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Editors

Ammar Sakaji

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Quantum Computing Through Quaternions

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Abstract: Using quaternions, we study the geometry of the single and two qubit states of quantum computing. Through the Hopf fibrations, we identify geometric manifestations of the separability and entanglement of two qubit quantum systems.

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1. Introduction

Ever since the invention of “quaternions [1-6]” in 1843 by Sir William Hamilton to model the three dimensional motion of rigid bodies, these magic numbers have fascinated mathematicians and physicists worldwide with application growing by the day. Quaternions have provided a successful and elegant means for the representation of three dimensional rotations, Lorentz transformations of special relativity, robotics, computer vision, problems of electrical engineering and so on. Quaternionic Quantum Mechanics has also shown potential of possible unification with General Relativity. In fact, there is belief in some schools of thought that the conventional quantum mechanics in complex spacetime is an asymptotic version of the Quaternionic Quantum Mechanics.

In this paper, an attempt is made to apply these “quaternions” in quantum information processing.

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2. What are “Quaternions [1-6]”

We summarize below the salient properties of the “quaternion algebra” to facilitate completeness and continuity in this article.

The “quaternions” are generalized complex numbers of the form $q = w + xi + yj + zk$ with $w, x, y, z \in \mathbb{R}$, the set of real numbers and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being imaginary units that satisfy the quaternionic algebra $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$.

Furthermore, $Re q = \frac{1}{2}(q + \bar{q}) = w$, $Im q = \frac{1}{2}(q - \bar{q}) = xi + yj + zk$, where $\bar{q} = Re q - Im q$ is the conjugate of $q = Re q + Im q$.

Quaternionic multiplication is associative and distributive but not commutative. In fact, we have, for any two quaternions $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $y = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$

$$xy = (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + (x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)\mathbf{i} \\ + (x_0y_2 + x_2y_0 + x_3y_1 - x_1y_3)\mathbf{j} + (x_0y_3 + x_3y_0 + x_1y_2 - x_2y_1)\mathbf{k}$$

which can be succinctly expressed as $xy = x_0y_0 - \mathbf{x} \cdot \mathbf{y} + x_0\mathbf{y} + \mathbf{x}y_0 + \mathbf{x} \times \mathbf{y}$. For pure quaternions i.e. quaternions with $Re q = 0$, this simplifies to $xy = -\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \times \mathbf{y}$. Furthermore, since $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$, we also have $\frac{1}{2}(xy + yx) = x_0y_0 - \mathbf{x} \cdot \mathbf{y} + x_0\mathbf{y} + \mathbf{x}y_0$, $\frac{1}{2}(xy - yx) = \mathbf{x} \times \mathbf{y}$ with the corresponding values for pure quaternions being $\frac{1}{2}(xy + yx) = -\mathbf{x} \cdot \mathbf{y}$, $\frac{1}{2}(xy - yx) = \mathbf{x} \times \mathbf{y}$. The product of two quaternions is again a quaternion being the sum of a real number ($\mathbf{x} \cdot \mathbf{y}$) and a pure quaternion ($\mathbf{x} \times \mathbf{y}$). The cross product $\mathbf{x} \times \mathbf{y}$ also satisfies the Jacobi identity that makes the vector space \mathbb{R}^3 with the bilinear map $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \times \mathbf{y}$ into a Lie algebra.

We define the norm of a quaternion as $N(q) = \|q\| = (q\bar{q})^{1/2} = w^2 + x^2 + y^2 + z^2$. The inverse of a quaternion is naturally defined by $q^{-1} = \frac{\bar{q}}{\|q\|^2}$.

Writing the quaternions as $q = Re q + Im q$, we can split the quaternion algebra \mathbb{Q} into the direct sum of two orthogonal subspaces $\mathbb{Q} \equiv \mathbb{R} \oplus \mathbb{R}^3$ where the real part of the quaternion maps onto the straight line \mathbb{R} and the imaginary part maps onto the orthogonal three dimensional real plane.

The quaternion algebra also provides a representation of the group of symplectic transformations $Sp(1)$ (defined as the group of all linear quaternion transformations ϕ that leave the origin unchanged and preserve the real valued scalar product defined below) [7].

For this purpose, we define, in the quaternion algebra, a real valued symmetric scalar product as $\langle x | y \rangle = Re x\bar{y}$ which coincides with the conventional dot product of vectors i.e. $\langle x | y \rangle = \sum_{i=0}^3 x_i y_i$ as is easily verified. To explicitly set out the representation of the symplectic group $Sp(1)$, we identify the quaternion algebra \mathbb{Q} with the complex space \mathbb{C}^2 by writing an arbitrary quaternion $q \in \mathbb{Q}$ as $q = (q_0 + q_1i) + j(q_2 - q_3i) = q_\alpha + j\bar{q}_\beta$ with $q_\alpha = (q_0 + q_1i)$, $q_\beta = (q_2 + q_3i) \in \mathbb{C}$. Under this canonical identification, the quaternion valued form $\langle x | y \rangle_{\mathbb{Q}} = x\bar{y}$, $x, y \in \mathbb{Q}$ becomes $\langle x | y \rangle_{\mathbb{Q}} = x\bar{y} = (x_\alpha\bar{y}_\alpha + x_\beta\bar{y}_\beta) + (x_\beta y_\alpha - x_\alpha y_\beta)j = \langle x | y \rangle_{\mathbb{C}} + (x, y)_{\mathbb{C}}$ with the former form being hermitian and the latter skew-symmetric. It can be shown that a transformation that preserves the scalar product

$\langle x | y \rangle = Re x \bar{y} = Re \langle x | y \rangle_{\mathbb{Q}}$ also preserves the scalar product $\langle x | y \rangle_{\mathbb{Q}} = x \bar{y}$ and vice versa. This follows from the fact that a transformation preserving $\langle x | y \rangle_{\mathbb{Q}} = x \bar{y}$ would, obviously, preserve the real and imaginary components of the scalar product separately. Conversely, let a quaternionic transformation $\phi \in Sp(1)$ preserve the real valued product so that $\langle \phi x | \phi y \rangle = \langle x | y \rangle = Re \langle x | y \rangle_{\mathbb{Q}}$. Since this expression holds for quaternionic vectors of the form $|ix\rangle$ as well, we have $Re \langle ix | y \rangle_{\mathbb{Q}} = Re \langle \phi(ix) | \phi y \rangle_{\mathbb{Q}}$. Now, since, for the transformation $\phi \in Sp(1)$, we have $\phi(ix) = i\phi(x)$ so that $Re \langle ix | y \rangle_{\mathbb{Q}} = Re \langle i\phi(x) | \phi y \rangle_{\mathbb{Q}}$ which implies that the i th component of the quaternionic product is preserved if the real part is preserved by the transformation $\phi \in Sp(1)$. Similarly, the j, k th components can also be shown to be preserved. It follows that if a quaternionic transformation $\phi \in Sp(1)$ preserves the real product, then it also preserves the imaginary part and hence the complete quaternionic product.

With the identification of \mathbb{Q} with \mathbb{C}^2 , the group $Sp(1)$ is embedded as a subgroup in $U(2)$. This follows from the fact that every quaternion transformation $\phi \in Sp(1)$ preserves the quaternionic product $\langle x | y \rangle_{\mathbb{Q}} = \langle x | y \rangle_{\mathbb{C}} + (x, y)_{\mathbb{C}}$ therefore, such transformation must necessarily preserve the hermitian complex form $\langle x | y \rangle_{\mathbb{C}}$ and also the skew symmetric form $(x, y)_{\mathbb{C}}$. Hence, $\phi \in Sp(1)$ is a unitary transformation in \mathbb{C}^2 and so it belongs to $U(2)$.

Any element $\phi \in Sp(1)$ can, therefore, be written as a 2×2 unitary matrix, say $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, the unitary and symplectic nature of $\phi \in Sp(1)$ translate to the constraints $\phi \mathbf{E} \phi^T = \mathbf{E}$, $\phi^\dagger = \bar{\phi}^T = \phi^{-1}$ or $\phi \mathbf{E} = \mathbf{E}(\phi^T)^{-1} = \mathbf{E}(\phi^{-1})^T = \mathbf{E} \phi^*$ where $\mathbf{E} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that $b = -\bar{c}$, $d = \bar{a}$, with a, b being determined from the unitarity conditions $a\bar{a} + b\bar{b} = 1$ and $ab = ba$. In the case of an infinitesimal $\phi \in Sp(1)$, we can

write it in the neighborhood of the identity transformation as $\phi = \mathbf{I} + \varepsilon \begin{pmatrix} \alpha & \beta \\ \chi & \gamma \end{pmatrix}$. The

constraints on the transformation $\phi \in Sp(1)$ translate into the following constraints on $\alpha, \beta, \chi, \gamma$ viz. $\gamma = \bar{\alpha}$, $\chi = -\bar{\beta}$ and $\bar{\alpha} = -\alpha$.

The fact that the group of quaternions is isomorphic to $Sp(1)$ and also to the sphere S^3 in \mathbb{R}^4 , then follows from the fact that elements of the group $Sp(1)$ act on the space \mathbb{Q} of quaternions as $\phi q = q\bar{a}$ for $q \in \mathbb{Q}$ and $a \in \mathbb{Q}$ being determined by the transformation $\phi \in Sp(1)$. Since $\phi \in Sp(1)$ preserves the quaternionic product, we have $\langle x | y \rangle_{\mathbb{Q}} = x \bar{y} = x \bar{a} a \bar{y} = \|a\|^2 x \bar{y}$ whence $\|a\| = 1$. Since the identity $\|ab\| = \|a\| \|b\|$ holds for all quaternions, it follows that the group $Sp(1)$ is isomorphic to the group of unit quaternions that form a sphere S^3 in \mathbb{R}^4 for $1 = \|a\|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

3. The Geometry of a Single Qubit

The “quantum bit” or “qubit” plays the role of a “bit” in quantum computing [8] and constitutes a unit of quantum information [8-9]. It is represented by a state vector of a two-level quantum system. The representation space is, therefore, a two dimensional Hilbert space of the complex numbers and the basis vectors are usually chosen as $|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T$ and $|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T$, being the eigenvectors of the “spin” operator σ_3 in the direction of the z axis.

The fundamental difference between the “classical bit” and the “qubit” is that the former can have only two possible values viz. 0,1. The “qubit”, on the other hand, can occur in an infinite number of states being the superposition of the “pure states” represented by the basis vectors. We can, therefore, express a qubit as a linear combination of the two basis states as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. $\alpha, \beta \in \mathbb{C}$ are the probability amplitudes whose squares provide a measure of the probability of the qubit being in state $|0\rangle$ and state $|1\rangle$ respectively. We must, therefore, have $|\alpha|^2 + |\beta|^2 = 1$

The state space of a single qubit quantum register admits a geometrical representation as a Bloch sphere [10]. This is established as follows:-

The state space of a two level quantum system is conventionally taken as the Hilbert space $H \equiv \mathbb{C} \otimes \mathbb{C}$ [11]. Now, if two physical states $|\psi\rangle, |\phi\rangle$ that differ merely by a phase i.e. a complex number of unit magnitude i.e. $|\psi\rangle = e^{i\omega} |\phi\rangle$, then they represent the same physical state. It follows, therefore, that the proper space for a two level quantum system is the above Hilbert space $H \equiv \mathbb{C} \otimes \mathbb{C}$ quotiented by the equivalence relation $|\psi\rangle \sim |\phi\rangle$ iff $|\psi\rangle = e^{i\omega} |\phi\rangle$. It will, thus, be the projective Hilbert space created by this equivalence relation and may be defined as $\Pi(H) = H / \sim$. Sets of points in H differing only in phase (i.e. the same quantum ray) will be mapped onto the same point in $\Pi(H)$. Thus, $\psi \mapsto \Pi(\psi) =: \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$. Now, the complex space \mathbb{C}^2 has already been identified with the algebra of quaternions \mathbb{Q} through the symplectic decomposition of an arbitrary quaternion $q \in \mathbb{Q}$ as $q = (q_0 + q_1i) + j(q_2 - q_3i) = q_\alpha + j\bar{q}_\beta$ $q_\alpha = (q_0 + q_1i), q_\beta = (q_2 + q_3i) \in \mathbb{C}$. The set of normalized quaternions i.e. quaternions with unit modulus get mapped into a sphere S^3 embedded in \mathbb{R}^4 . It, therefore, follows that normalized state vectors in \mathbb{C}^2 can also be canonically identified with the sphere S^3 embedded in \mathbb{R}^4 . Quotienting \mathbb{C}^2 by the equivalence relation $|\psi\rangle \sim |\phi\rangle$ iff $|\psi\rangle = e^{i\omega} |\phi\rangle$ to get the projective Hilbert space $\Pi(H) = H / \sim$, amounts to constructing the complex projective space $CP(1)$ i.e. $S^3/U(1)$ which yields the sphere S^2 usually referred to in the literature as the Bloch sphere. In other words, the geometry of the two level quantum system (qubits) can be conveniently represented by the Bloch sphere.

