hat{\Theta}_t + \hat{y}\,\sin\hat{\Theta}_t\right)\right]$$

where $\hat{\Theta}_t$ and $\hat{\nu}_t$ are the transmitted angle and frequency perceived by the observer at rest in vacuum, related to the transmitted angle ${\Theta'}_t$ and frequency ν' in the co-moving LRS by $\nu'\,\sin{\Theta'}_t = \hat{\nu}_t\,\sin\hat{\Theta}_t$ and $\cos{\Theta'}_t = \frac{\cos\hat{\Theta}_t - \beta}{1 - \beta\,\cos\hat{\Theta}_t} = \gamma\,\frac{\hat{\nu}_t}{\nu'}\,\left(\cos\hat{\Theta}_t - \beta\right)$; on the moving interface, since $\underline{x} = \hat{x} = \beta\,x^{\bar{0}} + d_V = \beta\,\hat{x}^0 + d_V$, one has for the transmitted field on the interface perceived by the observer at rest in vacuum:

$$\vec{\hat{E}}_t = \underline{\hat{E}_{0t}}\,\exp\left\{-\frac{2\,i\,\pi\,\hat{\nu}_t}{c}\,\left[\hat{x}^0\,\left(1 - \beta\,\cos\hat{\Theta}_t\right) - \hat{y}\,\sin\hat{\Theta}_t\right]\right\}\,\left(-\sin\hat{\Theta}_t\,\vec{e}_{\bar{x}} + \cos\hat{\Theta}_t\,\vec{e}_{\bar{y}}\right)$$

$$\vec{\hat{B}}_t = \frac{\underline{\hat{E}_{0t}}}{c}\,\exp\left\{-\frac{2\,i\,\pi\,\hat{\nu}_t}{c}\,\left[\hat{x}^0\,\left(1 - \beta\,\cos\hat{\Theta}_t\right) - \hat{y}\,\sin\hat{\Theta}_t\right]\right\}\,\vec{e}_{\bar{z}}$$

with $\underline{\hat{E}_{0t}} = \hat{E}_{0t}\,\exp\left(\frac{2\,i\,\pi\,\hat{\nu}_t}{c}\,d_V\,\cos\hat{\Theta}_t\right)$

since Eq. (79) must be verified, one obtains the components of the transmitted field in the co-moving LRS of the particle bound to the interface in vacuum:

$$\hat{B}_t^{x'} = \hat{B}_t^{\underline{x}} = \hat{B}_t^{\underline{y}} = \hat{B}_t^{y'} = 0 \quad \hat{E}_t^{z'} = \hat{E}_t^{\underline{z}} = 0$$

$$\hat{E}_t^{x'} = \hat{E}_t^{x} = -\ \hat{E}_{0t} \sin \hat{\Theta}_t \ e^{-i\,\hat{\Psi}_t} = -\ \hat{E}_{0t} \frac{\nu'}{\hat{\nu}_t} \sin \Theta'_t \ e^{-i\,\hat{\Psi}_t}$$

for the phase, it comes easily after calculation:

$$\hat{\Psi}_t = \tfrac{2\pi\,\hat{\nu}_t}{c}\left[x^{\bar{0}} - \left(\underline{x}\cos\hat{\Theta}_t + \underline{y}\sin\hat{\Theta}_t\right)\right] = \tfrac{2\pi\nu'}{c}\left[\bar{x}^{0'} - (\bar{x}'\cos\Theta'_t + \bar{y}'\sin\Theta'_t)\right] - \tfrac{2\pi\,\hat{\nu}_t}{c}\, d_V \cos\hat{\Theta}_t$$
$$= \Psi'_t - \tfrac{2\pi\,\hat{\nu}_t}{c}\, d_V \cos\hat{\Theta}_t$$

and: $\hat{E}_t^{x'} = -\underline{\hat{E}_{0t}}\, \frac{\nu'}{\hat{\nu}_t} \sin\Theta'_t\ e^{-i\,\Psi'_t}$

doing so for the two other non zero components leads to:

$$\hat{E}_t^{y'} = \gamma\left(\hat{E}_t^{y} - c\,\beta\,\hat{B}_t^{z}\right) = \gamma\,\hat{E}_{0t}\left(\cos\hat{\Theta}_t - \beta\right) e^{-i\,\hat{\Psi}_t} = \underline{\hat{E}_{0t}}\,\tfrac{\nu'}{\hat{\nu}_t}\cos\Theta'_t\ e^{-i\,\Psi'_t}$$
$$\hat{B}_t^{z'} = \gamma\left(\hat{B}_t^{z} - \tfrac{\beta}{c}\,\hat{E}_t^{y}\right) = \gamma\,\tfrac{\hat{E}_{0t}}{c}\left(1 - \beta\cos\hat{\Theta}_t\right) e^{-i\,\hat{\Psi}_t} = \underline{\underline{\tfrac{\hat{E}_{0t}}{c}}}\,\tfrac{\nu'}{\hat{\nu}_t}\ e^{-i\,\Psi'_t}$$

The electromagnetic field expressed in the co-moving LRS of a particle inside the moving medium *and* on the moving plane interface has the following form:

$$\vec{E'}_i = E'_0 \exp\left[-\tfrac{2\,i\,\pi\,\nu'}{c}\left(x^{0'} - n\,y'\sin\Theta'_i\right)\right]\begin{pmatrix} -\sin\Theta'_i \\ \cos\Theta'_i \\ 0 \end{pmatrix}$$
$$\vec{B'}_i = \tfrac{n\,E'_0}{c}\exp\left[-\tfrac{2\,i\,\pi\,\nu'}{c}\left(x^{0'} - n\,y'\sin\Theta'_i\right)\right]\vec{e}_{z'}$$

since $x' = 0$ on the interface, where $\vec{E'}_i$ and $\vec{B'}_i$ are the incident fields on the interface; when the incident wave impinges the interface with incident angle Θ'_i, a reflected wave appears in the medium and a transmitted one appears in the vacuum, such that the total fields are:

* in the medium:

$$\vec{E'}_T = \vec{E'}_i + \vec{E'}_r = E'_0\exp\left[-\tfrac{2\,i\,\pi\,\nu'}{c}\left(x^{0'} - n\,y'\sin\Theta'_i\right)\right]\begin{bmatrix} -\left(1 + r_{||}\right)\sin\Theta'_i \\ \left(1 - r_{||}\right)\cos\Theta'_i \\ 0 \end{bmatrix}$$
$$\vec{B'}_T = \vec{B'}_i + \vec{B'}_r = \tfrac{n\,E'_0}{c}\left(1 + r_{||}\right)\exp\left[-\tfrac{2\,i\,\pi\,\nu'}{c}\left(x^{0'} - n\,y'\sin\Theta'_i\right)\right]\vec{e}_{z'}$$

* in vacuum:

$$\vec{E'}_t = E'_0\,t_{||}\exp\left[-\tfrac{2\,i\,\pi\,\nu'}{c}\left(x^{0'} - y'\sin\Theta'_t\right)\right]\left(-\sin\Theta'_t\,\vec{e}_{x'} + \cos\Theta'_t\,\vec{e}_{y'}\right)$$
$$\vec{B'}_t = \tfrac{E'_0}{c}\,t_{||}\exp\left[-\tfrac{2\,i\,\pi\,\nu'}{c}\left(x^{0'} - y'\sin\Theta'_t\right)\right]\vec{e}_{z'}$$

since the reflected angle equals the incident one and where the frequency remains unchanged threw the interface in the co-moving LRS, Θ'_t is the transmitted angle, $r_{||}$ and $t_{||}$ the parallel amplitude reflection and transmission factors, since by construction the matter co-ordinates $x^{0'}$, y' and associated vectors of the co-moving frame remain unchanged from the medium to vacuum; hence the continuity relations for the fields through an interface lead to $n \sin \Theta'_i = \sin \Theta'_t$ which is the classical Descartes' law, $r_{||} = \frac{\cos \Theta'_i - n \cos \Theta'_t}{\cos \Theta'_i + n \cos \Theta'_t}$ and $t_{||} = \frac{2\, n \cos \Theta'_i}{\cos \Theta'_i + n \cos \Theta'_t}$; then one deduces from what precedes that $E'_0\, t_{||} = \frac{\nu'}{\hat{\nu}_t} \underline{\hat{E}_{0t}}$ and the incident electromagnetic field inside the medium relatively to the reference observer at rest in the medium is such that Eq. (79) is verified, that is after a simple calculation:

$$\vec{E}_i = E_0\, e^{-i\,\Psi_i} \left(-\sin \Theta_i\, \vec{e}_x + \cos \Theta_i\, \vec{e}_y\right) \qquad \vec{B}_i = \frac{n\,E_0}{c}\, e^{-i\,\Psi_i}\, \vec{e}_z \qquad \text{where } E_0 = \frac{\nu_i}{\nu'}\, E'_0 \exp\left(\frac{2\,i\,\pi\,\nu_i}{c}\, n\, d_M \cos \Theta_i\right)$$

and $\Psi_i = \frac{2\,\pi\,\nu_i}{c}\left[\tilde{x}^0 - n\,(\tilde{x} \cos \Theta_i + \tilde{y} \sin \Theta_i)\right]$,

hence: $E_0 = \frac{\nu_i}{\hat{\nu}_t}\frac{\hat{E}_{0t}}{t_{||}} \exp\left[\frac{2\,i\,\pi}{c}\left(\hat{\nu}_t\, d_V \cos \hat{\Theta}_t + n\, \nu_i\, d_M \cos \Theta_i\right)\right]$

while the phase continuity implies for all y that: $\hat{\nu}_t \sin \hat{\Theta}_t = n\, \nu_i \sin \Theta_i$; then from what precedes, one has:

$$\hat{E}_{0i} = \frac{\hat{\nu}_i}{\hat{\nu}_t}\frac{\hat{E}_{0t}}{t_{||}} \exp\left[-\frac{2\,i\,\pi}{c}\, d_V \left(\hat{\nu}_i \cos \hat{\Theta}_i - \hat{\nu}_t \cos \hat{\Theta}_t\right)\right]$$

and the apparent incident field perceived by the reference observer at rest in vacuum is:

$$\vec{\hat{E}}_i = n_e\left(\hat{\Theta}_i\right) N_e\left(\hat{\Theta}_i\right) \hat{E}_{0i}\, e^{-i\,\hat{\Psi}_i} \left(-\frac{\sin \hat{\Theta}_i}{n_\perp^2}\, \vec{e_{\bar{x}}} + \frac{\cos \hat{\Theta}_i}{n_{||}^2}\, \vec{e_{\bar{y}}}\right)$$
$$\vec{\hat{B}}_i = \frac{N_e\left(\hat{\Theta}_i\right)}{c}\, \hat{E}_{0i}\, e^{-i\,\hat{\Psi}_i}\, \vec{e_{\bar{z}}}$$

with $\hat{\Psi}_i = \frac{2\,\pi\,\hat{\nu}_i}{c}\left\{\hat{x}^0 - n_e\left(\hat{\Theta}_i\right)\left[\hat{x}\left(\hat{\Theta}_i\right) \cos \hat{\Theta}_i + \hat{y}\left(\hat{\Theta}_i\right) \sin \hat{\Theta}_i\right]\right\}$

