

# Multiboundary Algebra as Pregeometry

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**Abstract:** It is well known that the Clifford Algebras, and their quaternionic and octonionic subalgebras, are structures of fundamental importance in modern physics. Geoffrey Dixon has even used them as the centerpiece of a novel approach to Grand Unification. In the spirit of Wheeler’s notion of “pregeometry” and more recent work on quantum set theory, the goal of the present investigation is to explore how these algebras may be seen to emerge from a simpler and more primitive order. In order to observe this emergence in the most natural way, a pregeometric domain is proposed that consists of two different kinds of boundaries, each imposing different properties on the combinatory operations occurring between elements they contain. It is shown that a very simple variant of this kind of “multiboundary algebra” gives rise to Clifford Algebra, in much the same way as Spencer-Brown’s simpler single-boundary algebra gives rise to Boolean algebra.

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## 1. Introduction

John Wheeler, in his classic text *Gravitation* (1973), suggested that propositional logic might potentially serve as a kind of “pregeometry,” in the sense that physical geometric structures might be derivable from statistical patterns in large numbers of logical propositions. He enlarged on this idea in later writings using the catch-phrase “It from Bit” (Wheeler, 1993). This interesting suggestion of Wheeler’s has not yet directly borne fruit, however, the goal of this paper is to demonstrate that something conceptually similar may well yield interesting results and merit further pursuit.

G. Spencer-Brown, in his work *Laws of Form* (1969), presented a simple formalism equivalent to Boolean algebra, but using only one mark, which he called the “mark of distinction” and which serves as a sort of bracketing or boundary operation. The mark of distinction is different from the Peirce/Sheffer stroke (Peirce, 1880), which is an alternate way to reduce Boolean logic to a single operation. The Brownian formalism has the feel of a pregeometry to it, and Spencer-Brown presents philosophical arguments as to why the mark of distinction might serve as a foundation for describing the structures of the physical world (though without reference to the specifics of physical theory). But as with

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Wheeler’s suggestion about propositional logic, Spencer-Brown’s suggestion in this regard has not yet led anywhere interesting, although a significant literature on the mathematics and philosophy of Brownian algebra and logic has arisen (see e.g. Kauffmann, 2006; Varela, 1978).

The central insight presented in the present paper is that if one generalizes Brown’s simple algebra into a related algebra involving not one but two separate “marks of distinction” (or as I call them here, two “boundary types”), obeying different rules – then one starts to make some headway toward deriving physical structure from pregeometric, propositional-logic-like considerations. Specifically, what I show here is that a simple two-boundary algebra, generalizing Brownian algebra, gives rise in a simple way to the Discrete Clifford Group. According to standard mathematics, this in turn gives rise to Discrete Clifford Algebra, which gives rise to general Clifford Algebra – and as Dixon (1994) among others has argued, once one has Clifford Algebra structure, one has the core of modern physics.

Certainly, the simple results presented here are very far from constituting a complete fulfillment of Wheeler’s grand vision of a pregeometric derivation of physical reality. However, we believe they set a better foundation for work in this direction than anything prior. Finkelstein’s (1993) quantum set theory might be identified as the closest competitor in the physics literature, but that is a different sort of approach which in our view lacks the elemental simplicity Wheeler was looking for in his “It from Bit” conception. The formalism described here seems to be the simplest set-theoretic/logic-type formalism from which the algebraic structures key to modern physics emerge in a natural way. Also, the study of multiboundary algebras as mathematical objects is new and may potentially lead to interesting theoretical and applied conclusions, when pursued further.

## 2. Ons and Boundaries

This section presents a conceptual framework for pregeometry, which is called “Ons Algebra” and which uses the language of multiboundary forms. The following section presents rigorous algebraic rules building on the ideas presented here.

The basis of Ons algebra is the On, to be denoted  $N$ . An On is a truly foundational and elementary “particle.” There may be many Ons, but all are identical. Note that this “particle” is truly atomic, in the manner of a single bit of information: by construction, there is no way to decompose it into smaller pieces.

The universe is then interpreted to consist of Ons grouped and separated by boundaries.

First, for any  $X$ , a boundary separating  $X$  from everything else is denoted  $[X]$ . The division of the boundary symbol into two parts,  $[$  and  $]$ , is an artifact of typographical notation; in a two-dimensional symbolism, a boundary separating off  $X$  would be denoted by a box or circle drawn around  $X$ .