4. The Hopf Map

The identification of S^3 in \mathbb{R}^4 with the Bloch sphere (S^2) is done through the well studied Hopf map. As a by product of the Hopf analysis, one also recovers the association between the geometry of qubits [12-15] and quaternions. To construct the Hopf map, we recall that

the sphere S^3 is the group manifold of the special unitary group of matrices $SU(2)$ i.e. matrices with unit determinant that is isomorphic to the symplectic group $Sp(1)$ of transformations that preserve the quaternionic form. Elements on S^3 can be expressed in terms of quaternions $q \equiv (z_\alpha, z_\beta)$ through the symplectic decomposition $q = z_\alpha + j\bar{z}_\beta$,

$z_\alpha, z_\beta \in \mathbb{C}$ or equivalently by matrices $q_m = \begin{pmatrix} z_\alpha & z_\beta \\ -\bar{z}_\beta & \bar{z}_\alpha \end{pmatrix}$ with $z_\alpha\bar{z}_\alpha + z_\beta\bar{z}_\beta = 1$ for, writing

$z_\alpha = q_0 + iq_1$, $z_\beta = q_2 + iq_3$, we obtain $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. confirming that $q \equiv (z_\alpha, z_\beta)$ lies on the sphere S^3 .

To obtain explicit expressions for the Hopf map, we make use of the canonical representation of the quaternion units by the well known Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 =$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $i \equiv -i\sigma_1$, $j \equiv -i\sigma_2$, $i \equiv -i\sigma_3$. In terms of these matrices,

acting as the basis, the Hopf mapping is defined by $\mathbf{x} = \pi(q) = \begin{pmatrix} \bar{z}_\alpha & \bar{z}_\beta \end{pmatrix} \sigma \begin{pmatrix} z_\alpha & z_\beta \end{pmatrix}^T$ yielding

$$\mathbf{x} = (\bar{z}_\beta z_\alpha + \bar{z}_\alpha z_\beta, i(\bar{z}_\beta z_\alpha - z_\beta \bar{z}_\alpha), |z_\alpha|^2 - |z_\beta|^2)$$

$$= (2(q_0q_2 + q_1q_3), 2(q_0q_3 - q_1q_2), q_0^2 + q_1^2 - q_2^2 - q_3^2).$$

Let us take an element of the unitary group $U(1)$, say, $\varphi = \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix} = \lambda\mathbf{I} + \mu\sigma_3$. We,

then, have $\pi(q\varphi) = (q\varphi)^\dagger \sigma q\varphi = \varphi^\dagger \mathbf{x} \varphi = \mathbf{x}$ confirming, thereby that $\pi(q) = \pi(q\varphi)$ for $\varphi \in U(1)$ and hence, establishing the projective nature of the Hopf map taking all elements of S^3 connected through a unitary transformation to a single image. The image set is confirmed to be S^2 since $\mathbf{x}^2 = 1$ as can be easily verified. Thus, the Hopf map creates a principal bundle structure for S^3 with the base manifold being S^2 and the fibres being circles S^1 (members of the unitary group $U(1)$).

To obtain the local charts and the transition functions for the Hopf map, we parameterize the sphere S^3 by the stereographic projection coordinates. Let (X, Y) be the stereographic projection coordinates of a point in the southern hemisphere U_S of S^2 from the North Pole. Consider a complex plane that contains the equator of S^2 . Then, $Z = X + iY$ lies within the circle of unit radius on the plane. Further, from the standard expressions for stereographic coordinates, we have $Z = \frac{x_1 + ix_2}{1 - x_3} = \frac{q_0 - iq_1}{q_2 - iq_3} = \frac{\bar{z}_\alpha}{\bar{z}_\beta}$. The projective nature of the Hopf map again manifests itself here as the invariance of Z under the transformation $(z_\alpha, z_\beta) \rightarrow (\lambda z_\alpha, \lambda z_\beta)$ for $|\lambda| = 1$. Similarly, the stereographic coordinates of (U, V) of a point in the northern hemisphere U_N with respect to the South Pole will be given by $W = U + iV = \frac{\bar{z}_\beta}{\bar{z}_\alpha}$.

We can, now, define the fibre bundle structure of the Hopf map. The local trivializations in the northern and southern hemisphere are respectively given by:-

$$(i) \phi_N^{-1} : \pi^{-1}(U_N) \rightarrow U_N \times U(1) \text{ by } (z_\alpha, z_\beta) \mapsto \left(\frac{\bar{z}_\beta}{\bar{z}_\alpha}, \frac{z_\alpha}{|z_\alpha|} \right)$$

$$(ii) \phi_S^{-1} : \pi^{-1}(U_S) \rightarrow U_S \times U(1) \text{ by } (z_\alpha, z_\beta) \mapsto \left(\frac{\bar{z}_\alpha}{\bar{z}_\beta}, \frac{z_\beta}{|z_\beta|} \right)$$

(Both these trivializations are well defined on the respective charts for, in the northern hemisphere $z_\alpha \neq 0$ and in the southern hemisphere $z_\beta \neq 0$).

(iii) On the equator, $x_3 = 0$ so that $|z_\alpha| = |z_\beta| = 2^{-1/2}$, whence, on the equator, the local trivializations become $\phi_N^{-1} : (z_\alpha, z_\beta) \mapsto \left(\frac{\bar{z}_\beta}{\bar{z}_\alpha}, \sqrt{2}z_\alpha \right)$ and $\phi_S^{-1} : (z_\alpha, z_\beta) \mapsto \left(\frac{\bar{z}_\alpha}{\bar{z}_\beta}, \sqrt{2}z_\beta \right)$ leading to the equatorial transition function $t_{NS} = \frac{z_\alpha}{z_\beta}$.

5. The Geometry of Two Qubit States & Quantum Entanglement

The Hopf map described above can easily be generalized to $\pi : S^7 \rightarrow S^4$. This motivates us to examine the geometry of a two qubit quantum state using the formalism of the Hopf map. However, when addressing multiple qubit states, one needs to carefully consider the issue of quantum entanglement. The “quaternions” again come in handy in studying the two qubit state.

The Hilbert space for the compound system H will be the tensor product of the individual Hilbert spaces H_A, H_B of the two qubits and the basis vectors will be the direct product of the bases of the two spaces. We can, therefore, write a pure state of a two qubit system as $|\Phi\rangle = \alpha|00\rangle + \beta|01\rangle + \chi|10\rangle + \delta|11\rangle$ where $|ij\rangle \equiv |i\rangle \otimes |j\rangle$, $|i\rangle \in H_A$, $|j\rangle \in H_B$, $\alpha, \beta, \chi, \delta \in \mathbb{C}$, $\alpha = \alpha_{Re} + i\alpha_{Im}$, $\beta = \beta_{Re} + i\beta_{Im}$, $\chi = \chi_{Re} + i\chi_{Im}$ and $\delta = \delta_{Re} + i\delta_{Im}$, $|\alpha|^2 + |\beta|^2 + |\chi|^2 + |\delta|^2 = 1$. This normalization condition translates to a sphere S^7 embedded in \mathbb{R}^8 . Now, if the two qubit state is a composition of two one qubit states, then it should be possible to write the composite state as the tensor product of the two single qubit states. Writing $|\phi\rangle_A = a_1|0\rangle_A + a_2|1\rangle_A$, $|\phi\rangle_B = b_1|0\rangle_B + b_2|1\rangle_B$, we have, for separable states $|\Phi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B = a_1b_1|00\rangle + a_1b_2|01\rangle + a_2b_1|10\rangle + a_2b_2|11\rangle$ whence, the separability condition can be inferred as $\alpha\delta - \beta\chi = 0$.

To introduce the Hopf fibration $\pi : S^7 \rightarrow S^4$ through the quaternions, we write the probability amplitudes $\alpha, \beta, \chi, \delta \in \mathbb{C}$ in the form of two quaternions using the symplectic decomposition as $q_1 = \alpha_{Re} + \alpha_{Im}\mathbf{i} + \beta_{Re}\mathbf{j} + \beta_{Im}\mathbf{k}$ and $q_2 = \chi_{Re} + \chi_{Im}\mathbf{i} + \delta_{Re}\mathbf{j} + \delta_{Im}\mathbf{k}$. Obviously, the normalization condition implies that $|q_1|^2 + |q_2|^2 = 1$. Parametrizing the sphere S^4 as $\sum_{l=1}^5 \xi_l^2 = 1$, we obtain the Hopf map $\pi : S^7 \rightarrow S^4$ by the mapping $\xi_1 = Q_0$, $\xi_2 = Q_1$, $\xi_3 = Q_2$, $\xi_4 = Q_3$ and $\xi_5 = \sqrt{(1 - |Q|^2)}$ where $\pi(q_1, q_2) = Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k} = 2(\overline{q_1 q_2})$. Explicit computation using the values of the quaternions q_1 and q_2 yield

$$\xi_1 = 2(\alpha_{Re}\chi_{Re} + \beta_{Re}\delta_{Re} + \alpha_{Im}\chi_{Im} + \beta_{Im}\chi_{Im})$$

$$\xi_2 = 2(\alpha_{Re}\chi_{Im} - \alpha_{Im}\chi_{Re} + \beta_{Re}\delta_{Im} - \beta_{Im}\delta_{Re})$$

$$\xi_3 = 2(\alpha_{Re}\delta_{Re} - \alpha_{Im}\delta_{Im} - \beta_{Re}\chi_{Re} + \beta_{Im}\chi_{Im})$$

$$\xi_3 = 2(\alpha_{Re}\delta_{Im} + \alpha_{Im}\delta_{Re} - \beta_{Re}\chi_{Im} - \beta_{Im}\chi_{Re})$$

$$\xi_5 = 1 - 2|q_1q_2|$$

The Hopf map $\pi : S^7 \rightarrow S^4$ is equivalent to the mapping of S^7 onto a fibre bundle with the base space being the unit sphere S^4 and the fibres being spheres S^3 (this is evidenced by the invariance of this map under the transformation $(q_1, q_2) \mapsto (\lambda q_1, \lambda q_2), |\lambda| = 1$)

A perusal of the above expressions reveals an intriguing feature of the Hopf map. If the two qubit states are separable i.e. $\alpha\delta - \beta\chi = 0$, then $\xi_3 = \xi_4 = 0$ and the base space reduces to S^2 which is the Bloch sphere discussed in the earlier section of this manuscript. This Bloch sphere (the base space) constitutes the state space of one of the qubits of the two qubit separable system. The obvious question to be posed, then is – What about the state space of the other qubit of this separable system? A possible solution is to introduce a second Hopf map that fibres out the fibrings of the first Hopf map. As mentioned earlier the fibres of the map $\pi : S^7 \rightarrow S^4$ consist of spheres S^3 attached to the base space S^4 . By means of another Hopf map $\pi' : S^3 \rightarrow S^2$ we can further, fibrate the fibres of the first map into a base space (the two sphere S^2) and fibres (being the one dimensional sphere). This creates another Bloch sphere that can be considered as the state space of the second qubit in the two qubit separable composite system. It needs be emphasized here that such a construction is not permissible in an entangled system because of the non vanishing of the coordinates ξ_3, ξ_4 .

Conclusion

It is shown that the “quaternions” provide an attractive and efficient machinery to study the geometry of the one qubit and two qubit systems. One is led to the conclusion, through the Hopf map $\pi : S^3 \rightarrow S^2$, that the one qubit system has a geometrical representation as the Bloch sphere S^2 which the base space of a principal bundle with fibres consisting of the one dimensional sphere S^1 . In the case of the two qubit composite system, a similar over fibration $\pi : S^7 \rightarrow S^4$ implies that the system has the geometry of a fibre bundle with the base space being the four dimensional sphere S^4 fibres consisting of S^3 . As a fallout of the Hopf map analysis, we also find that unentangled two qubit systems admit a geometry as a direct product of two Bloch spheres as is intuitively to be expected. However, the Bloch sphere corresponding to one of the qubits in an unentangled system must be extracted from the S^3 fibres of the $\pi : S^7 \rightarrow S^4$ by invoking a second Hopf fibration of these S^3 fibres as $\pi : S^3 \rightarrow S^2$.

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Constructible Models of Orthomodular Quantum Logics

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Abstract: We continue in this article the abstract algebraic treatment of quantum sentential logics [39]. The Notions borrowed from the field of Model Theory and Abstract Algebraic Logic - *AAL* (i.e., consequence relation, variety, logical matrix, deductive filter, reduced product, ultraproduct, ultrapower, Frege relation, Leibniz congruence, Suszko congruence, Leibniz operator) are applied to quantum logics. We also proved several equivalences between state property systems (Jauch-Piron-Aerts line of investigations) and *AAL* treatment of quantum logics (corollary 18 and 19). We show that there exist the uniquely defined correspondence between state property system and consequence relation defined on quantum logics. We also signalize that a metalogical property - Lindenbaum property does not hold for the set of quantum logics.