since on the moving interface $\underline{x} = n_e\left(\hat{\Theta}_i\right) \hat{x}\left(\hat{\Theta}_i\right) = \beta\, x^{\bar{0}} + d_V = \beta\, \hat{x}^0 + d_V$ and $n_e\left(\hat{\Theta}_i\right) \hat{y}\left(\hat{\Theta}_i\right) = n\,\hat{y}$, one has for the apparent incident field on the interface perceived by the observer at rest in vacuum:

$$\vec{\hat{E}}_i = n_e\left(\hat{\Theta}_i\right) N_e\left(\hat{\Theta}_i\right) \hat{E}_{0i} \exp\left(\frac{2\,i\,\pi\,\hat{\nu}_i}{c}\, d_V \cos \hat{\Theta}_i\right)$$
$$\times \exp\left\{-\frac{2\,i\,\pi\,\hat{\nu}_i}{c}\left[\hat{x}^0\left(1 - \beta \cos \hat{\Theta}_i\right) - n\,\hat{y} \sin \hat{\Theta}_i\right]\right\}\left(-\frac{\sin \hat{\Theta}_i}{n_\perp^2}\, \vec{e_{\bar{x}}} + \frac{\cos \hat{\Theta}_i}{n_{||}^2}\, \vec{e_{\bar{y}}}\right)$$
$$\vec{\hat{B}}_i = \frac{N_e\left(\hat{\Theta}_i\right)}{c}\, \hat{E}_{0i} \exp\left(\frac{2\,i\,\pi\,\hat{\nu}_i}{c}\, d_V \cos \hat{\Theta}_i\right) \exp\left\{-\frac{2\,i\,\pi\,\hat{\nu}_i}{c}\left[\hat{x}^0\left(1 - \beta \cos \hat{\Theta}_i\right) - n\,\hat{y} \sin \hat{\Theta}_i\right]\right\} \vec{e_{\bar{z}}}$$

the phase continuity for the fields implies $\hat{\nu}_t\left(1 - \beta \cos \hat{\Theta}_t\right) = \hat{\nu}_i\left(1 - \beta \cos \hat{\Theta}_i\right)$ and $\hat{\nu}_t \sin \hat{\Theta}_t = n\,\hat{\nu}_i \sin \hat{\Theta}_i$ which is obviously verified, and the apparent transmission factors are defined such that:

$$\vec{\hat{E}}_t = \overline{\overline{T}}_{||,E}\, \vec{\hat{E}}_i \quad \vec{\hat{B}}_t = T_{||,B}\, \vec{\hat{B}}_i$$

with:

$$T_{||,B} = \frac{\left\|\vec{\hat{B}}_t\right\|}{\left\|\vec{\hat{B}}_i\right\|} = \frac{\hat{E}_{0t}}{N_e\left(\hat{\Theta}_i\right) \hat{E}_{0i}} \exp\left[-\frac{2\,i\,\pi}{c}\, d_V \left(\hat{\nu}_i \cos \hat{\Theta}_i - \hat{\nu}_t \cos \hat{\Theta}_t\right)\right]$$
$$= \frac{\hat{E}_{0t}}{N_e\left(\hat{\Theta}_i\right) \hat{E}_{0i}} \exp\left[-\frac{2\,i\,\pi\, d_V}{c\,\beta}\left(\hat{\nu}_i - \hat{\nu}_t\right)\right] = \frac{\hat{\nu}_t}{\hat{\nu}_i}\frac{t_{||}}{N_e\left(\hat{\Theta}_i\right)}$$

and:

$$\overline{\overline{T}}_{||,E} = \begin{pmatrix} T^{xx}_{||,E} \; T^{xy}_{||,E} \; 0 \\ T^{yx}_{||,E} \; T^{yy}_{||,E} \; 0 \\ T^{zx}_{||,E} \; T^{zy}_{||,E} \; T^{zz}_{||,E} \end{pmatrix} \Rightarrow \begin{array}{l} -\, n_{||}^2 \; T^{xx}_{||,E} \sin \hat{\Theta}_i + n^2 \; T^{xy}_{||,E} \cos \hat{\Theta}_i = -\frac{n^2 \; n_{||}^2}{n_e\left(\hat{\Theta}_i\right)} \; T_{||,B} \sin \hat{\Theta}_t \\ -\, n_{||}^2 \; T^{yx}_{||,E} \sin \hat{\Theta}_i + n^2 \; T^{yy}_{||,E} \cos \hat{\Theta}_i = \frac{n^2 \; n_{||}^2}{n_e\left(\hat{\Theta}_i\right)} \; T_{||,B} \cos \hat{\Theta}_t \\ T^{zx}_{||,E} \sin \hat{\Theta}_i = \frac{n^2}{n_{||}^2} \; T^{zy}_{||,E} \cos \hat{\Theta}_i \end{array}$$

for the amplitudes, $T_{||,E} = \frac{\left\| \vec{\hat{E}}_t \right\|}{\left\| \vec{\hat{E}}_i \right\|} = N_e\left(\hat{\Theta}_i\right) \; T_{||,B}$ and:

$$n_e^2\left(\hat{\Theta}_i\right) \; N_e^2\left(\hat{\Theta}_i\right) \left[\left(-\frac{T^{xx}_{||,E} \sin \hat{\Theta}_i}{n^2} + \frac{T^{xy}_{||,E} \cos \hat{\Theta}_i}{n_{||}^2} \right)^2 + \left(-\frac{T^{yx}_{||,E} \sin \hat{\Theta}_i}{n^2} + \frac{T^{yy}_{||,E} \cos \hat{\Theta}_i}{n_{||}^2} \right)^2 \right] = T^2_{||,E}$$