The construct  $[N]$  is interpreted as a Chronon, a particle of time. For convenience, a Chronon may be denoted  $C = [N]$ ; this is solely for the purpose of having one symbol

instead of three.

The Void, aka emptiness or absence, will be denoted  $V$ . I.e.,

$$V =$$

Next, a permeable or "soft" boundary separating off  $X$  is denoted  $\langle X \rangle$ . This denotes a boundary that entities can pass easily through. By contrast to  $\langle X \rangle$ ,  $[X]$  will be called a "hard" boundary around  $X$

A boundary with nothing inside it is taken to be Void, i.e.

$$[] = \langle \rangle = V$$

The "space" that an entity lives in is the hard or soft boundary surrounding it. The universe itself may be considered an implicit hard boundary, defining the space in which unbounded expressions such as  $X$ ,  $X+Y$ , etc., live.

Coexistence of  $X$  and  $Y$  in a single "space" is denoted  $X + Y$ . This operation is commutative, and furthermore a sequence of entities connected by +'s in sequence is invariant with respect to permutations, e.g.  $X + Y + W + M = M + W + Y + X$ , etc.

$N$  is an annihilator with respect to coexistence; i.e.

$$N + N = V$$

Inside a hard boundary, hard boundaries are associative, i.e.

$$[[X] + [Y]] = [[X + Y]]$$

Inside a soft boundary, associativity of hard boundaries fails, and we instead have the rule

$$\langle [C + X] + X \rangle = \langle V \rangle = V =$$

which is incompatible with associativity.

Temporal comparison or "temporization" of two entities  $X$  and  $Y$  is denoted by  $X \wedge Y$ . Temporal comparison tells which of  $X$  and  $Y$  occurs first. The concept is: if  $X$  occurs before  $Y$ ,  $X \wedge Y = X$ ; otherwise  $X \wedge Y = Y$ .

Interpenetration of  $X$  and  $Y$  is denoted by  $A * B$ .  $A * B$  is defined by

$$\begin{aligned} X * Y &= X + Y + X \wedge Y \\ [A+B] * [C+D] &= [A + B + C + D + A \wedge C + A \wedge D + B \wedge C + B \wedge D] \\ &\text{etc.} \end{aligned}$$

In general, if  $G$  and  $H$  are any two collections of coexisting entities,  $G * H$  is the collection of the following mutually coexisting entities: all entities in  $G$ , all entities in  $H$ , and all pairs  $X \wedge Y$  where  $X$  is in  $G$  and  $Y$  is in  $H$ .

The main goal of this paper is to show that, if one conceptualizes Ons as has been done above, this leads naturally to the mathematical rules of Clifford Algebra, which are core to modern theoretical physics. Thus Ons seem a viable candidate for fundamental pregeometry.

### 3. Formal Rules of Ons Algebra

Following on the simple conceptual ideas of the prior section, in this section I describe the mathematical formalism called “Ons Algebra.”

The main originality of Ons algebra in purely formal terms is the introduction of two kinds of boundaries, ”hard” and ”soft” boundaries. Normally in algebra we have only one kind of boundary, represented by parentheses  $()$ , and various operations that act on bounded and unbounded entities. In constructing Ons algebra, it has seemed most simple to restrict the number of operations to three, but to introduce two types of boundaries  $[]$  and  $\langle \rangle$ , which are acted on by the operations in different ways. Basically, a soft boundary  $\langle \rangle$  is a boundary that operations can distribute through, and addition can associate through; whereas a hard boundary  $[]$  prevents distribution and additive associativity, but has some special ”collapse” rules of its own.

The primitive elements of On algebra are given as follows. Some of the elements have been given evocative verbal names: not too much significance should be attached to these names, they are simply used to make textual reference simpler.