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Keywords: Abstract Algebraic Logic (*AAL*); Model Theory; Consequence Relation; Logical Matrix; Sasaki Deductive Filter; State of Experimental Provability; State Property System; Orthogonality Relation; Lindenbaum Property

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1. Introduction

Quantum logics (just like classical logic) can be considered as a kind of propositional logic. A set of formulae of quantum sentential logics constitutes a complete formal description of physical systems. They describe the quantum entity in the terms of its actual and potential properties – or dually – in terms of its states [1].

The general idea of quantum logics is based on the isomorphism relation between the set of self-adjoint projection operators defined on a Hilbert space and the set of properties of physical system. The set of all self-adjoint projection operators defined on

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a Hilbert space form – in the algebraic terms – the orthomodular lattice. Above idea can be traced back to the work of von Neumann and G. Birkhoff [2]. In our considerations concerning the foundations of quantum mechanics, we will follow the approach developed by *Geneva-Brussels School* of quantum logic.

There exist two different and competitive ways of understanding the notion of logic. Historically speaking, the old style is to understand a logic as a set of valid formulae (these formulae are also forced to satisfy certain presupposed conditions, for instance the invariance under substitutions). In this case one can identify a logic \mathcal{S} with a set of theorems [20]. The second manner of conceiving logic \mathcal{S} is to define this concept as a consequence relation between sets of formulae and the formula denoted by $\vdash_{\mathcal{S}}$. In this case, a set of formulae is also forced to fulfill a set of certain specific conditions, for example the invariance under substitutions or finitariness. The consequences of the empty set of assumptions are called theorems and they constitute a logic in the old style. Above sketched second definition of logic is called *Tarski style* and belongs to the heritage of *Lvov-Warsaw School of Logic* [20]. This view constitutes the basis for the development of the so-called *Abstract Algebraic Logic* [21]. This kind of research is preferred especially by algebraically oriented logicians. In this paper, we follow this path of investigations.

Modern scientists – mainly theoretic physicists – are interested not only in one description of quantum (or cosmological) phenomena, but they are going to construct a whole set of possible models which correspond to possible pathway of the evolution of the investigated system. This model-theoretic approach is widespread among contemporary scientists and is advised by methodologists and philosophers of science [14, 15]. Basing on above hints concerning the qualitative face of investigations, one can get the complete knowledge indicating the possible ways of the evolution of the investigated physical system. Above methodological requirements prompted us to use the model-theoretic approach in the investigating of the realm of quantum logics.

This article tries to explore the models of quantum logics. In case of classical logic, and more popular and widespread non-classical logics (e.g., intuitionistic, modal and many-valued logics), the model-theoretic problems are well understood and deeply elaborated. However in the case of quantum logics, our knowledge concerning the possible models of this sentential logics is very poor [39]. This article is planned to bridge this gap.

Firstly, we define quantum sentential logic as an absolutely free algebra (section 2). We also define structural consequence operations on this algebra (section 2). The main results of this paper are included in section 3 – 7 where we construct several models of quantum logics and give main theorems characterizing these models. The section 8 is devoted to concluding remarks.

2. Preliminary Remarks

All algebras which are considered in this paper have the signature $\langle \mathbf{A}, \leq, \cap, \cup, \mathbf{0}, \mathbf{1} \rangle$ and are of similarity type $\langle 2, 2, 1, 0, 0 \rangle$. All abstract algebras, such as algebraic structures, are labeled with a set of boldface complexes of letters beginning with a capitalized Latin

characters, e.g., $\mathbf{A}, \mathbf{B}, \mathbf{Fm}, \dots$ and their universes by the corresponding lightface characters A, B, Fm, \dots . All our classes of algebra are varieties (varieties of algebras are defined as an equationally definable classes of algebras closed under formation of Cartesian products, ultraproducts, subalgebras and homomorphic images [3]). The fact that the given class of algebras \mathbf{K} is equationally definable means that there exists a set of equations Σ which are satisfied by all members of the class \mathbf{K} [21].

Definition 1. An orthomodular lattice is an algebraic structure $\mathcal{U} = \langle \mathbf{A}, \leq, \cap, \cup, (\cdot)', \mathbf{0}, \mathbf{1} \rangle$ if it satisfies the following conditions:

1) $\langle \mathbf{A}, \leq, \cap, \cup, \mathbf{0}, \mathbf{1} \rangle$ is a bounded lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$.

2) $(\cdot)'$ is a unary antitone and an idempotent operator (called orthocomplementation) on \mathbf{A} which satisfies the following conditions:

a) for any $x \in A, x'' = x$

b) for any $x, y \in A, \text{if } x \leq y \text{ then } y' \leq x'$

c) for any $x \in A, x \cap x' = \mathbf{0}$

3) orthomodular law.

We also supposed that all orthomodular lattices considered here are complete. When one removes the orthomodular law from the above definition, one gets the definition of ortholattice. All classes of algebras we mention here are varieties being subvarieties of \mathbf{OL} (the variety of all ortholattices). One can symbolically depict the relation between algebraic structures which are mentioned in this paper as follows:

$$\mathbf{BA} \subseteq \mathbf{MOL} \subseteq \mathbf{OML} \subseteq \mathbf{OL}.$$

Above abbreviations mean: \mathbf{BA} – the variety of all Boolean algebras, \mathbf{MOL} – the variety of all modular ortholattices, \mathbf{OML} – the variety of all orthomodular lattices, \mathbf{OL} – the variety of all ortholattices.

Undoubtedly, one can define many other subvarieties of \mathbf{OL} , but these algebraic structures are not mentioned here.

In our investigations, we work in the frame of binary orthologic introduced by Goldblatt [24]. The definition of binary orthologic corresponding to the \mathbf{OL} variety can be found in our previous paper [39]. The reader can also find there the listed axiom schemes and inference rules for this logic. The definitions of orthomodular logics (\mathbf{OML}) and the modular orthologic (\mathbf{MOL}) are also included in [39].

In our investigation of different models of quantum propositional logics, we follow the path taken by algebraically oriented logicians. We use the definition of the sentential language as an absolutely free algebra [38, 40, 41, 39]. \mathbf{Fm} denotes the algebra of formulae which is supposed to be absolutely free algebra of type \mathbf{L} over a denumerable set of generators $Var = \{p, q, r, \dots\}$. The set of free generators is identical with the infinite countable set of propositional variables. Inductive definition of formula describing quantum entities can be found in [39].

The algebra of terms \mathbf{Fm} is endowed with finitely many finitary operations (in sentential language – connectives) F_1, F_2, \dots, F_n . The structure $Fm = \langle \mathbf{Fm}, F_1, F_2, \dots, F_n \rangle$ is called the algebra of formulae – or equivalently – the algebra of terms [40, 41, 21]. It was stated explicitly in our previous paper that the notion of quantum logic can be identified with the structural consequence operation [29, 39]. The concept of logic or – more generally – the concept of deductive system in the language of type \mathbf{L} is defined as a pair $\mathcal{S} = \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ where \mathbf{Fm} is the algebra of formulae of type \mathbf{L} , and $\vdash_{\mathcal{S}}$ is a substitution-invariant consequence relation on \mathbf{Fm} . More precisely, the consequence relation is defined as a: $\vdash_{\mathcal{S}} \mathcal{P}(Fm) \times Fm$ satisfying the formal conditions stated in [38, 40, 41, 39]. ($\mathcal{P}(Fm)$ denotes the power set of Fm). We also explicitly postulate that for every $X \subseteq Fm$ and every $\alpha \in Fm$, the subsequent equivalence holds:

$$X \vdash_{Cn} \alpha \text{ iff } \alpha \in Cn(X).$$

In our paper it is supposed that all considered logics are finite, i.e., structural consequence operations are finitary [39].

By a model for quantum sentential logics we mean a couple $\mathcal{M} = \langle \mathbf{A}, F \rangle$ where \mathbf{A} is an algebra of the same similarity type as the algebra of terms of a given propositional language, and F is a subset of the universe of the algebra \mathbf{A} , i.e., $F \subseteq A$, and F is called the set of designated elements of \mathcal{M} . The structure $\mathcal{M} = \langle \mathbf{A}, F \rangle$ is termed logical matrix and can be understood as a semantical model of the given sentential logic. The notion of logical matrix is regarded as a fundamental notion of *Abstract Algebraic Logic* [38, 40, 41, 21]. Every logical matrix consists of an algebra which is homomorphic with the algebra of formulae of a considered propositional logic. Logical matrices adequate (see part 5) for quantum logics are formed of a variety of \mathbf{OL} or \mathbf{OML} . These varieties are considered as canonical classes of homomorphic algebras forming logical matrices. To every formula φ of the language of quantum logic, one can ascribe a unique interpretation in the algebra \mathbf{A} which depends on the values in \mathbf{A} that are assigned to variables of this formula [38, 40, 41].

Since \mathbf{Fm} is absolutely free algebra freely generated by a set of variables (i.e., the set of free generators) and \mathbf{A} is an algebra of the same similarity type as \mathbf{Fm} , then there exists a function $f : Var \rightarrow A$ and exactly one function $h^f : Fm \rightarrow A$ which is the extension of the function f , i.e., $h^f(p) = f(p)$ for each $p \in Var$. Above function is the homomorphism from the algebra of the terms into the algebra \mathbf{A} constituting logical matrix $\mathcal{M} = \langle \mathbf{A}, F \rangle$ [40, 41, 21].

Using logical matrix as a basic tool in the algebraic treatment of logic, one can identify the interpretation of a given formula φ of Fm with $h(\varphi)$ where h is a homomorphism from \mathbf{Fm} to \mathbf{A} that maps each variable of φ into its algebraic counterpart, i.e., into its assigned value. If we represent a formula of quantum logic in the form $\varphi(x_0, x_1, \dots, x_{n-1})$ in order to indicate that each of its variables occur in the list x_0, x_1, \dots, x_{n-1} then $\varphi^A(a_0, a_1, \dots, a_{n-1})$ denotes the algebraic translation of this formula for a given homomorphism $h(\varphi)$ such

that $h(x_i) = a_i$ for all $i < \omega$. Considering a quantum logic \mathcal{S} in the language of the type \mathbf{L} , we can say that matrix $\mathcal{M} = \langle \mathbf{A}, F \rangle$ is a semantic model of \mathcal{S} iff for every $h \in \text{Hom}_{\mathcal{S}}(\mathbf{Fm}, \mathbf{A})$ and every $\Gamma \cup \{\varphi\}$:

$$\text{If } h[\Gamma] \subseteq F \text{ and } \Gamma \vdash_{\mathcal{S}} \varphi \text{ then } h(\varphi) \in F.$$

In this case, the set F is called a *deductive filter* of the logic \mathcal{S} – or alternatively – *Sasaki deductive filter* of this logic [35, 38, 40, 41, 4]. By $h \in \text{Hom}_{\mathcal{S}}(\mathbf{Fm}, \mathbf{A})$ we mean a homomorphism from the algebra of formulae into the variety of algebra constituting logical matrix for quantum logics. For a given quantum logics one can define a whole set of Sasaki deductive filters. This set is partially ordered (by the set-theoretic relation of inclusion) and is denoted by $\mathcal{F}i_{\mathcal{S}}\mathbf{A}$. The class of logical models (i.e., logical matrices) for quantum propositional logics is denoted by \mathbf{ModS} [21, 39].

As a starting point of our investigations in this paper we assume the corollaries included in [39]. The strong version of the consequence operation is determined by the class of models of quantum logics as follows [24]:

$$\Gamma \vdash_{\mathcal{S}} \varphi \text{ iff } \forall \mathbf{A} \in \mathbf{OML}, \forall h \in \text{Hom}(\mathbf{Fm}, \mathbf{A}), \forall a \in A \text{ if } a \leq h(\beta), \forall \beta \in \Gamma \text{ then } a \leq h(\varphi).$$

Corollary 2. *The class of matrices:*

$$\mathbf{ModS} = \{ \langle \mathbf{A}, [a] \rangle : \mathbf{A} \in \mathbf{ModS}, a \in A \}.$$

is a matrix semantics for the strong version of quantum logic. $[a]$ is a principal filter of the form $\{x \in A : x \geq a\}$ [24].

In this paper our attention will be focused mainly on above defined Sasaki deductive filters. If it is not stated otherwise F denotes Sasaki deductive filter of the form $[a] = \{x \in A : x \geq a\}$ [29, 39].

Corollary 3. *The class of matrices:*

$$\mathbf{ModS} = \{ \langle \mathbf{A}, \{1\} \rangle : \mathbf{A} \in \mathbf{ModS} \}.$$

is a matrix semantics for the weak version of quantum logic [26, 29, 39].