$$\Rightarrow \frac{n_e^2\left(\hat{\Theta}_i\right)}{n^4 \; n_{||}^4} \begin{bmatrix} n^4 \left(T^{xy\;2}_{||,E} + T^{yy\;2}_{||,E} \right) \cos^2 \hat{\Theta}_i + n_{||}^4 \left(T^{xx\;2}_{||,E} + T^{yx\;2}_{||,E} \right) \sin^2 \hat{\Theta}_i \\ -\, 2 \; n^2 \; n_{||}^2 \left(T^{xx}_{||,E} \; T^{xy}_{||,E} + T^{yy}_{||,E} \; T^{yx}_{||,E} \right) \sin \hat{\Theta}_i \cos \hat{\Theta}_i \end{bmatrix} = T^2_{||,B}$$

then it is efficient to choose $T^{xx}_{||,E} \; T^{xy}_{||,E} + T^{yy}_{||,E} \; T^{yx}_{||,E} = 0$ and the transmission matrix can be diagonal, with:

$$T^{xx}_{||,E} \sin \hat{\Theta}_i = \frac{n^2}{n_e\left(\hat{\Theta}_i\right)} \; T_{||,B} \sin \hat{\Theta}_t \quad T^{yy}_{||,E} \cos \hat{\Theta}_i = \frac{n_{||}^2}{n_e\left(\hat{\Theta}_i\right)} \; T_{||,B} \cos \hat{\Theta}_t$$

For the reflected fields, relatively to the two reference observers, the situation is slightly different, since if in the co-moving LRS the reflected angle and frequency equal the incident ones, it is not the case for the reference observer located inside the medium and for the one located in vacuum; in the co-moving LRS, the reflected unit wave vector is

$$\vec{\Omega'}_r = \begin{pmatrix} -\cos \Theta'_i \\ \sin \Theta'_i \\ 0 \end{pmatrix} = \begin{bmatrix} \cos\left(\pi - \Theta'_i\right) \\ \sin\left(\pi - \Theta'_i\right) \\ 0 \end{bmatrix} = \begin{pmatrix} \cos \Theta'_r \\ \sin \Theta'_r \\ 0 \end{pmatrix}$$

applying the angle and frequency transformation relatively to the observer located in the medium leads to:

$$\nu_r \sin \Theta_r = \nu' \sin \Theta'_r = \nu' \sin \Theta'_i = \nu_i \sin \Theta_i$$
$$\cos \Theta_r = \frac{\cos \Theta'_r + \bar{\beta}}{1 + \bar{\beta} \cos \Theta'_r} = \frac{\cos\left(\pi - \Theta'_i\right) + \bar{\beta}}{1 + \bar{\beta} \cos\left(\pi - \Theta'_i\right)} = \frac{\bar{\beta} - \cos \Theta'_i}{1 - \bar{\beta} \cos \Theta'_i}$$

but $\cos \Theta'_i = \frac{\cos \Theta_i - \bar{\beta}}{1 - \bar{\beta} \cos \Theta_i}$, from which one obtains:

$$\cos \Theta_r = \frac{2\,\bar{\beta} - \left(1 + \bar{\beta}^2\right) \cos \Theta_i}{1 + \bar{\beta}^2 - 2\,\bar{\beta} \cos \Theta_i} \neq -\cos \Theta_i \Rightarrow \Theta_r \neq \pi - \Theta_i$$

hence one has the fundamental result: relatively to the observer located in the moving medium, and similarly relatively to the one located in vacuum, the reflected angle is not the incident angle, the latter result being true if and only if $\Theta_i = 0$ or $\bar{\beta} = 0$; then the relations between the reflected and incident angles and frequencies are:

$$\cos\,\Theta_r = \frac{2\,\bar{\beta} - \left(1+\bar{\beta}^2\right)\,\cos\,\Theta_i}{1+\bar{\beta}^2 - 2\,\bar{\beta}\,\cos\,\Theta_i} \quad \sin\,\Theta_r = \frac{\sin\,\Theta_i}{\bar{\gamma}^2\left(1+\bar{\beta}^2 - 2\,\bar{\beta}\,\cos\,\Theta_i\right)}$$
$$\nu_r = \bar{\gamma}^2\,\nu_i\,\left(1 + \bar{\beta}^2 - 2\,\bar{\beta}\,\cos\,\Theta_i\right)$$

one easily verifies that $\cos^2\,\Theta_r + \sin^2\,\Theta_r = 1$; relatively to the reference observer located in vacuum, it comes:

$$\cos\,\hat{\Theta}_r = \frac{\beta-\bar{\beta}+\left(1-\beta\,\bar{\beta}\right)\,\cos\,\Theta_r}{1-\beta\,\bar{\beta}+\left(\beta-\bar{\beta}\right)\,\cos\,\Theta_r} = \frac{\beta+\bar{\beta}-\left(1+\beta\,\bar{\beta}\right)\,\cos\,\Theta_i}{1+\beta\,\bar{\beta}-\left(\beta+\bar{\beta}\right)\,\cos\,\Theta_i} = \frac{2\,\beta-\left(1+\beta^2\right)\,\cos\,\hat{\Theta}_i}{1+\beta^2-2\,\beta\,\cos\,\hat{\Theta}_i}$$
$$\sin\,\hat{\Theta}_r = \frac{\sin\,\Theta_r}{\gamma\,\bar{\gamma}\left[1-\beta\,\bar{\beta}+\left(\beta-\bar{\beta}\right)\,\cos\,\Theta_r\right]} = \frac{\sin\,\Theta_i}{\gamma\,\bar{\gamma}\left[1+\beta\,\bar{\beta}-\left(\beta+\bar{\beta}\right)\,\cos\,\Theta_i\right]} = \frac{\sin\,\hat{\Theta}_i}{\gamma^2\left(1+\beta^2-2\,\beta\,\cos\,\hat{\Theta}_i\right)}$$