- $V$ , referred to as the Void
- $N$ , referred to as an On
- $[X]$ , a hard boundary demarcating some entity  $X$
- $\langle X \rangle$ , a soft boundary demarcating some entity  $X$
- $C = [N]$ , referred to as a Chronon
- $+$ , a binary operation denoting coexistence
- $\wedge$ , a binary operation denoting temporal precedence (note that the reference to time here is metaphorical: this is a kind of “pregeometric” time which is different from ordinary physical time)
- $*$ , a binary operation denoting interpenetration

Next we articulate some notation and terminology related to the above entities, boundaries and operations:

- An ”entity” is a legal expression in On algebra
- $X, Y$ , etc. are variables, referring to any entities
- A legal expression is
  - $V$  or  $N$
  - $X + Y$ ,  $X * Y$ , or  $X \wedge Y$  where  $X$  and  $Y$  are legal expressions
  - $[X]$  or  $\langle X \rangle$  where  $X$  is a legal expression
- $X(i), Y(j)$ , etc., are also variables; the  $($  and  $)$  in these expressions have no fundamental meaning in On algebra but are merely a notational device for denoting indices  $i, j$ , etc.
- $S(X(i); i=1, \dots, n) = X(1) + X(2) + \dots + X(n)$ ; the  $S$  or ”summation” operator has no fundamental meaning in On algebra but is merely a notational device

Next, the following are the basic rules of Ons algebra:

- $[ ] = \langle \rangle = \langle V \rangle = V =$

- $V + X = X$
  - $N + N = V + V = V$
  - $\langle [X + C] + [X] + Z \rangle = \langle V + Z \rangle$
  - $[ [X] + [Y] + Z ] = [ [X+Y] + Z ]$
  - $+ =$
  - $X * =$
  - $X \wedge Y = C$  or  $X \wedge Y = V$ , for any arguments  $X$  and  $Y$
  - $S( X(P(j)); j=1,\dots,n)$  is identical for all functions  $P()$  that permute  $\{1,\dots,n\}$
  - $X * Y = X + Y + X \wedge Y$
  - if  $Q = S(X(i); i=1,\dots,n)$ ,  $R = S(X(i); i=1,\dots,m)$ ? then
- (1)  $* [Q]*[R] = [Q + R + S(X(i) \wedge Y(j); i=1,\dots,n; j=1,\dots,m)]$
- (2)  $* [Q] \wedge [R] = [ S(X(i) \wedge Y(j); i=1,\dots,n; j=1,\dots,m) ]$

Finally, the following further terminology will be useful:

- If  $X \wedge Y = C$  then we may say equivalently that  $X > Y$
- A "temporal domain" is an pair  $(G, \wedge)$  where  $G$  is a set of entities and  $\wedge$  obeys transitivity and antisymmetry and nonidentity over  $G$ , so that
  - if  $X \wedge Y = C$ , then  $Y \wedge X = V$
  - if  $X \wedge Y = C$  and  $Y \wedge Z = C$  then  $X \wedge Z = C$
  - $X \wedge X = V$
- The  $\wedge$  operator may be defined in such a way as to make the set of all entities a temporal domain. Then it is called a "linear time operator."
- Given an entity  $G = S(X(i); i=1,\dots,m)$ , the "hard powerset" of  $G$ ,  $h(G)$ , is the collection of all entities of the form  $[ S(X(i(k)); k=1,\dots,M) ]$ , where each  $0 \leq i(k) \leq n$ .
- Similarly, the "soft powerset" of  $G$ ,  $s(G)$ , is the collection of all entities of the form  $\langle S(X(i(k)); k=1,\dots,M) \rangle$ , where each  $0 \leq i(k) \leq n$ .

## 4. Review of Discrete Clifford Algebras

This section reviews two finite algebras: the Discrete Clifford Group  $DCLG(n)$  and the integer lattice of the Clifford Algebra,  $DCL(n)$ . The material presented in this section is well-known mathematics (see e.g. Simon, 1996) and is not original in its basic content, but it constitutes a somewhat nonstandard formulation of these algebras, specifically intended to make the relationship with Ons algebra more apparent, as will be utilized in the theorems of the following section.

Firstly,  $DCLG(n)$  is defined in terms of a basis of elements  $\{e(1), \dots, e(n)\}$ , and the subsets of  $\{1,\dots,n\}$ . Where  $A$  is a subset of  $\{1,\dots,n\}$  and  $e(i)$  is a basis element, the elements of  $DCLG(n)$  are of the form  $+e(A)$  or  $-e(A)$ , where  $e(A)$  is the Clifford element  $e(j(1))e(j(2))\dots e(j(k))$ , and  $j(1), j(2), \dots, j(k)$  are the elements of the subset  $A$ , arranged in increasing order (the use of a standard order is important). Since we have used  $+$  and  $-$ , this yields  $2n+1$  elements in  $DCLG(n)$ .