The Sasaki deductive filters defined by these versions of quantum logics are one-element subsets of \mathbf{OML} , i.e., $F = \{1\}$ [26]. This kind of Sasaki deductive filters will be mentioned only occasionally.

3. Simple Models for Quantum Sentential Logics

By a simple model for quantum sentential logics we mean an ordered pair $\mathcal{M} = \langle \mathbf{A}, F \rangle$ where \mathbf{A} denotes variety of algebra associated with algebra of formulae of quantum logics, i.e., the variety of **OL** or **OML**, and F denotes the Sasaki deductive filter of this algebra (also called a deductive filter). It was mentioned in the previous section that by a logic one can understand a structural consequence operation. This is a purely logical definition of a deductive system. Nevertheless, in case of quantum logics (identified with structural consequence operations defined on an algebra of formulae expressing properties of a given quantum system), there exists also the physical interpretation of such conceived notion of logic (i.e., the structural consequence operation).

In the realm of quantum mechanics, the rays of a Hilbert space are understood as a mathematical representation of (pure) states of a physical system [18]. In this paper we supposed that there exists a bijection between rays of the Hilbert space (formal representation of quantum entity) and the structural consequence operations defined on a corresponding orthomodular lattice (i.e., the lattice of the properties of quantum entity). Basing on the excellent paper of K. Engesser and D. M. Gabbay, [18] one can assume that the physical state of a quantum system can be understood as a “*state of provability*” or more adequately as a “*state of experimental provability*”. Such conceived correspondence between the logical notion of structural consequence operation and physical concept of state can be simply illustrated. Let x denote pure state, A and B denote two observables, for instance energy and momentum of an elementary particle. It is not supposed that observables must be “*sharp*” in x . It is said that a given observable with a value λ in the state x is sharp if a measurement yields the value λ with probability equal 1. Our considerations are conducted in the language containing atomic formulae which have the following meaning: $A = \lambda, B = \rho, \dots$. It is supposed that observable A is not sharp in state x . By α we denote the proposition $A = \lambda$, and by β the proposition $B = \rho$. We measure A and the outcome is equal λ . If we end up our measurement (experiment), then the quantum entity is in a state y ($x \xrightarrow{A} y$) in which observable A is sharp (projection postulate of quantum mechanics). In the state y , observable B is sharp with value ρ (subsequent assumption). Shortly, it can be said “*if in the state x a measurement of A yields λ , then, after measurement, the system is in a state in which observable B is sharp with value ρ* ” [18]. Symbolically it can be expressed: $\alpha \vdash_x \beta$.

The relation \vdash_x is considered as a consequence relation since it has all formal properties of consequence operator (the whole example is borrowed from [18]).

4. Model-Theoretic Operations on Single Models

As it was explained in the first part of this article, by a simple model for quantum sentential logics we mean an ordered pair $\mathcal{M} = \langle \mathbf{A}, F \rangle$ where \mathbf{A} is a homomorphic algebra with regard to a quantum sentential language, and F is a Sasaki deductive filter of this algebra. Basing on the classical results from *Model Theory* obtained by Tarski,

Malcev, Robinson, Łoś, Chang, Keisler and other ([38, 28, 36, 27, 7]) one can define several different constructible models adequate for quantum sentential logics and operation on them.

Let $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ and $\mathcal{N} = \langle \mathbf{OML}, G \rangle$ be similar matrices. Suppose that F and G are two Sasaki deductive filters. A mapping $h : \mathcal{M} \rightarrow \mathcal{N}$ is called a matrix homomorphism from \mathcal{M} into \mathcal{N} , symbolically $h \in Hom_S(\mathcal{M}, \mathcal{N})$, when:

$$\text{if } a \in OML \text{ and } a \in F \text{ then } h(a) \in G.$$

One-to-one matrix homomorphisms are called isomorphic embeddings. When an isomorphic embedding h is onto, then h is termed an isomorphism. If $h \in Hom_S(\mathcal{M}, \mathcal{N})$ is onto, then \mathcal{N} is called a matrix homomorphic image under h . We use the notation $\mathcal{M} \cong \mathcal{N}$ when matrices \mathcal{M} and \mathcal{N} are isomorphic.

Let $\mathcal{M} = \langle \mathbf{OML}^{\mathcal{M}}, F \rangle$ and $\mathcal{N} = \langle \mathbf{OML}^{\mathcal{N}}, G \rangle$ be similar matrices. \mathcal{M} is said to be a *submatrix* (*submodel*) of \mathcal{N} (in symbol $\mathcal{M} \subseteq \mathcal{N}$) if $\mathbf{OML}^{\mathcal{M}}$ is a subalgebra of $\mathbf{OML}^{\mathcal{N}}$ and $\mathcal{M} = \mathbf{OML}^{\mathcal{M}} \cap \mathcal{N}$.

Let $\mathcal{M}_i = \langle \mathbf{OML}_i, F_i \rangle, i \in I$, be a family of similar matrices. By the *direct product* of matrices $\mathcal{M}_i, i \in I$, we understand the matrix $\prod_{i \in I} \mathcal{M}_i = \langle \mathbf{OML}, F \rangle$ where $OML =$

$$\prod_{i \in I} OML_i \text{ is the direct product of algebras, } i \in I, \text{ and } F = \prod_{i \in I} F_i, \text{ i.e., is the direct}$$

product of Sasaki deductive filters. The elements of the set $\prod_{i \in I} OML_i$ are denoted by

$$\langle f(i) : i \in I \rangle, \langle g(i) : i \in I \rangle \text{ if all matrices } \mathcal{M}_i \text{ are the same, then } \prod_{i \in I} \mathcal{M}_i \text{ is called a } \textit{direct}$$

power of \mathcal{M} . It is denoted by \mathcal{M}^I .

Considering classes of algebras and classes of logical matrices we may introduce the standard class operator symbols $\mathbb{I}, \mathbb{H}, \overleftarrow{\mathbb{H}}, \mathbb{S}, \mathbb{P}, \mathbb{P}_S, \mathbb{P}_U, \mathbb{P}_R, \mathbb{P}_{R_m}$. They means, respectively, for the formation of isomorphic and homomorphic images, homomorphic counterimages, subalgebras, direct and subdirect products, and ultraproducts. $\mathbb{P}_R, \mathbb{P}_{R_\sigma}$ stand for the reduced products and σ -reduced products, respectively, where σ is a regular cardinal number [34]. The class of all matrix/algebraic homomorphic counterimages of member of \mathbf{K} (i.e., the class of algebras or logical matrices) is defined:

$$\mathcal{M} \in \overleftarrow{\mathbb{H}}(\mathbf{K}) \text{ iff there exists a matrix } \mathcal{N} \in \mathbf{K} \text{ and a matrix homomorphism } h : \mathcal{M} \rightarrow \mathcal{N}.$$

Additionally, the class operator $\mathbb{U} = \mathbb{U}_{Var}$ is defined:

$$\mathbb{U}(\mathbf{K}) = \{ \mathbf{A} : \text{ every subalgebra of } \mathbf{A} \text{ generated by } \leq |Var| \text{ free generators belongs to } \mathbf{K} \}.$$

Definition 4. The class of **OML** algebras is termed **ISP**–*class* if it is closed under \mathbb{I} , \mathbb{S} and \mathbb{P} [40, 41, 34].

ISP–class is termed a **UISP**–class if it is closed under \mathbb{U} . This is a quasivariety if it is closed under \mathbb{P}_U and a variety if it is closed under \mathbb{H} . For every class **OML** of the orthomodular algebras it follows that:

$$\mathbb{ISP}(\mathbf{OML}) \subseteq \mathbb{UISP}(\mathbf{OML}) \subseteq \mathbb{ISPP}_U(\mathbf{OML}) \subseteq \mathbb{HSP}(\mathbf{OML})$$

These symbols stand for the smallest **ISP**–class, the smallest **UISP**–class, the smallest quasivariety and the smallest variety containing **OML**, respectively.

Applying the standard procedure of the models' construction, one can also define *reduced products* of elementary matrices. If $\{\mathcal{M}_i\}_{i \in I}$ is an indexed family of matrices of the same type, $\mathcal{M}_i = \langle \mathbf{OML}_i, F_i \rangle$ and ∇ is a filter (proper filter) over the set of indexes I , then the reduced product of matrices $\{\mathcal{M}_i\}_{i \in I}$ modulo ∇ is denoted by

$$\prod_{i \in I} \mathcal{M}_i / \nabla$$

More precisely, the matrix $\prod_{i \in I} \mathcal{M}_i$ is defined in the following manner. On the Cartesian product $\mathcal{C} = \prod_{i \in I} \mathbf{OML}_i$, we define the relation $=_{\nabla}$ of ∇ –equivalence by the condition: for $f, g \in \mathcal{C}$, $f =_{\nabla} g$ iff $\{i \in I : f(i) = g(i)\} \in \nabla$. The relation of ∇ –equivalence is a congruence of the algebra $\prod_{i \in I} \mathbf{OML}_i$. It follows from the definition that $\prod_{i \in I} \mathcal{M}_i / \nabla = \langle \mathbf{OML}_{\nabla}, F_{\nabla} \rangle$ where $\mathbf{OML}_{\nabla} = \prod_{i \in I} \mathbf{OML}_i / \nabla$ and $F_{\nabla} = \prod_{i \in I} F_i / \nabla$. The members of $\prod_{i \in I} \mathcal{M}_i$ are denoted by f_{∇}, g_{∇} or $\langle f(i) : i \in I \rangle_{\nabla}, \langle g(i) : i \in I \rangle_{\nabla}$. If $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, then the reduced product may be written $\prod_{i \in I} \mathcal{M} / \nabla$ or simply \mathcal{M}^I / ∇ and is called the reduced power of $\{\mathcal{M}_i\}_{i \in I}$ modulo ∇ .

If filter ∇ is an *non-principal ultrafilter* over I , denoted by \mathcal{U} , then $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ is termed the ultraproduct of matrices $\{\mathcal{M}_i\}_{i \in I}$. If $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, then the ultraproduct may be written $\prod_{i \in I} \mathcal{M} / \mathcal{U}$ or simply $\mathcal{M}^I / \mathcal{U}$. We suppose that this ultrafilter is non-principal and countably incomplete. The ultrafilter \mathcal{U} defined on the set of natural numbers is termed *countably incomplete* if there is a sequence of elements of \mathcal{U} satisfying for every $J \in \mathcal{U}$:

$$J_1 \supseteq J_2 \supseteq \dots, \bigcap_{k=1}^{\infty} J_k = \emptyset.$$

Theorem 4 (cf. [40, 41]). For each standard consequence operation defined on the quantum sentential language, the class $\text{Matr}(C)$ is closed under \mathbb{I} , \mathbb{S} , \mathbb{P} , \mathbb{H} , \mathbb{H}^C , \mathbb{P}_R and \mathbb{P}_U .

Proof: see [40, 41]. \square

In above theorem, the symbol $Matr(C)$ denotes the algebraic semantics for quantum logics. A good behaviour of a given logic – from semantical point of view – is often indicated by stipulation that this logic must satisfy the so-called *Czelakowski's theorem* ([40, 12]).

Theorem 5 ([12, 40]). *Let Cn be a standard consequence operation, and let $Cn = C_{\mathcal{M}}$ for some matrix semantics \mathbf{K} . Then:*

$$Matr(C) = \overleftarrow{\mathbb{H}}\mathbb{H}SP_{R_\sigma}(\mathbf{K}).$$

Moreover, if Cn is finitary and $\sigma = \aleph_0$ then

$$Matr(C) = \mathbb{H}_C\mathbb{H}SP_R(\mathbf{K}) = \overleftarrow{\mathbb{H}}\mathbb{H}SP_U(\mathbf{K}).$$

Proof: see [40, 41]. \square

5. Adequacy of Single Logical Matrices for Quantum Logics

As it was stated in the author's previous paper, all logical matrices constituting a model for quantum sentential logics determine not only the set of their own tautologies, but mainly the so-called matrix consequence operation – $C_{\mathcal{M}}$ [39].