obviously the relations $\nu_i\,\sin\,\Theta_i = \nu_r\,\sin\,\Theta_r = \hat{\nu}_i\,\sin\,\hat{\Theta}_i$ are verified, and one obtains after calculation the expected result $\hat{\nu}_r = \gamma^2\,\hat{\nu}_i\,\left(1 + \beta^2 - 2\,\beta\,\cos\,\hat{\Theta}_i\right)$ from which $\hat{\nu}_r\,\sin\,\hat{\Theta}_r = \hat{\nu}_i\,\sin\,\hat{\Theta}_i$; furthermore, from the definition of the reflected angle, it is easy to obtain :

$$\cos\,\Theta'_r = \frac{\cos\,\hat{\Theta}_r - \beta}{1 - \beta\,\cos\,\hat{\Theta}_r} = \gamma\,\frac{\hat{\nu}_r}{\nu'}\,\left(\cos\,\hat{\Theta}_r - \beta\right) = \frac{\cos\,\Theta_r - \bar{\beta}}{1 - \bar{\beta}\,\cos\,\Theta_r} = \bar{\gamma}\,\frac{\nu_r}{\nu'}\,\left(\cos\,\Theta_r - \bar{\beta}\right)$$

then the apparent reflection factors on the interface are defined such that:

$$\vec{\hat{E}}_r = \overline{\overline{R}}_{||,E}\,\vec{\hat{E}}_i \quad \vec{\hat{B}}_r = R_{||,B}\,\vec{\hat{B}}_i,\ \text{with}\ R_{||,B} = \frac{\left\|\vec{\hat{B}}_r\right\|}{\left\|\vec{\hat{B}}_i\right\|}\ \text{and}\ \overline{\overline{R}}_{||,E} = \begin{pmatrix} R^{xx}_{||,E} & 0 & 0 \\ 0 & R^{yy}_{||,E} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the total apparent electromagnetic field on the interface is given by:

$$\vec{\hat{E}}_T = \hat{E}_{0i}\,\exp\left(\frac{2\,i\,\pi\,\hat{\nu}_i}{c}\,d_V\,\cos\,\hat{\Theta}_i\right)\,n_e\left(\hat{\Theta}_i\right)\,N_e\left(\hat{\Theta}_i\right)\,e^{-i\,\underline{\hat{\Psi}_i}}\,\left(\overline{\overline{I}} + \overline{\overline{R}}_{||,E}\right)\,\left(-\,\frac{\sin\hat{\Theta}_i}{n_\perp^2}\,\vec{e_{\bar{x}}} + \frac{\cos\hat{\Theta}_i}{n_{||}^2}\,\vec{e_{\bar{y}}}\right)$$
$$\vec{\hat{B}}_T = \frac{\hat{E}_{0i}}{c}\,N_e\left(\hat{\Theta}_i\right)\,\exp\left(\frac{2\,i\,\pi\,\hat{\nu}_i}{c}\,d_V\,\cos\,\hat{\Theta}_i\right)\,\left(1 + R^B_{||}\right)\,e^{-i\,\underline{\hat{\Psi}_i}}\,\vec{e_{\bar{z}}}$$

where $\underline{\hat{\Psi}_i} = \frac{2\,\pi\,\hat{\nu}_i}{c}\,\left[\hat{x}^0\,\left(1 - \beta\,\cos\,\hat{\Theta}_i\right) - n\,\hat{y}\,\sin\,\hat{\Theta}_i\right]$; then the continuity of the fields implies:

* for the magnetic field: $1 + R_{||,B} = T_{||,B} \Rightarrow 1 + R_{||,B} = \frac{\hat{\nu}_t\,n\left(1+r_{||}\right)}{\hat{\nu}_i\,N_e\left(\hat{\Theta}_i\right)}$

* for the tangential component of the electric field:

$$T^{yy}_{||,E} = 1 + R^{yy}_{||,E} \quad \left(1 + R^{yy}_{||,E}\right)\,\frac{\cos\,\hat{\Theta}_i}{n_{||}^2} = \frac{T_{||,B}}{n_e\left(\hat{\Theta}_i\right)}\,\cos\,\hat{\Theta}_t$$

* for the normal component of the electric induction (when the interface is assumed free of charges and currents):

$$1 + R^{xx}_{||,E} = \frac{T^{xx}_{||,E}}{n^2} \quad \left(1 + R^{xx}_{||,E}\right) \sin\hat{\Theta}_i = \frac{T_{||,B}}{n_e\left(\hat{\Theta}_i\right)} \sin\hat{\Theta}_t$$

One may notice that all these expressions (for parallel polarisation) are valid for an uniaxial negative crystal, with $\beta = \frac{n_\perp - n_{||}}{n_\perp \, n_{||} - 1}$ where $n_\perp$ and $n_{||}$ are the principal refractive indices of the crystal.

It is important to note here that the refractive index such that $d\hat{s} = \frac{\sqrt{n^2\cos^2\hat{\Theta} + n_{||}^2 \sin^2\hat{\Theta}}}{n\, n_{||}} d\,\hat{x}^0$ is not an effective index but only an apparent index, and the refractive phenomena which occur inside the moving medium have to examined from the reference observer located inside the medium point of view; hence, when the reference observer located in vacuum perceives an incoming intensity emerging from the moving medium, he knows its transmitted direction and frequency, from which he can deduce from what precedes the apparent incident direction and frequency, directly related to the real incident direction and frequency perceived by the reference observer at rest in the medium: then the initial problem, that is to find an invariant form of the radiative transfer equation has to be examined from the internal observer's point of view.