The multiplication rule for  $DCLG(n)$  (where multiplication is denoted by adjacency)

is given by

$$(x(A) e(a)) (x(B) e(b)) = x(A, B) e(C)$$

where  $C$  is the symmetric difference of the sets  $A$  and  $B$ , and  $x(A)$ ,  $x(B)$ , and  $x(A, B)$  are  $+1$  or  $-1$ . The sign  $x(A, B)$  of the product is determined by an algorithm using the rules.

- $e(i) e(i) = +1$
- $e(i) e(j) = - e(j)e(i)$  for  $i \neq j$

The algorithm works as follows. First, one creates a sequence of elements  $e(X)$  by concatenating the sequence representations  $e(A)$  and  $e(B)$  (placing  $e(B)$  at the end of  $e(A)$ ). Then one performs a series of operations on  $eX$ , using the rule  $e(i)e(j) = - e(j)e(i)$  to move each of the  $B$ -elements to the left until it either:

- meets a similar element, in which case it is cancelled with the similar element using  $e(i) e(i) = +1$ , or
- finds a place in between two  $A$ -elements in the proper order.

According to this algorithm, each time a  $B$ -element is moved to the left, the sign of  $e(X)$  changes. If the total number of moves is even, then the sign of  $e(X)$  is unchanged by the algorithm, and  $x(A, B) = x(A) x(B)$ . If the total number of moves is odd, then the sign of  $e(X)$  is reversed by the algorithm, and  $x(A, B) = - x(A)x(B)$ . Elegantly, the algorithm also performs the symmetric difference operation: the only elements left in  $e(X)$  after the transformation process is complete, are the ones that are present in both  $e(A)$  and  $e(B)$ .

Some simple examples may be helpful for understanding this algorithm. For doing example, I will adopt a simpler notation, and write  $e(i) = ei$ , e.g.  $e(1) = e1$ . Suppose we have

$$A = e1 e2 e3$$

$$B = e2 e3 e4$$

and we wish to compute the product

$$A * B = e1 e2 e3 * e2 e3 e4$$

According to the DCLG algorithm, we move the elements of  $B$  over, one by one, until they meet their respective elements of  $A$ . each time we move an element of  $B$  past an element of  $A$ , we switch the sign of the product from its initial positive value. So to compute  $A*B$ , we have

$$\begin{aligned} eX &= e1 e2 e3 e2 e3 e4 = \\ &= -e1 e2 e2 e3 e3 e4 = \\ &= -e1 e4 \end{aligned}$$

The sign is negative. And what is left over after cancellation,  $e(1) e(4)$ , is exactly the symmetric difference of the sets  $A$  and  $B$ .

Or, consider:

$$e1 e2 e5 * e1 e2 e4 =$$

$$\begin{aligned}
 & -e_1 e_2 e_1 e_5 e_2 e_4 = \\
 & e_1 e_1 e_2 e_5 e_2 e_4 = \\
 & -e_2 e_2 e_5 e_4 = \\
 & e_4 e_5
 \end{aligned}$$

Again, each switch does a sign change, and the resultant is the signed symmetric difference of the multiplicands.

The step from DCLG( $n$ ) to the full Clifford algebra  $CL(n)$ , finally, is not a large one. Clifford algebra is obtained by considering the  $M = 2^{(N-1)}$  elements  $\{V(1), \dots, V(M)\}$  of the DCLG to combine, not only by DCLG multiplication, but also by summation and scalar multiplication, according to standard vector space laws. Clifford algebra, as normally studied, is a continuous construction, involving either the reals or the complex numbers as a scalar field. But for many purposes, one does not require the full continuous Clifford Algebra  $Cl(N)$ , only the discrete lattice subset  $DCL(N)$  obtained by considering  $\{V_1, \dots, V_M\}$  as a vector space with scalar coefficients drawn from the integers.

To make the relation between DCLG and Ons algebra more apparent, it is useful to go through the intermediary representation of bit strings, strings of 0's and 1's. Suppose one writes the expansions of DCLG elements in terms of elements as bit strings, sequences of zeroes and ones, where . The length of the bit strings is  $n$ , the number of basis elements. In an algebra with four basis elements, for example, the element  $e(1)$  is represented 1000; the element  $e(3)$  is represented 0010; the element  $e(2) e(3)$  is represented 0110; the element  $e(1)e(2)e(3)e(4)$  is represented 1111.