For all logical matrices $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ and for arbitrary $X \subseteq Fm$ and $\alpha \in Fm$, the operation $C_{\mathcal{M}}$ is defined:

$$\alpha \in C_{\mathcal{M}}(X) \leftrightarrow (h(X) \subseteq F \rightarrow h(\alpha) \in F) \text{ where } h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML}).$$

Such operator $C_{\mathcal{M}}$ can be understood as a structural consequence operation. Basing on above considerations, one can generalize the notion of $C_{\mathcal{M}}$ and introduce the operator $C_{\mathbf{K}}$. The symbol \mathbf{K} denotes the class of matrices. The operator $C_{\mathbf{K}}$ is defined: for arbitrary $X \subseteq Fm$ and for arbitrary $\alpha \in Fm$ it is the case that:

$$\alpha \in C_{\mathbf{K}}(X) \text{ iff } \forall \mathcal{M} \in \mathbf{K} (\alpha \in C_{\mathcal{M}}(X)).$$

Above introduced operator $C_{\mathbf{K}}$ is named the consequence operator determined by the class \mathbf{K} of matrices. The consequence operators $C_{\mathcal{M}}$ constitute the complete lattice. In the lattice-theoretic term, the operator $C_{\mathbf{K}}$ can be defined as follows:

$$C_{\mathbf{K}} = \inf\{C_{\mathcal{M}} : \mathcal{M} \in \mathbf{K}\}.$$

Definition 6 ([27, 40, 41]). The class \mathbf{K} of matrices is termed *adequate* for sentential calculus iff for arbitrary $X \subseteq Fm$ and $\alpha \in Fm$ subsequent conditions are satisfied:

$$\alpha \in Cn(X) \text{ iff for every } \alpha \in C_{\mathbf{K}}(X).$$

or shortly:

$$Cn = C_{\mathbf{K}}$$

Definition 7 ([40, 41]). Logical matrix is termed *Cn-matrix* if for every set of formulae $X \subseteq Fm$ it follows that:

$$Cn(X) \subseteq C_{\mathcal{M}}(X).$$

Such matrix \mathcal{M} is called *Cn-matrix* if the consequence operator determined by this matrix - $C_{\mathcal{M}}$ - is not weaker than consequence operator Cn . Symbolically:

$$Cn \leq C_{\mathcal{M}}.$$

In [39] several algebraic and semantical conditions were presented in the form of theorems so that the subsequent equality for quantum logic was satisfied:

$$Cn = C_{\mathcal{M}}$$

Such posed question concerning the sentential logics belongs to the core problems of *Abstract Algebraic Logic* and was studied from the early beginnings of this branch of logic. In modern terminology, above sketched problem can be expressed as follows: To give necessary and sufficient conditions (having syntactical and algebraic characters) which must be satisfied by a given logic $\langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$ in order to indicate a single matrix which is *strongly adequate* for this logic. A matrix is termed *strongly adequate* for a given logic if the following equality is satisfied:

$$Cn = C_{\mathcal{M}}.$$

The first investigations into the problem of the adequacy of logical matrices for quantum sentential logics were carried out by Malinowski [29]. Historically speaking, the problem of the existence of strongly adequate models for a given propositional logics can be regarded as a generalization of the problem of the so-called *weak adequacy* for logics. This topic constitutes the problem of finding a single matrix – for a given structural consequence operation Cn – so that the subsequent equality was satisfied :

$$Cn(\emptyset) = E(\mathcal{M}).$$

In this notation $E(\mathcal{M})$ is a set of logical tautologies determined by a given logical matrices. Matrix which satisfied above equality is termed *weakly adequate* matrix for a given sentential logic. Every weakly adequate logical matrix for quantum sentential logic determines the set of tautologies of this logic. The formula which are satisfied in a matrix under the given homomorphism are denoted by $Sat_h(\mathcal{M})$.

In the case of quantum logic, subsequent equalities take place:

$$\alpha \in Sat_h(\mathcal{M}) \leftrightarrow h(\alpha) \in F.$$

$$Sat_h(\mathcal{M}) = h^{-1}(F).$$

The set of formulae which are satisfied under the homomorphism h is a counterimage of a set of designated values (in this case, the only designated value is $\mathbf{1}$) with regards to this homomorphism. The tautologies of quantum logics are identified with a set of formulae which are satisfied for every valuations (i.e., for every homomorphisms) of sentential variables of the term algebra – \mathbf{Fm} . Above set is designated by $E(\mathcal{M})$. The following equality takes place:

$$E(\mathcal{M}) = \bigcap_h Sat_h(\mathcal{M}) \text{ where } h \in Hom_S(\mathbf{Fm}, \mathbf{OML}).$$

Basing on *Geneva-Brussels* approach to the foundation of quantum mechanics, the subsequent theorem can be deduced:

Theorem 8. *In the case of quantum sentential logics weakly adequate matrices (i.e., the sets of formulae determined by these matrices) can be identified with the so-called trivial question.*

Definition 9 ([31, 37]). *Trivial question* in the framework of *Geneva-Brussels* paradigm is identified with the following definite experimental procedure: "Do whatever you wish with the system and assign the response "yes"" [31].

Above experimental situation also encompasses doing nothing with the physical system. We can call this experimental procedure certain iff the physical entity exists. The trivial question is true always when we are certain of obtaining the positive answer (i.e., "yes") were we to perform this question. The only condition – and indeed, *ontological (existential) condition* – of the trivial question is that we have a physical system to begin with.

Proof of the theorem 8. From the definition of weakly adequate matrices it follows that for all $\alpha \in Sat_h(M) \leftrightarrow h(\alpha) \in F$ for all $\alpha \in Sat_h(\mathcal{M}) \longleftrightarrow h(\alpha) \in F$ for all

$h \in Hom_S(\mathbf{Fm}, \mathbf{OML})$. Now, consider two arbitrarily chosen Sasaki deductive filters determined by the strong version of quantum logic, i.e., they have the form:

$F_1 = [a_1] = \{x_1 \in OML : x_1 \geq a_1\}$ and $F_2 = [a_2] = \{x_2 \in OML : x_2 \geq a_2\}$ (Corollary 2). From these definitions of filters it is obvious that they must have *at least* one common element, i.e., top element of \mathbf{OML} . This top element is identified with $\mathbf{1}$. Hence, in order to be sure that a given formula is always true we choose such homomorphisms $h \in Hom_S(\mathbf{Fm}, \mathbf{OML})$ that $h(\alpha) = \mathbf{1} \in F_i$ for arbitrary i . Such defined formula α is a trivial question in the sense of definition 9. \square

Theorem 10. *There must exists at least one quantum entity.*

Proof : Above theorem belongs to the so-called ontological presuppositions of quantum logics. The tautologies (i.e., $Cn(\emptyset) = E(\mathcal{M})$) of classical logic are satisfied even in the empty domain. Since tautologies of quantum logics are satisfied under the presuppositions that there exists *at least* one quantum entity to answer the trivial question positively, i.e., $h(\alpha) \in F$. \square

The problem of ontological assumptions of quantum logics will be discussed in full elsewhere.

In this article it is only signalized that such model-theoretic constructions as reduced products and ultraproducts can be used to describe not only separated quantum entities but also entangled ones.

6. Pasting of Single Models of Quantum Logics

Subsequent model-theoretic construction useful in the studying of quantum sentential logics is the so-called $\{0, 1\}$ -pasting (Bruns and Kalmbach 1971, 1972, Ptak and Pulmannova 1991, Miyazaki 2005)[5, 6, 33, 30].

Definition 11 ([30]). Let $\mathcal{A} = \langle \mathbf{A}, \leq, \cap, \cup, (\cdot)', \mathbf{0}_{\mathcal{A}}, \mathbf{1}_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle \mathbf{B}, \leq, \cap, \cup, \mathbf{0}_{\mathcal{B}}, \mathbf{1}_{\mathcal{B}} \rangle$ are two non-trivial orthomodular lattices. By $\{0, 1\}$ -pasting of these lattices one

understands the structure $\mathcal{A} + \mathcal{B} = \langle (\mathbf{A} \cup \mathbf{B}) / \equiv, \leq, \cap, \cup, \mathbf{0}, \mathbf{1} \rangle$ where \equiv is an equivalence relation defined $\equiv =_{df} \{(x, x) \mid x \in A \cup B\} \cup \{(\mathbf{0}_A, \mathbf{0}_B), (\mathbf{0}_B, \mathbf{0}_A), (\mathbf{1}_A, \mathbf{1}_B), (\mathbf{1}_B, \mathbf{1}_A)\}$. The relation of order \leq and other operations $\cap, \cup, (\cdot)'$ are inherited from original orthomodular lattices \mathcal{A} and \mathcal{B} .

In the literature one can find two alternative concepts naming this construction: $\{0, 1\}$ -pasting ([5, 6]) and the term ‘horizontal sum’ ([33]). It is a well known facts, from the theory of orthomodular lattices that $\{0, 1\}$ -pasting of Boolean algebras is an orthomodular lattice [30]. The horizontal sum of finitely or infinitely many ortholattices or orthomodular lattices - $\sum \mathcal{A}_i$ - is defined in a similar way. Basically, for given orthomodular lattices \mathcal{A} and \mathcal{B} , $\{0, 1\}$ -pasting $\mathcal{A} + \mathcal{B}$ is an orthomodular lattice where $\mathbf{0}_A$ and $\mathbf{0}_B$ are identical to the new smallest element $\mathbf{0}_{A+B}$ and $\mathbf{1}_A$ and $\mathbf{1}_B$ are identical to the new largest element $\mathbf{1}_{A+B}$. Other elements are the same as in the original lattices \mathcal{A} and \mathcal{B} .

From the definition of variety it does not follow that variety must be closed with regards to the operation of $\{0, 1\}$ -pasting. In fact, there are varieties which are closed under this operation and varieties which are not closed with regards to horizontal sum.

Observation 12 ([30]). *The variety OML is closed under $\{0, 1\}$ -pasting. As it can be deduced from the figure below, neither variety MOL nor BA is closed under $\{0, 1\}$ -pasting.*

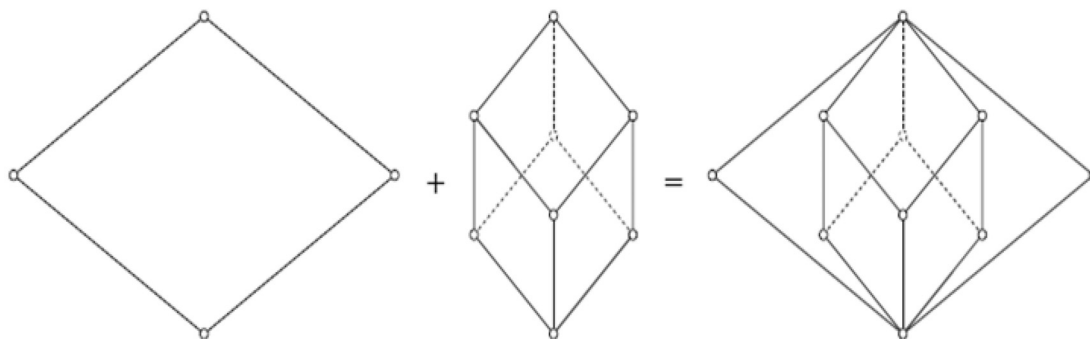


Fig. 1 $\{0, 1\}$ -pasting of two Boolean algebras ([30]).

7. Congruences of Orthomodular Lattices and State Property System

The set of all congruences of an algebra \mathbf{A} is denoted by $Co\mathbf{A}$. Considering the set of all congruences of the orthomodular lattices $CoOML$ it is stated that this set constitutes a distributive and Brouwerian lattices [8].

Definition 13 ([12, 22]). Let $\theta \in CoOML$ and $F \subseteq OML$. It is defined that θ is compatible with F , symbolically $\theta comp F$ when for all $a, b \in F$, if $\langle a, b \rangle \in \theta$ and $a \in F$ then $b \in F$.

The congruence θ is compatible with F iff F is a union of equivalence classes of θ [12, 21, 22]. Above relationship can be also expressed using the projection mapping $\pi : \mathbf{OML} \rightarrow \mathbf{OML}/\theta$, it is the case that $\theta \text{comp} F$ iff $F = \pi^{-1}[G]$ for some $G \subseteq A/\theta$ [21, 22]. Congruences of an algebra \mathbf{OML} compatible with F are also termed congruences of the matrix $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ (or alternatively – the strict congruences of $\mathcal{M} = \langle \mathbf{OML}, F \rangle$). Above introduced the projection mapping π is canonical surjective homomorphism. When θ is compatible with F it can be assumed that:

$$F = \{a/\theta : a \in F\}.$$

The largest congruence of \mathbf{OML} which is compatible with F can be always indicated. This congruence is called the *Leibniz congruence* of the matrix $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ and is denoted by $\Omega_{\mathbf{OML}}F$ (the notion of the Leibniz congruence belongs to the field of Abstract Algebraic Logic and is also used when other algebras (logics) are considered) [12, 21, 22]. In the general case, the Leibniz congruence is denoted by $\Omega_{\mathbf{A}}F$. The congruences of the matrix constitute the *principal ideal* of the lattice *CoOML* generated by $\Omega_{\mathbf{OML}}F$. The matrix $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ is called *reduced* – or *Leibniz reduced* – when its Leibniz congruence is the identity on \mathbf{OML} , i.e., $\Omega_{\mathbf{OML}} = id$. For an arbitrary matrix $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ its *reduction* is equivalent to its quotient by its Leibniz congruence, i.e., the matrix of the form $\mathcal{M}^* = \langle \mathbf{OML}/\Omega_{\mathbf{OML}}F, F/\Omega_{\mathbf{OML}}F \rangle$. The definition of the Leibniz congruence is absolutely independent of any logic (i.e., structural consequence operation). It is *intrinsic* to \mathbf{OML} and F [12, 21, 22].