4. Derivation of the Radiative Transfer Equation

Let us now pay attention to the derivative operator $P^\alpha\,\partial_\alpha$ along a photon path inside the medium, using the two sets of mater co-ordinates relatively to the observer located inside the medium; from (72), one easily obtains that

$$\begin{array}{l} \frac{\partial}{\partial x^{0'}} = \bar{\gamma}\left(\frac{\partial}{\partial\tilde{x}^0} + \frac{\bar{\beta}}{n}\frac{\partial}{\partial\tilde{x}}\right) \quad \frac{\partial}{\partial x'} = \bar{\gamma}\left(\bar{\beta}\, n\frac{\partial}{\partial\tilde{x}^0} + \frac{\partial}{\partial\tilde{x}}\right) \\ \frac{\partial}{\partial y'} = \frac{\partial}{\partial\tilde{y}} \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial\tilde{z}} \end{array}, \qquad (80)$$

Performing the calculation of the different contravariante components of the 4-impulsion vector leads to

$$\begin{array}{l} P^{0'} = n^2\,\frac{h\,\nu'}{c} = n^2\,\frac{h\,\bar{\gamma}\,\nu}{c}\left(1 - \bar{\beta}\,\mu\right) \\ P^{x'} = \frac{h\,\nu'\,n\,\mu'}{c} = \frac{h\,\nu\,n}{c}\,\bar{\gamma}\left(\mu - \bar{\beta}\right) \\ P^{y'} = \frac{h\,\nu'\,n\,\sin\Theta'\,\cos\Phi'}{c} = \frac{h\,\nu\,n\,\sin\Theta\,\cos\Phi}{c} \quad P^{z'} = \frac{h\,\nu'\,n\,\sin\Theta'\,\sin\Phi'}{c} = \frac{h\,\nu\,n\,\sin\Theta\,\sin\Phi}{c} \end{array}, \qquad (81)$$

from which one obtains

$$\begin{array}{l} P^{0'}\,\frac{\partial}{\partial x^{0'}} + P^{x'}\,\frac{\partial}{\partial x'} = \frac{h\nu n^2}{c}\,\bar{\gamma}^2\left\{\left[1 - \bar{\beta}\mu + \bar{\beta}\left(\mu - \bar{\beta}\right)\right]\frac{\partial}{\partial\tilde{x}^0} + \frac{1}{n}\left[\bar{\beta}\left(1 - \bar{\beta}\mu\right) + \mu - \bar{\beta}\right]\frac{\partial}{\partial\tilde{x}}\right\} \\ = \frac{h\nu n}{c}\left(n\frac{\partial}{\partial\tilde{x}^0} + \mu\frac{\partial}{\partial\tilde{x}}\right) = P^{\tilde{0}}\,\frac{\partial}{\partial\tilde{x}^0} + P^{\tilde{x}}\,\frac{\partial}{\partial\tilde{x}} \end{array}, \qquad (82)$$

and easily deduces that

$$P^{y'}\,\frac{\partial}{\partial y'}\;+\;P^{z'}\,\frac{\partial}{\partial z'}\;=\;\frac{h\,\nu\,n}{c}\,\left(\sin\,\Theta\,\cos\,\Phi\,\frac{\partial}{\partial\tilde{y}}\;+\;\sin\,\Theta\,\sin\,\Phi\,\frac{\partial}{\partial\tilde{z}}\right)\;=\;P^{\tilde{y}}\,\frac{\partial}{\partial\tilde{y}}\;+\;P^{\tilde{z}}\,\frac{\partial}{\partial\tilde{z}},\tag{83}$$

Then one has the final and important result

$$P^{0'}\,\frac{\partial}{\partial\,x^{0'}}\;+\;P^{x'}\,\frac{\partial}{\partial x'}\;+\;P^{y'}\,\frac{\partial}{\partial y'}\;+\;P^{z'}\,\frac{\partial}{\partial z'}\;=\;P^{\tilde{0}}\,\frac{\partial}{\partial\,\tilde{x}^{0}}\;+\;P^{\tilde{x}}\,\frac{\partial}{\partial\tilde{x}}\;+\;P^{\tilde{y}}\,\frac{\partial}{\partial\tilde{y}}\;+\;P^{\tilde{z}}\,\frac{\partial}{\partial\tilde{z}}$$

rewritten under the more compact form

$$P^{\alpha'}\,\partial_{\alpha'}\;=\;P^{\tilde{\alpha}}\,\partial_{\tilde{\alpha}},\tag{84}$$

which reveals that the derivative operator along a photon path is an invariant quantity, unless one uses the fundamental mater variables $\left(x^{0'},\,x',\,y',\,z'\right)$ and $\left(\tilde{x}^{0},\,\tilde{x},\,\tilde{y},\,\tilde{z}\right)$ associated to the fundamental basis $\left(\vec{e_{0'}},\,\vec{e_{x'}},\,\vec{e_{y'}},\,\vec{e_{z'}}\right)$ and $\left(\vec{\tilde{e}_{0}},\,\vec{\tilde{e}_{x}},\,\vec{\tilde{e}_{y}},\,\vec{\tilde{e}_{z}}\right)$; note that

$$\begin{aligned}P^{\alpha'}\,\partial_{\alpha'}\;&=\;\tfrac{h\,\nu'\,n}{c}\,\left[n\,\tfrac{\partial}{\partial\,x^{0'}}\;+\;\cos\,\Theta'\,\tfrac{\partial}{\partial x'}\;+\;\sin\,\Theta'\,\left(\cos\,\Phi'\,\tfrac{\partial}{\partial y'}\;+\;\sin\,\Phi'\,\tfrac{\partial}{\partial z'}\right)\right]\\&=\;\tfrac{h\,\nu'\,n}{c}\,\left(n\,\tfrac{\partial}{\partial\,x^{0'}}\;+\;\vec{\Omega}'\,\vec{grad}\right)\;=\;\tfrac{h\,\nu'\,n}{c}\,\left(n\,\tfrac{\partial}{\partial\,x^{0'}}\;+\;\tfrac{\partial}{\partial s'}\right)\end{aligned}$$