Consider again the first example above,  $e_1 e_2 e_3 * e_2 e_3 e_4 = -e_1 e_4$ . In bit string notation, this result is represented

$$1110 * 0111 = -1001$$

To compute the sign using only bit strings, one reasons as follows. First, one lines up the two multiplicands, one under the other:

$$\begin{array}{r}
 1110* \\
 0111
 \end{array}$$

Then, for each 1 in the second multiplicand (B), one adds up the number of 1's in the first multiplicand (A) that come strictly after it (moving from left to right) For instance, for the 1 in the second place in B in this example, we have the one 1 in the third place in A. For the 1 in the third place in B in this example, we have no ones coming after it in A; and likewise for the 1 in the fourth place in B in this example.) Then, one takes the sum one has gotten by this procedure, and uses it to determine the sign of the product. If this sum is odd, the sign of the product is negative. If the sum is even, the sign of the product is positive.

In this example, we have moved through the 1 bits of the representation of the second multiplicand B, summing for each one the number of 1 bits of A that come after it.

Equivalently, one can sum over all the 1's in the first multiplicand, A, summing for each one of these, the number of 1's in B that occur strictly before it. The two procedures always give the same result.

Similarly, the second example above,  $e_1 e_2 e_5 * e_1 e_2 e_4$ , translates to

$$11001*$$

$$11010$$

Here, for the 1 in the first place of B, we have two 1's in A coming after it; for the 1 in the second place of B, we have one 1 in A coming after it; for the 1 in the fourth place of B, we have one 1 in A coming after it. So, the sum is four, and the sign of the product is plus.

## 5. Connecting Ons Algebra with Clifford Algebra

Now we expound some basic theory regarding Ons algebra, leading up to a theoretical description of the relationship between Ons algebra and DCLG.

**Theorem 1:** For any entity X,

$$[X + X] = [V] = V =$$

(i.e.,  $X+X=V$  within a hard boundary)

**Proof:**

The proof is by induction on the number n of symbols in the expression X.

We know the claim is true for  $n=1$ , as the only legal expression of length 1 is N, and  $[N+N] = [V] = V =$ .

Now, suppose the claim is true for all 0

X must be of one of the following forms: [Y], ,  $Y+Z$ ,  $Y^{\wedge}Z$ ,  $Y^*Z$ , where Y and Z necessarily include fewer symbols than X, and can hence be assumed by the inductive assumption to obey the claim of the theorem.

Suppose  $X=[Y]$ . Then,  $[[Y]+[Y]] = [[Y+Y]] = [[V]] = [V] = V =$ . This shows e.g. that  $[C+C] = V$ .

Suppose  $X =$ . Then,  $[+] = [] = [] = [V] = V =$

Suppose  $X = Y+Z$ . Then,  $[ [Y+Z] + [Y+Z] ] = [ [ Y + Z + Y + Z ] ] = [[Y + Y + Z + Z]] = [[ V + V ] ] = [[V]] =$

Suppose  $X = Y^*Z$ . Then,  $[ [Y^*Z] + [Y^*Z] ] = [ [ Y + Z + Y^{\wedge}Z ] + [Y + Z + Y^{\wedge}Z ] ] = [[ Y + Y + Z + Z + Y^{\wedge}Z + Y^{\wedge}Z ] ] = [[Y^{\wedge}Z + Y^{\wedge}Z]]$ . Now,  $Y^{\wedge}Z =$  either C or V, yielding  $[[Y^{\wedge}Z + Y^{\wedge}Z]] = [[C + C]] = [V] = V$  or  $[[Y^{\wedge}Z + Y^{\wedge}Z]] = [[V + V]] = [[V]] = V$

Finally, suppose  $X=Y^{\wedge}Z$ . Then, by reasoning analogous to that of the previous sentence, the claim holds.

QED ■

**Corollary:** For any entity G, the hard superset  $h(G)$  is finite

Now, building on Theorem 1 and on the the representation of the DCLG product sign rule given in the prior section, we proceed to prove that the hard powerset of a temporal domain of entities is isomorphic to DCLG.

**Theorem 2.** Consider a temporal domain  $(G, \hat{\phantom{a}})$  where  $G$  contains  $n$  entities, none of which is a chronon  $C$ . Then the algebra  $(h(G), *)$  is isomorphic to the Discrete Clifford Algebra  $DCLG(n)$

**Proof**

Let us identify the elements of  $G$  with the basis elements of a Discrete Clifford Algebra  $DCLG(n)$ , supposing that the ordering of the  $DCLG$  basis elements is taken to be consistent with the linear time operator  $\hat{\phantom{a}}$ .