The class of reduced matrix models of a logic \mathcal{S} is symbolized by $\mathbf{Mod}^*\mathcal{S}$. The class of algebraic reducts of the reduced models of \mathcal{S} i.e., the class of algebras that is associated with a logic \mathcal{S} is denoted by $\mathbf{OML}^*\mathcal{S}$ (in a general case : $\mathbf{A} \text{lg}^*\mathcal{S}$) [12, 21, 22]. Formally, the class $\mathbf{OML}^*\mathcal{S}$ can be defined as follows:

$$\mathbf{OML}^*\mathcal{S} = \{ \mathbf{OML} : \exists F \in \text{Fi}_{\mathcal{S}}\mathbf{OML} \text{ and } \Omega_{\mathbf{OML}}F = id \}.$$

The subsequent useful tool to study logical matrices for quantum logics is a *Frege relation* of a matrix $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ relative to the logic \mathcal{S} [12, 21, 22]. This relation is denoted by $\Lambda_{\mathcal{S}}^{\mathbf{OML}}F$ and is defined on \mathbf{OML} by:

$$\Lambda_{\mathcal{S}}^{\mathbf{OML}}F = \{ \langle a, b \rangle \in \mathbf{OML} \times \mathbf{OML} : \forall G \in \text{Fi}_{\mathcal{S}}\mathbf{OML}, F \subseteq G \rightarrow (a \in G \leftrightarrow b \in G) \}.$$

Above relation means that $\langle a, b \rangle \in \Lambda_{\mathcal{S}}^{\mathbf{OML}}F$ iff a and b belong to the same Sasaki deductive filter of an algebra \mathbf{OML} which include F . Alternatively, it can be expressed:

$$\langle a, b \rangle \in \Lambda_{\mathcal{S}}^{\mathbf{OML}}F \text{ iff } \text{Fi}_{\mathcal{S}}^{\mathbf{OML}}(F \cup \{a\}) = \text{Fi}_{\mathcal{S}}^{\mathbf{OML}}(F \cup \{b\}).$$

One can also introduce the notion of the *Suszko congruence* of the matrix $\mathcal{M} = \langle \mathbf{OML}, F \rangle$ relative to \mathcal{S} – it is the largest congruence included in $\Lambda_{\mathcal{S}}^{\mathbf{OML}} F$. The Suszko congruence is denoted by $\tilde{\Omega}_{\mathcal{S}}^{\mathbf{OML}}$. Formally, the Suszko congruence for every Sasaki deductive filter F on the algebra \mathbf{OML} is defined:

$$\tilde{\Omega}_{\mathcal{S}}^{\mathbf{OML}} F = \bigcap \{ \Omega_{\mathbf{OML}} G : G \text{ is a Sasaki deductive filter of } \mathbf{OML} \text{ and } F \subseteq G \}.$$

Contrary to the Leibniz congruence, the notion of Suszko congruence *is not intrinsic* to \mathbf{OML} and F but it depends on the whole logic \mathcal{S} , i.e., all Sasaki deductive filters embracing a given F [12, 13, 21, 22]. Explicitely, it can be expressed that the Suszko congruence relative to a logic \mathcal{S} of a matrix $\langle \mathbf{OML}, F \rangle \in \mathbf{Mod}\mathcal{S}$ depends not only on the Sasaki deductive filter F but also on the whole family of these filters which include F :

$$[F]_{\mathcal{S}} = \{ G \in \mathcal{F}i_{\mathcal{S}}\mathbf{OML} : F \subseteq G \}.$$

This collection of the Sasaki deductive filters is a closure system (or closed-set system) on the universe of \mathbf{OML} . The Suszko congruence can be conceived as a function of the family of models for the quantum logics, i.e., $\{ \langle \mathbf{OML}, G \rangle : G \in [F]_{\mathcal{S}} \}$, or equivalently of the pair $\langle \mathbf{OML}, [F]_{\mathcal{S}} \rangle$ [21, 22].

In the operational approach to algebraic logic, the notion of Leibniz operator is introduced [12, 21, 22]. The Leibniz operator Ω is a function which assigns to each Sasaki deductive filter the largest congruence θ of the term algebra compatible with F . Compatibility of the largest congruence of \mathbf{OML} with arbitrary Sasaki deductive filter is defined as a congruence of \mathbf{OML} such that for all $a \in \mathbf{OML}$ we have:

$$\text{either } a/\theta \subseteq F \text{ or } (a/\theta) \cap F = \emptyset.$$

If two elements $a, b \in \mathbf{OML}$ are orthogonal (written $a \perp b$), i.e., $a \leq b'$, then they can not simultaneously belong to the same Sasaki deductive filter. Alternatively, there does not exist deductive filter (i.e., \mathcal{S} -theory) which embraces these two elements.

Theorem 14. *Let $a, b \in \mathbf{OML}$. It follows that:*

if $a \perp b$ then $\neg \exists F$ such that $a, b \in F$ where F is an arbitrary Sasaki deductive filter.

In words, there does not exist the deductive filter i.e., the logical theory, which simultaneously realizes two orthogonal properties.

Proof: Simple, from definitions. If $a \perp b$ then there does not exist congruence relation θ such that $\langle a, b \rangle \in \theta$ and θ would be compatible with F . \square

In the considerations of the foundation of quantum mechanics the problem of the orthogonal states arises. In the operational approach to orthogonality relation developed mainly in the *Geneva-Brussels School* the following definition can be formulated ([1]):

Definition 15 [37]. Two quantum states $p, q \in \Sigma$ are *orthogonal*, written $p \perp q$, if there exists a definite experimental project α such that α is certain for p and impossible for q .

In the seminal works of Aerts it was stated explicitly that the complete description of the quantum particle (the quantum entity) can be identified with the so-called *state property system*, i.e., the ordered triple $(\Sigma, \mathcal{L}, \xi)$ [1]. In this notation, Σ denotes the set of states, \mathcal{L} is a set of properties (the so-called property lattice) and ξ is a function from Σ to $\mathcal{P}(\mathcal{L})$. Conceptually, a state is an abstract name for a singular realization of the particular physical system [31]. Equivalently, a state (or more precisely a state of provability or a state of experimental provability, cf. section 3 of this article) can be identified with the consequence operation defined on the property lattice. Above equivalence can be deduced from the fact that according to the *Geneva-Brussels School*, a state is a dual notion with regards to the concept of property. To each state p we can associate the family $\xi(p)$ of all of its actual properties, and conversely, to each property a we can associate the family $\kappa(a)$ of all states in which this property is actual. In order to formulate above sketched duality, one can introduce the mapping $\xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$. The set of all properties which are actual in a given state $p \in \Sigma$ are denoted by $\xi(p) \in \mathcal{P}(\mathcal{L})$. Dually, if $(\Sigma, \mathcal{L}, \xi)$ is a state property system, then its *Cartan map* is the mapping $\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$ defined ([1]):

$$\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma) : a \rightarrow \kappa(a) = \{p \in \Sigma \mid a \in \xi(p)\}.$$

Theorem 16. *If the property lattice \mathcal{L} is atomistic and orthomodular, then to each state $p \in \Sigma$ one can attribute the unique Sasaki deductive filter (of above lattice). The mapping between the definite state p and F defined on the property lattice is injective.*

Proof and comments : Basing on the mapping $\xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$ in the *Geneva-Brussels School* notation, it can be deduced that a given state $p \in \Sigma$ is identified with the unique subset of property lattice, i.e., with the set of all properties which are actual in this state - $\xi(p)$. In our terminology, the set of all actual properties (or more precisely – their algebraic counterparts in orthomodular lattices) is identified with the Sasaki deductive filter defined on \mathcal{L} . Hence, for the definite state $p \in \Sigma$ there exists the unique Sasaki deductive filter of \mathcal{L} . Above defined correspondence is not surjective since not all possible Sasaki deductive filters must be realized by the considered quantum entity T . The injectivity of this correspondence derives from the fact that there does not exist two non-equivalent states p_1, p_2 such that $p_1 \neq p_2$ and $\xi(p_1) = \xi(p_2)$. \square

Corollary 17. *Any state p of the quantum entity T can be uniquely represented as a particular Sasaki deductive filter F defined on \mathcal{L} . Equivalently, any state p can be identified with a particular Sasaki deductive filter F defined on a term algebra. It can be expressed by the following equality:*

$$\xi(p) = F \subseteq OML.$$

Let us recall that a \mathcal{S} -theory of quantum logic is an arbitrary set of sentences describing the quantum entity of the fixed language. If this set is closed under a consequence operation C , i.e., if $X = C(X)$, or equivalently if $X = C(Y)$ for some Y , then X is called a \mathcal{S} -theory of C [38, 39]. In equivalent terminology, $C(X)$ is also called a deductive system or, simply, a system of C . $C(X)$ is the least \mathcal{S} -theory of C that contain X and $C(\emptyset)$ and is the system of all logically provable or - equivalently speaking - logically valid sentences of C . It can be stated that :

$$\text{if } \varphi \in C(X) \text{ then } \varphi \in X.$$

One can say that the *deductively closed set* $C(X)$ is termed a \mathcal{S} -theory. The set of all \mathcal{S} -theories of a given quantum logic is denoted by \mathbf{ThS} . This set of all \mathcal{S} -theories defined on one given logic is ordered by set-theoretic inclusion and it constitutes a complete lattice $\mathbf{ThS} = \langle \mathbf{ThS}, \cap, \cup \rangle$ [38, 12, 11, 39]. Considering an algebraic semantics for this quantum logic (i.e., logical matrices - $\mathcal{M} = \langle \mathbf{OML}, F \rangle$) it can be stated that any \mathcal{S} -theory has its algebraic counterpart in the form of Sasaki deductive filters $\{F_i\}_{i \in I}$ defined on \mathbf{OML} , i.e., property lattice \mathcal{L} . The different \mathcal{S} -theories correspond to different Sasaki deductive filters. Hence, if F has the form $[a] = \{x \in OML : x \geq a\}$ (corollary 2) then it corresponds to one \mathcal{S} -theory $C(X)$ defined in an orthomodular quantum logic \mathcal{S} . The set of all Sasaki deductive filters is denoted by $\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$ and if these filters have the form $[a]$ then their set constitute a complete lattice. If $X \subseteq OML$ then one can always indicates the least Sasaki deductive filter of \mathbf{OML} which contains X . This filter is *generated* by X and is denoted by $\mathcal{F}i_{\mathcal{S}}^{\mathbf{OML}}(X)$. The largest \mathcal{S} -theory is the set Fm of all formulae - and dually - the smallest \mathcal{S} -theory is the set of all \mathcal{S} -theorems (i.e., the formulae φ such that $\vdash_{\mathcal{S}} \varphi$, where $\vdash_{\mathcal{S}}$ means that $\emptyset \vdash_{\mathcal{S}} \varphi$). For any two \mathcal{S} -theories \mathcal{T}, \mathcal{S} we have $\mathcal{T} \cup \mathcal{S} = \bigcap \{\mathcal{R} \in \mathbf{ThS} : \mathcal{T} \cup \mathcal{S} \subseteq \mathcal{R}\}$. So it is possible to define a deductive system as the pair $\langle Fm, \mathbf{ThS} \rangle$.

Corollary 18. *The lattice of all \mathcal{S} -theories $\mathbf{ThS} = \langle \mathbf{ThS}, \cap, \cup \rangle$ defined on \mathbf{Fm} (i.e., on the term algebra describing quantum entity) is isomorphic to the lattice of all Sasaki deductive filters $\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$.*

Summing up above considerations (sections 3 and 7) one can claim that every quantum state (p) can be identified with the particular set of its actual properties i.e., $\xi(p)$.

These sets of actual properties are proper subsets of the property lattice \mathcal{L} . By an identification of \mathcal{L} with an algebraic semantics for orthomodular quantum logics, i.e., OML, one can deduce that above proper subsets of \mathcal{L} are exactly the Sasaki deductive filters of an algebra constituting above mentioned algebraic semantics for these logics. The deductive filters in Abstract Algebraic Logic (AAL) correspond to the deductively closed sets named \mathcal{S} -theories. Every deductively closed set, i.e., every \mathcal{S} -theory, corresponds to a quantum consequence operation C defined on a term algebra of quantum logics \mathcal{S} deeply studied in [24, 18, 29, 39]. An algebraic treatment of logical systems gives a general and uniform understanding of the deductive relationship between different terms and between sets of these terms [38, 28, 40, 41, 19, 21, 12, 39].