where $\frac{\partial}{\partial s'}$ is the habitual curvilinear spatial derivative, that is the propagation unit vector so as the gradient vector (expressed with the mater co-ordinates) are given in the vacuum basis; similarly one has

$$\begin{aligned}P^{\tilde{\alpha}}\,\partial_{\tilde{\alpha}}\;&=\;\tfrac{h\,\nu\,n}{c}\,\left[n\,\tfrac{\partial}{\partial\,\tilde{x}^{0}}\;+\;\cos\,\Theta\,\tfrac{\partial}{\partial\tilde{x}}\;+\;\sin\,\Theta\,\left(\cos\,\Phi\,\tfrac{\partial}{\partial\tilde{y}}\;+\;\sin\,\Phi\,\tfrac{\partial}{\partial\tilde{z}}\right)\right]\\&=\;\tfrac{h\,\nu\,n}{c}\,\left(n\,\tfrac{\partial}{\partial\,\tilde{x}^{0}}\;+\;\vec{\Omega}\,\vec{grad}\right)\;=\;\tfrac{h\,\nu\,n}{c}\,\left(n\,\tfrac{\partial}{\partial\,\tilde{x}^{0}}\;+\;\tfrac{\partial}{\partial\tilde{s}}\right)\end{aligned}$$

hence from Eq. (83) one may rewrite the derivative operator transformation under the useful form

$$\nu\,\left(\frac{n}{c}\,\frac{\partial}{\partial\tilde{t}}\;+\;\frac{\partial}{\partial\tilde{s}}\right)\;=\;\nu'\,\left(\frac{n}{c}\,\frac{\partial}{\partial t'}\;+\;\frac{\partial}{\partial s'}\right),\tag{85}$$

It is now time to focus on the energetic invariant quantity, namely the specific intensity; let us first remind the obtaining of the specific intensity when the system to be considered is vacuum, following the steps developed by Mihalas [1]: if N is the photons number passing through a surface element perpendicular to the particle speed vector at a given frequency and a given propagation direction, this number in the comobile LRS is expressed as

$$N\;=\;\frac{L'\;\vec{\Omega}'\;\vec{dS}\;d\nu'\,d\Omega'\,dt'}{h\,\nu'}\;=\;\frac{L'\;\cos\,\Theta'\,d\nu'\,d\Omega'\,dS\,dt'}{h\,\nu'},\tag{86}$$

relatively to the observer LRS, this number is

$$N\;=\;\frac{L\,d\nu\,d\Omega\;\vec{dS}\;dt}{h\,\nu}\,\left(\vec{\Omega}\;-\;\vec{\beta}\right)\;=\;\frac{L\,d\nu\,d\Omega\,dS\,dt}{h\,\nu}\,\left(\cos\,\Theta\;-\;\beta\right),\tag{87}$$

from which one immediately deduces that

$$\frac{L' \cos \Theta' \, d\nu' \, d\Omega'}{\nu'} = \gamma \, \frac{L \, d\nu \, d\Omega}{\nu} \, (\cos \Theta - \beta) \,, \tag{88}$$

since the proper times are related with $dt' = \frac{dt}{\gamma}$; moreover, using the optical aberration and frequency Doppler shift transformation, $\cos \Theta - \beta = \frac{\nu' \cos \Theta'}{\gamma \, \nu}$, hence, using the solid angle conservation $\nu \, d\nu \, d\Omega = \nu' \, d\nu' \, d\Omega'$, one obtains

$$\frac{L'}{\nu'^3} = \frac{L}{\nu^3}, \tag{89}$$

which justifies that $I = \frac{L}{\nu^3}$ is the specific intensity for an unit refractive index medium; if the refractive index is not one, analogously to what precedes, the specific intensity is simply $\frac{L'}{n^2 \, \nu'^3} = \frac{L}{n^2 \, \nu^3}$ since relatively to the observer at rest inside the medium, the moving medium remains isotropic of refractive index n; then since at rest the intensity is $\frac{L}{n^2}$, well-known result [7], one deduces the previous result; then the left handed side term of Eq. (1) obeys to the following relation

$$\nu \left[\frac{n}{c} \frac{\partial}{\partial \tilde{t}} \left(\frac{L}{n^2 \, \nu^3} \right) + \frac{\partial}{\partial \tilde{s}} \left(\frac{L}{n^2 \, \nu^3} \right) \right] = \nu' \left[\frac{n}{c} \frac{\partial}{\partial t'} \left(\frac{L'}{n^2 \, \nu'^3} \right) + \frac{\partial}{\partial s'} \left(\frac{L'}{n^2 \, \nu'^3} \right) \right], \tag{90}$$