Let us call the mapping from entities into Clifford basis elements "cliff", so that e.g.  $w = \text{cliff}(W)$ . For instance, if  $G = [W + X + Y + Z]$ , and according to the operator  $\hat{\phantom{a}}$  we have  $W < X < Y < Z$ , then we use the ordering

$$(\text{cliff}(W), \text{cliff}(X), \text{cliff}(Y), \text{cliff}(Z))$$

for the  $DCLG$ .

To extend cliff into a mapping from  $h(G)$  into general  $DCLG$  elements, we use the following two rules

- 1) a rule which holds so long as none of the  $X(i)$  in the formula is equal to  $C$ :

$$\begin{aligned} \text{cliff}([S(X(i); i = 1, \dots, n)]) = \\ \text{cliff}(X(i(1)) \text{cliff}(i(2)) \dots \text{cliff}(i(n))) \end{aligned}$$

where the function  $i()$  is a permutation of  $\{1, \dots, n\}$ , defined to enforce the ordering arrangement  $X(i(k)) < X(i(k+1))$ .

- 2) a rule for dealing with  $C$ , namely

$$\text{cliff}([C + X]) = -\text{cliff}(X)$$

What needs to be shown is that cliff is an isomorphism, i.e. that where  $Q$  and  $R$  are elements of  $h(G)$ ,

$$\text{cliff}(Q) * \text{cliff}(R) = \text{cliff}(Q * R)$$

where the right-hand  $*$  is the Ons interpenetration operator and the left-hand  $*$  is the  $DCLG$  operator.

In general, Theorem 1 guarantees that the isomorphism will hold where sign is not considered. To see this, consider the operation

$$\begin{aligned} [S(X(i); i = 1, \dots, n)] +' [S(Y(i); i = 1, \dots, n)] = \\ [S(X(i); i = 1, \dots, n) + S(Y(i); i = 1, \dots, n)] \end{aligned}$$

and the mapping  $\text{set}([S(X(i); i=1, \dots, n)]) = \{X(i); i=1, \dots, n\}$ . Note that the mapping  $\text{set}()$  is one-to-one because of Theorem 1: if there are multiple copies of some  $X(i)$  within  $([S(X(i); i=1, \dots, n)])$ , then they will cancel out, leaving at most a single copy of each one. It is clear that

$$\text{set}(Q +' R) = \text{set}(Q) \Delta \text{set}(R)$$

where  $\Delta$  denotes the symmetric difference operator. This is because, if some element occurs in both  $Q$  and  $R$ , it will cancel out in the sum  $Q +' R$  by Theorem 1. The only things left in the sum will be those things that are in either  $Q$ , or  $R$ , but not both.

This shows that the operation  $+$ ' is isomorphic to symmetric difference; but it is already known that the DCLG multiplication operation  $e(A) * e(B)$ , without signs considered, is identical to symmetric difference on the set of basis elements making up DCLG elements.

Next, to see how the sign works out, consider the example where  $G = [W + X + Y + Z]$ , and consider the product,

$$\begin{aligned} [X + Y + Z] * [X + Y + W] = \\ [X + Y + Z + X + Y + W + X^{\wedge}X + X^{\wedge}Y + X^{\wedge}W \\ + Y^{\wedge}X + Y^{\wedge}Y + Y^{\wedge}W \\ + Z^{\wedge}X + Z^{\wedge}Y + Z^{\wedge}W] \end{aligned}$$

In accordance with the above observations, by Theorem 1,  $X+X$  and  $Y+Y$  cancel out. Given the ordering  $W < X < Y < Z$

$$\begin{aligned} [X + Y + Z] * [X + Y + W] = \\ [W + Z + V + V + C \\ + C + V + C \\ + C + C + C] \end{aligned}$$

Since  $C+C=V$  inside a hard boundary, the result of this computation is  $[W]$ .

Now, suppose we did this computation using the DCLG rule.