Hence, there exist the well-defined one-to-one correspondence between the different consequence operations defined on OML (which determine the Sasaki deductive filters - corollary 1 and 2) and the different \mathcal{S} -theories (i.e., different deductively closed sets on OML). Any structural consequence operation can be understood as a separate sentential logic. It brings about that one can say that there exist plenty of quantum logics on the same OML. Any quantum logic is identified with a separate deductive Sasaki filter. A consequence operation C defined on OML is additionally termed a structural consequence operation if C also satisfies the following condition:

$$e(C(X)) \subseteq C(e(X)) \text{ for } X \subseteq Fm.$$

Here, e denotes any substitution in the universe of the term algebra \mathbf{Fm} . From a purely algebraic point of view a substitution in quantum logic can be regarded as a function:

$$e : Var \rightarrow Fm.$$

Based on above fact and assuming that the algebra of terms is the free algebra this function e can be extended to an endomorphism:

$$h^e : Fm \rightarrow Fm.$$

Or under assumption that there exist the set of homomorphisms $h : Fm \rightarrow OML$ one can introduce a composition of two functions namely $h \circ e$ which is defined:

$$h \circ e : Fm \rightarrow OML.$$

In the author's opinion the process of identification of a single quantum state with one consequence operation on OML- or alternatively - with one Sasaki deductive filter

(or with one \mathcal{S} -theory) is a fundamental concept bringing together the logical notion of provability or deducibility with the physical notion of a quantum state. One can see that there exist the uniquely determined one-to-one correspondence between the Geneva-Brussel approach to the foundation of quantum theory and the above algebraic treatment of quantum sentential logic. In a common opinion the notion of logic understood as a structural consequence operation is the most important logical concept. Regarding logic as a structural consequence operation is the contribution of Lvov-Warsaw school of logic and initiates the development of the so-called abstract algebraic logic (AAL) and model theory of propositional logic.

One can state two following theorems:

Corollary 18. *A logical matrix (i.e., a logical model) constituting of **OML** and of Sasaki deductive filter F , i.e., $\mathcal{M} = \langle \mathbf{OML}, F \rangle$, can be understood as a particular realization of one quantum state $p \in \Sigma$.*

Corollary 19. *Following conditions are equivalent:*

a) *there is a one-to-one correspondence between the set of all quantum states Σ which are allowed for one quantum entity T and the family of all logical matrices $\mathcal{M}_i = \langle \mathbf{OML}, F_i \rangle$ (where $i \in I$) adequate for quantum logic describing this entity.*

b) *the family of all Sasaki deductive filters $\{F_i\}_{i \in I}$ is in a one-to-one correspondence with the set of all quantum states Σ which are allowed for this quantum entity T .*

c) *the set of all theories of quantum logic \mathcal{S} denoted by **ThS** is in a one-to-one correspondence with the set of all quantum states Σ which are allowed for this quantum entity T .*

d) *the set of all theories of quantum logic \mathcal{S} denoted by **ThS** is in a one-to-one correspondence with the closed set system defined on the property lattice \mathcal{L} .*

e) *if there exist a sequence (finite or infinite) of quantum states p_1, p_2, p_3, \dots describing the evolution of quantum entity then there exist the corresponding sequence of logical matrices (i.e., the models) $\mathcal{M}_1 = \langle \mathbf{OML}, F_1 \rangle, \mathcal{M}_2 = \langle \mathbf{OML}, F_2 \rangle, \mathcal{M}_3 = \langle \mathbf{OML}, F_3 \rangle, \dots$ which differ by their Sasaki deductive filters.*

f) *the lattices $\mathcal{F}_{i\mathcal{S}}\mathbf{OML}$ and $\mathbf{ThS} = \langle \mathbf{ThS}, \cap, \cup \rangle$ are isomorphic.*

Using above sketched formalism the theorem characterizing the orthogonal quantum states can be formulated (cf. definition 15):

Theorem 20. *If two quantum states $p, q \in \Sigma$ are orthogonal (i.e., $p \perp q$) then two Sasaki deductive filters F_p and F_q which correspond to these states have at most one common element. It means that their intersection constitute of a one-element set. This one-element set is $\mathbf{1}$ - the top element of **OML** lattice. Formally,*

$$F_p \cap F_q = \{\mathbf{1}\}.$$

Proof: Two Sasaki deductive filters which correspond to the orthogonal quantum states have the form

$$F_p = [a_p] = \{x_p \in \mathbf{OML} : a_p \leq x_p \leq \mathbf{1}_p\} \text{ and } F_q = [a_q] = \{x_q \in \mathbf{OML} : a_q \leq x_q \leq \mathbf{1}_q\}$$

where $\mathbf{1}_p$ and $\mathbf{1}_q$ are the maximal elements of these filters. Basing on the Zorn lemma it is obvious that $\mathbf{1}_p = \mathbf{1}_q$. It means that these two quantum states answer in the same manner to definite experimental project consisting only of a trivial question. \square

The orthogonality relation is symmetric and antireflexive [1, 37]:

$$\text{If } p \perp q \text{ then } q \perp p \text{ and } p \neq q.$$

It can be easily observed that a physical condition of symmetricity of this relation can be translated into the language of quantum logics in the form of a *filter distributivity property* of these deductive systems. The filter distributivity property is a metalogical property deeply investigated in *AAL* [12, 21, 22].

Proposition 21. *If the orthogonality relation between two different states is symmetric then two Sasaki deductive filters corresponding to these states commutes. It can be alternatively stated that the lattice of all Sasaki deductive filters ($\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$) which can be defined on the same \mathbf{OML} is distributive. The logic with this property is termed a filter-distributive logic.*

The class of the filter-distributive logics is very wide and includes also all orthomodular quantum logics. This property is shared by all those logics which have a disjunction. The fact that the lattice $\mathcal{F}i_{\mathcal{S}}\mathbf{OML}$ is distributive has its purely algebraic counterpart in the observation that \mathbf{OML} has the property of congruence-distributivity.

The Lindenbaum property states that any semantically consistent set of terms admits a semantically consistent complete extension [23]. It is well known that a \mathcal{S} –theory in *AAL* can be equivalently considered as a set of non-contradictory formulae. A complete \mathcal{S} –theory (denoted by \mathcal{T}) is characterized by the following formal condition:

$$\forall\beta(\beta \in \mathcal{T} \text{ or } \neg\beta \in \mathcal{T}).$$

The *Lindenbaum property* asserts that any \mathcal{S} –theory can be extended to a complete, maximal non-contradictory \mathcal{S} –theory. Formally:

$$\mathcal{T} \subseteq \mathcal{T}_{\max}.$$

In [23] it was proved that the orthomodular quantum logics does not satisfies the Lindenbaum property. It means that there does not exist a complete, maximal \mathcal{S} –theory formulated in the quantum logic language. Here we give an alternative proof for this fact.

Basing on the previous corollaries it is known that any \mathcal{S} –theory can be equivalently represented as a Sasaki deductive filters defined on \mathbf{OML} . Supposing that any formula of quantum logic is uniquely represented as a single element of \mathbf{OML} , formally:

$$h(\varphi) = a \in OML \text{ where } h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML}).$$

then a complete, maximal \mathcal{S} –theory of quantum logic is represented as a Sasaki deductive filter which is an ultrafilter.

Claim 22. In the case of the orthomodular quantum logics the Lindenbaum property is equivalent to the Ultrafilter lemma.

The Ultrafilter lemma asserts that every filter on a set X can be extended to some ultrafilter on X .

Basing on the claim 22 it can be deduced that a complete, maximal \mathcal{S} –theory on \mathbf{OML} is represented by a *Sasaki deductive ultrafilter* on \mathbf{OML} .

Theorem 23. *In the orthomodular quantum logics the Lindenbaum property does not hold - or equivalently - there does not exist the Sasaki deductive ultrafilters on \mathbf{OML} .*

Proof and comments: From the definition of an ultrafilter \mathcal{U} defined on a set X it follows that if A is a subset of X then either A or $X \setminus A$ is an element of \mathcal{U} . In the language of *AAL* it is equal to the fact that for any formula φ , φ or $\neg\varphi$ has its algebraic counterpart, i.e., a or a' , belonging to \mathcal{U} defined on \mathbf{OML} . Formally, if \mathcal{S} –theory is complete and maximal then

$$\forall\varphi(\varphi \in \mathcal{T} \text{ or } \neg\varphi \in \mathcal{T}).$$

Algebraically such \mathcal{S} –theory corresponds to an ultrafilter \mathcal{U} defined on \mathbf{OML} . Suppose that if $h(\varphi) = a$ where $h \in Hom_{\mathcal{S}}(\mathbf{Fm}, \mathbf{OML})$ then

$$a \in OML \text{ or } a' \in OML.$$

From the theorem 14 concerning the orthogonal properties we know that if $a \perp b$ are two orthogonal properties represented by two elements $a, b \in OML$ then there does not exist the Sasaki deductive filter embracing these two elements (a and a' are trivially orthogonal). Now choose such $a, b \in OML$ which are in the relation of non-trivial orthogonality, i.e., $a \perp b$ and $a' \neq b$ then there does not exist the Sasaki deductive ultrafilter embracing these two orthogonal elements, i.e.,

$$\neg\exists\mathcal{U} \text{ such that if } a \perp b \text{ then } a, b \in \mathcal{U} \text{ where } a' \neq b.$$

Undoubtedly, if a and b are trivially orthogonal, i.e., $a' = b$, then it is trivially true that there does not exist such \mathcal{U} that $a, a' \in \mathcal{U}$. \square

Subsequent property which can be expressed in the language of *AAL* applied to quantum logics is the property of equivalent quantum states:

Definition 24. We call states $p, q \in \Sigma$ *equivalent* and denote them by $p \approx q$ iff $\xi(p) = \xi(q)$.

In the *AAL* treatment we know that any quantum state may be identified with a single Sasaki deductive filter (corollary 19), i.e., $\xi(p) = F_p \subseteq OML$.

Hence, if we suppose that two states are equivalent, i.e., $\xi(p) = \xi(q)$, then their Sasaki deductive filters must be equivalent, i.e., $F_p = F_q$. From the fact that a deductive filter correspond to a deductively closed set of formulae we obtain the following relationship:

$$\text{if } p \approx q \text{ then } C(X_p) = C(X_q).$$

Above equality means that two set of terms $X_p, X_q \in Fm$ describing the properties of quantum entity are equivalent with respect to a given logic C if their closures, i.e., $C(X_p)$ and $C(X_q)$ are equal.

Concluding Remarks

Contrary to other non-classical logics (just like many-valued logics, modal logics and intuitionistic logics) quantum logic is not well elaborated from the view point of Abstract Algebraic Logic (*AAL*). This article constitutes a author's second attempt to applying a machinery of *AAL* and Model Theory to quantum logic and inference rules encountered in this logic [39]. We also shown that there exist a one-to-one correspondence between approach based on the notion of state property system (Jauch-Piron-Aerts line of investigations) and our attitudes which use the sophisticated tools derived from two core branches of modern mathematical logic – *AAL* and Model Theory.

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Quantum Size Effect of Two Couple Quantum Dots

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Abstract: The quantum transport characteristics are studied for double quantum dots encountered by quantum point contacts. An expression for the conductance is derived using Landauer - Buttiker formula. A numerical calculation shows the following features: (i) Two resonance peaks appear for the dependence of normalized conductance, G , on the bias voltage, V_0 , for a certain value of the inter barrier thickness between the dots. As this barrier thickness increases the separation between the peaks decreases. (ii) For the dependence of, G , on, V_0 , the peak heights decrease as the outer barrier thickness increases. (iii) The conductance, G , decreases as the temperature increases and the calculated activation energy of the electron increases as the dimension, b , increases. Our results were found concordant with those in the literature.

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1. Introduction

Interest in low dimensional quantum confined structures has been fueled by the richness of fundamental phenomena therein and the potential device applications [1-4]. In particular, ideal quantum dots can provide three-dimensional carrier confinement and resulting discrete states for electrons and holes [5]. Interesting electronic properties related to the transport of carriers through the bound states and the trapping of quasi-particles can be used to realize a new class of devices such as the single electron transistor, multilevel logic element, memory element, etc. [6-8]. Because of its high switching speed, low power consumption, and reduced complexity to implement a given function, resonant tunneling

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diodes (RTDs) have candidates for digital circuit application [9]. Quantum dots can be built as single-electron transistors [10], a charge Quantum bit and double quantum charge qubit [11, 12]. They can serve as artificial atoms (quantum dots) and artificial molecules (coupled quantum dots) [13-15]. In parallel to technological efforts aimed toward searching compact circuit architecture, great deal of attention has been dedicated to model and simulate RTDs [16-18], as a way to optimize device design and fabrication [19] and also to understand mesoscopic transport properties of these devices. In the present paper, it is desired to investigate the quantum size effect on the electron transport of mesoscopic devices whose dimensions are less than the mean free path of electrons. Such a device will be modeled as two series-coupled quantum dots based semiconductor - heterostructure separated by an inner barrier of width, c . These quantum dots are coupled weakly to two conducting leads via quantum point contact. As we shall see from the treatment of this model, that such device will operate as a resonant tunneling device.