If one considers now the number of photons emitted by an elementary volume in an elementary solid angle around a given frequency interval, this number can be expressed as

$$N = \frac{\eta' \, d\nu' \, d\Omega' \, dV' \, d\, x^{0'}}{c \, h \, \nu'} = \frac{\eta \, d\nu \, d\Omega \, d\tilde{V} \, d\, \tilde{x}^0}{c \, h \, \nu}, \tag{91}$$

where η is the emissive power; due do the scalar density conservation, it comes that

$$\begin{aligned} &\sqrt{- Det \, \overline{\overline{\tilde{g}}}} \, d\, \tilde{x}^0 \; d\tilde{V} = \sqrt{- Det \, \overline{\overline{g'}}} \, d\, x^{0'} \; dV' \\ &\Rightarrow d\, \tilde{x}^0 \; d\tilde{V} = d\, x^{0'} \; dV' \end{aligned} , \tag{92}$$

hence one has from the previous result and with the help of Eq. (70)

$$\frac{\eta'}{\nu'^2} = \frac{\eta}{\nu^2}, \tag{93}$$

similarly the number of photons absorbed in the same conditions is

$$N = \frac{\kappa' \, L' \, d\nu' \, d\Omega' \, dV' \, d\, x^{0'}}{c \, h \, \nu'} = \frac{\kappa \, L \, d\nu \, d\Omega \, d\tilde{V} \, d\, \tilde{x}^0}{c \, h \, \nu}, \tag{94}$$

where κ is the absorption coefficient; from what precedes, one deduces that $\frac{\kappa' L'}{\nu'^2} = \frac{\kappa \, L}{\nu^2}$, and by definition of the specific intensity, one finally obtains

$$\kappa' \, \nu' = \kappa \, \nu, \tag{95}$$

so that the right handed side term of Eq. (1) obeys to the following relation

$$\frac{1}{n^2 \, \nu^2} \, (\eta - \kappa \, L) = \frac{1}{n^2 \, \nu'^2} \, (\eta' - \kappa' \, L') \,, \tag{96}$$

In the vacuum, the emissive power is simply the Planck function multiplied by the absorption coefficient $\kappa\ L^0$, while in a medium of refractive index n, it is $\eta\ =\ n^2\ \kappa\ L^0$, from which it comes the invariant forms of the RTE

$$\begin{aligned} \frac{n}{c}\frac{\partial}{\partial t'}\left(\frac{L'}{n^2\ \nu'^3}\right) + \frac{\partial}{\partial s'}\left(\frac{L'}{n^2\ \nu'^3}\right) &= \frac{\kappa'}{\nu'^3}\left(L^0 - \frac{L'}{n^2}\right) \\ \frac{n}{c}\frac{\partial}{\partial \tilde{t}}\left(\frac{L}{n^2\ \nu^3}\right) + \frac{\partial}{\partial \tilde{s}}\left(\frac{L}{n^2\ \nu^3}\right) &= \frac{\kappa}{\nu^3}\left(L^0 - \frac{L}{n^2}\right) \end{aligned}, \tag{97}$$

Hence, noting the total spatial derivative $\frac{d}{d\tilde{s}} = \frac{n}{c}\frac{\partial}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{s}}$, one finally obtains the usual form of the invariant RTE

$$\begin{aligned} &\frac{dI}{d\tilde{s}} + \kappa\, I = \kappa\, \frac{L^0}{\bar{\nu}^3} \\ &\Rightarrow I\left(\tilde{s}_f\right) = I\left(\tilde{s}_i\right)\,\exp\left[-\int_{\tilde{s}=\tilde{s}_i}^{\tilde{s}_f}\kappa\left(\tilde{s}\right)\,d\tilde{s}\right] + \int_{\tilde{s}=\tilde{s}_i}^{\tilde{s}_f}\frac{\kappa(\tilde{s})\ L^0(\tilde{s})}{\bar{\nu}^3(\tilde{s})}\,\exp\left[-\int_{\tilde{s}'=\tilde{s}}^{\tilde{s}_f}\kappa\left(\tilde{s}'\right)\,d\tilde{s}'\right]d\tilde{s} \end{aligned}, \tag{98}$$

Then the reference observer at rest in vacuum is able to determine the (real) radiative field inside the moving medium from the perceived emerging directional and spectral intensity field.

5. CONCLUSION

In this paper we described a way to construct a consistent "equivalent vacuum" and "matter" space bound to the observer after a diagonalisation of the metric tensor related to the Gordon's metric, due to the moving (with a constant speed) particles of the non unit refractive index semi-transparent medium; the construction of this space relatively to the observer allows then the calculation of the optical aberration and frequency transformation in the new fundamental co-ordinates attached to the observer space, and leads to the determination of the invariant specific intensity and the general form of the radiative transfer equation in this space, following the method developed by Mihalas in vacuum. We may expect to determine a more general formulation of this work by generalisation to the case of non constant speed moving particles in a semi-transparent medium of non constant refractive index.

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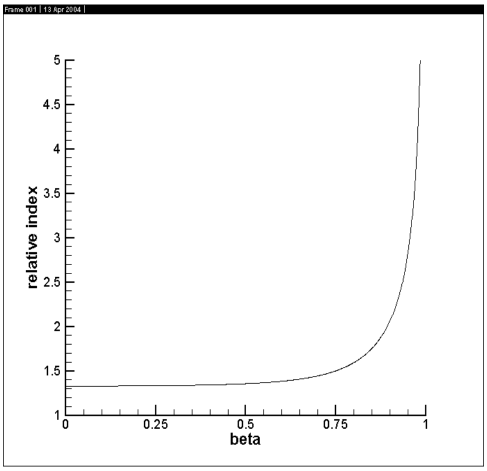
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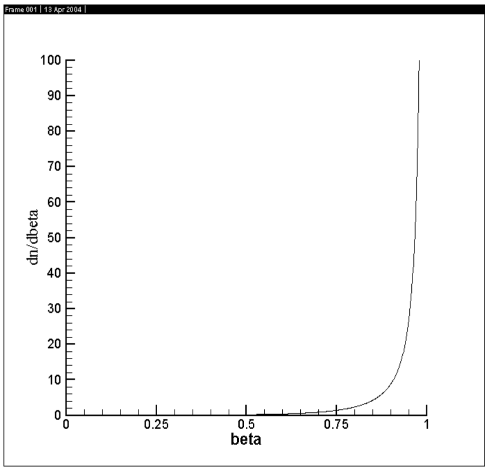
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