In the bit string version, we would have

$$\begin{array}{r} 0111 * \\ 1110 \\ \hline 1001 \end{array}$$

yielding the  $W$  and  $Z$ . Next the sign would be obtained as follows:

$$\begin{array}{r} 3 \text{ 1's in } 0111 \text{ coming after the } 1^{***} \text{ in } 1110 \\ 2 \qquad \qquad \qquad *1^{**} \\ 1 \qquad \qquad \qquad **1^* \\ \hline 6 \end{array}$$

Since this is even, we find that there is no negative sign, consistent with the fact that our answer in Ons algebra was  $[W]$  rather than  $[C + W]$ . It is easy to see that the same thing occurs more generally: the cancellation of  $C$ 's agrees with the DCLG sign rule, yielding an isomorphism even where sign is considered.

**QED ■**

**Theorem 3:**

Consider a temporal domain  $(G, \wedge)$  where  $G$  contains  $n$  entities, none of which is equal to  $C$ . Then, the algebra  $(s(h(G)), +, *)$  is isomorphic to the integer lattice  $DCL(n)$  of the real Clifford Algebra  $CL(n)$ .

**Proof:**

The elements of  $DCL(n)$  are finite sums of elements of  $DCLG(n)$ .

The first claim of the proof is that the elements of  $DCL(n)$  can be expressed as finite sums of elements of  $G$ . This follows because Theorem 2 shows that  $G$  is isomorphic to

DCLG(n) where the multiplication  $*$  is concerned; and the operation  $+$  in Ons algebra obeys the same rules as the operation  $+$  in DCL, i.e. it is commutative, associative and distributive with regard to the soft boundary operator  $\langle \rangle$ . (The behavior of the Ons algebra  $+$  with regard to the hard boundary operator is not relevant here, because only the soft boundary operator is used in computing products in  $(s(h(G)), +, *)$ . The hard boundary operator is used in computing the products  $X(i)*X(j)$  within the algebra  $G$ , but these products are a "given" in regard to DCL; they are the DCLG multiplication table.)

Next it must be shown that the elements of  $s(h(G))$  behave under  $*$  in the same way that the elements of DCL(n) do. But this follows from the distributive law for  $*$  under the soft boundary operator, e.g.

$$\begin{aligned} & \langle X + Y \rangle * \langle U + V \rangle = \\ & \langle \langle X + Y \rangle * U + \langle X + Y \rangle * V \rangle = \\ & X * U + X * V + Y * U + Y * V \end{aligned}$$

and from the cancellation law

$$\langle [X + C] + [X] \rangle = \langle V \rangle$$

which shows that the  $h(G)$  correlate of a DCLG element ( $[X]$ ) and its the  $h(G)$  correlate of its negative ( $[X+C]$ ) cancel out.

**QED ■**

## 6. Conclusion

Where do we go from here?

We have reconstructed Clifford Algebra from a simple pregeometric foundation, which is an interesting step toward a reconstruction of physics from pregeometry, but far from a completion of the task. There are many possible ways to move forward from here, and since none of them has been carried out in detail, to discuss any of them is to venture into the realm of speculation.

I will mention here only one of the many possible paths forward, which seems to me the most promising at the moment. This is to bring the Ons algebra together with the important recent work of Youssef (1994) on exotic probabilities. Youssef has shown that, in a sense, the only reasonable systems for managing uncertain truth values are real-number, complex-number, quaternionic or octonionic probabilities. He has also shown that the acceptance of complex-number probabilities leads on naturally to the basic dynamic equations of quantum theory. The possibility of quaternionic or octonionic probabilities has not been seriously explored yet. From an Ons perspective, this ties in with the possibility of generalizing Ons algebra from crisp boundaries to graded boundaries, so that each boundary may have a certain degree between 0 and 1. There may be interesting connections between graded Ons algebras and exotic probabilities, given the close algebraic

relationships between Clifford algebras, quaternions and octonions. This is interesting because it gives a potential path to get from Ons algebra to the dynamic equations of physics, building on the current work that derives algebraic structures of physics. But, there is much to be done to see whether this sort of vision can actually bear fruit.

On a more philosophical level, one interesting aspect of Ons algebra is the introduction of the  $\hat{\ } operator for pregeometric temporal ordering, which seems to be necessary in order to obtain the Clifford algebra structure. This pregeometric time is not the same as physical time (which is viewed, in the pregeometric perspective, as an emergent quality that is coupled with emergent geometry, rather than a pregeometric quality), but it seems to be conceptually tied to physical time. There may be a more general principle at play here, in terms of the need for a fundamental ordering operation to exist in the pregeometric domain, in order to obtain an emergent geometric domain that displays algebraic complexity appropriate to give rise to physical spacetime. This aspect of the current work also merits further investigation.$

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