2. The Model

The resonant tunneling device could be constructed as two series coupled quantum dots, each of diameter, a , and separated by an inner barrier of width, c . Also, these quantum dots are separated from two leads by outer tunnel barriers from the corresponding sides, each of width, b . Electron transport through these quantum dots could be affected by the phenomenon of Coulomb blockade [20]. An expression for the conductance, G , of the present device could be derived using Landauer- Buttiker formula [21]:

$$G = \frac{4e^2}{h} \int dE |\Gamma(E)| \left(-\frac{\partial f}{\partial E}\right) \sin \phi \quad (1)$$

Where $\Gamma(E)$ is the tunneling probability, ϕ is the phase angle of the tunneled electrons, h is Planck's constant and e is the electron charge. The derivative of the Fermi-Dirac distribution is given by:

$$-\frac{\partial f}{\partial E} = (4k_B T)^{-1} \cosh^{-2} \left[\frac{(E - E_F)}{2k_B T} \right] \quad (2)$$

Where E_F is the Fermi Energy, k_B is Boltzmann's constant and T is the absolute temperature. The tunneling probability, $\Gamma(E)$ of electrons through such device could be determined by using the method of a transfer matrix [22] as follows: The Schrödinger equation describing electron transport in the j^{th} region is given by [21]:

$$-\frac{\hbar^2}{2m^*} \frac{d^2 \psi_j}{dx^2} + \left(V_j + \frac{e^2 N^2}{4C} \right) \psi_j = E \psi_j \quad (3)$$

Where m^* is the effective mass of the electron, V_j is the potential energy of the j^{th} region, $e^2 N^2 / 4C$ is the charging energy of each quantum dot [18,20,21], in which C is its capacitance and N is the number of electrons entering each quantum dot. The eigenfunctions $\psi_j(x)$ in the j^{th} region corresponding to the Schrödinger equation (3) is expressed as [18, 21, and 23]:

$$\psi_j(x) = A_j \exp(ik_j \cdot x) + B_j \exp(-ik_j \cdot x) \quad (4)$$

Where:

$$k_j = \frac{[2m^*(E - V_j - e^2N^2/4C)]^{1/2}}{\hbar} \quad (5)$$

\hbar is Planck's constant divided by 2π . The coefficients A_j and B_j are determined by matching the wave functions ψ_j and their first derivatives at the subsequent interface. Now, using the transfer matrix, we get the coefficients A_j and B_j as:

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \prod_{j=1} R_j \begin{pmatrix} A_r \\ B_r \end{pmatrix} \quad (6)$$

Where the notations (l, r) denote left and, right regions. In Eq. (6), the coefficients R_j in the j^{th} region is given by:

$$R_j = \frac{1}{2k_j} \begin{pmatrix} (k_j + k_{j+1}) \exp[i(k_{j+1} - k_j)x_j] & (k_j - k_{j+1}) \exp[i(k_{j+1} + k_j)x_j] \\ (k_j - k_{j+1}) \exp[i(k_{j+1} + k_j)x_j] & (k_j + k_{j+1}) \exp[i(k_{j+1} - k_j)x_j] \end{pmatrix} \quad (7)$$

According to the present model of the device, the corresponding wave vectors in the regions where the barriers exist are given by:

$$\kappa = \frac{[2m^*(V_o + V_b + \frac{e^2N^2}{4C})]^{1/2}}{\hbar} \quad (8)$$

Where, V_b is the barrier height. And also, the wave vectors in the quantum dots are expressed as:

$$\kappa = \frac{[2m^*E]^{1/2}}{\hbar} \quad (9)$$

Now, the tunneling probability, $\Gamma(E)$, is given by solving Eq. (6) [22] and we get:

$$\Gamma(E) = \frac{1}{1 + A^2B^2} \quad (10)$$

Where:

$$A = \frac{(V_o + V_b + \frac{e^2N^2}{4C}) \cdot \sinh(\kappa b)}{(E(V_o + V_b + \frac{e^2N^2}{4C}) - E)^{1/2}} \quad (11)$$

and

$$B = D_1D_2 - \frac{\sinh(\kappa(2b - c))}{\sinh(\kappa b)} \quad (12)$$

in which the expression for D_1 and D_2 are:

$$D_1 = 2 \cosh(\kappa b) \cdot \cos(ka) - \frac{(2E - V_o - V_b - \frac{e^2N^2}{4C}) \sinh(\kappa b) \sin(ka)}{(E(V_o + V_b + \frac{e^2N^2}{4C}) - E)^{1/2}} \quad (13)$$

and

$$D_2 = 2 \cosh(\kappa c) \cos(ka) - \frac{\left(2E - V_o - V_b - \frac{e^2 N^2}{4C}\right) \sinh(\kappa c) \sin(ka)}{\left(E \left(V_o + V_b + \frac{e^2 N^2}{4C} - E\right)\right)^{1/2}} \quad (14)$$

Now, by substituting Eq. (10) for the tunneling probability, $\Gamma(E)$, into Eq. (1), taking into consideration eqs. (11-14), and performing the integration numerically (using Mathematica-4), one can calculate the conductance.

3. Numerical Calculation and Discussion

In order to show that the present mesoscopic junction operates as a resonant tunneling device, we perform a numerical calculation of the tunneling probability $\Gamma(E)$ (Eq.10).

1- Figure 1 shows the variation of the tunneling probability $\Gamma(E)$, with the bias voltage, V_o , in energy units (eV) which have two main resonant peaks at certain values of V_o voltage. These main peaks are due to the sequential resonant tunneling of the electron from the ground state of the 1st quantum dot to the 1st and the 2nd excited states of the adjacent quantum dot. It is noticed from figure (1) that the tunneling probability has different behaviors when the dimensions of the device [c, b, and a] are varied as follows:

- i) The peak separation decreases as the inter barrier width, c, increases and at certain value of c, we have only one peak, (see fig. 1-a).
- ii) The resonant peak heights decrease as the outer tunnel barrier width, b, increases without any shift in peak position (see fig. 1-b).
- iii) The peak heights decrease with an observable shift in peak positions to higher bias voltages as the diameter of the quantum dot, a, increases (see fig. 1-c). Behavior of the tunneling probability, $\Gamma(E)$, has been observed by other authors [16, 24].

2-a: Fig. 2 Shows the variation of the normalized conductance with the bias voltage, V_o , measured at different values of the inner barrier width between the two quantum dots, c. It is noticed from the figure that the peaks separation decreases as the barrier width, c, increases. At a certain value of, c, the two peaks becomes one. This may be attributed to the decrease in the degree of splitting of the conductance-energy state as the inner barrier width, c, increases and at a certain value of, c, the presented double quantum dot system becomes a system of two isolated quantum dots rather than coupled or superlattice system [25]. The same behavior is also noticed for the dependence of the conductance on the diameter of the dot, a, calculated at different values of the parameter, c.

b: The dependence of the conductance, G, on the bias voltage, V_o , at different values of the outer barrier width, b, between the reservoirs and the quantum dots is shown in fig. 3. No shift for the peak positions occurs, but the peak heights decrease as the value of the barrier width, b, increases. A similar behavior is also noticed for the dependence of the conductance on the diameter of the quantum dot, a, for different values of the outer barrier width, b.

c: The dependence of the normalized conductance, G, on the bias voltage, V_o , cal-

culated at different values of the diameter, a , is shown in fig. 4. It's noticed that the peak height is decreased as the diameter of the dot increases due to the coulomb blockade effect. Also, the peak heights shift to higher bias voltage as the diameter of the quantum dot increases. This behavior is concordant to that of the tunneling probability.

3: The dependence of the conductance, G , on the temperature, T , measured at different values of the barrier width, b , is shown in fig. 5. The conductance, G , decreases as the temperature increases. This agrees well with those published in literatures [26, 27, 28, and 29]. Also, the variation of, $\ln G$, versus, $1/T$, is plotted in fig. 6. By using the Arrhenius relation [$G = G_0 \exp(-E / k_B T)$], the activation energy of the electron is calculated and arranged in table 1. It's observed that the activation energy of the electron increases as the value of the dimension, b , increases. This increase in the activation energy is to overcome the increase in the resistance of the presented double quantum dot system as the value of, b , increases.

Table 1:

The activation energy of the electron (meV)	The value of b (nm)
0.314	1.0
0.385	1.10
0.405	1.15
0.410	1.20

It is seen from the results that the transmission spectrum is Lorentzian in shape for such present junction with multiple barrier structure. The features of confining effects at resonance levels are seen from our results. The dependence of the resonant level width on various parameters such as the quantum dot diameter and the two barrier width b , and c , are shown from our results which show the coupling effect between quantum dots. Our results are found concordant with those in the literature [30-32].

Conclusion

In this paper, we derived a formula for the conductance of two coupled quantum dots and analyzing its characteristic on the bias voltage, the barrier widths b , and c . We conclude from our results that this device operates as resonant tunneling device in the mesoscopic regime. Such quantum coherent electron device is promising for future high-speed nanodevices.

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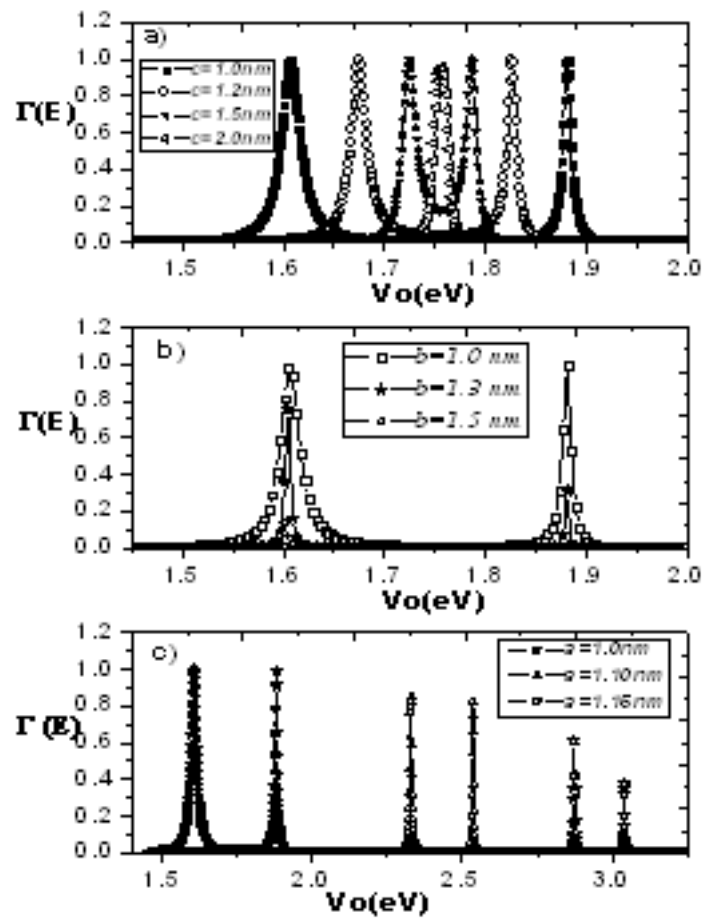


Fig. 1 The variation of the tunneling probability $\Gamma(E)$, with the bias voltage V_o (eV) at: a) different values of the inner barrier, c. b) different values of the outer barrier, b. c) different values of the diameter, a.

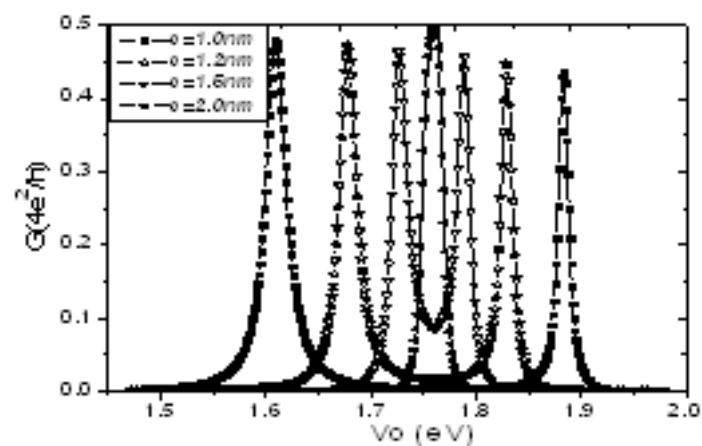


Fig. 2 The variation of the normalized conductance with the bias voltage, V_o (eV), at different values of the inner barrier width between the two dots, c.

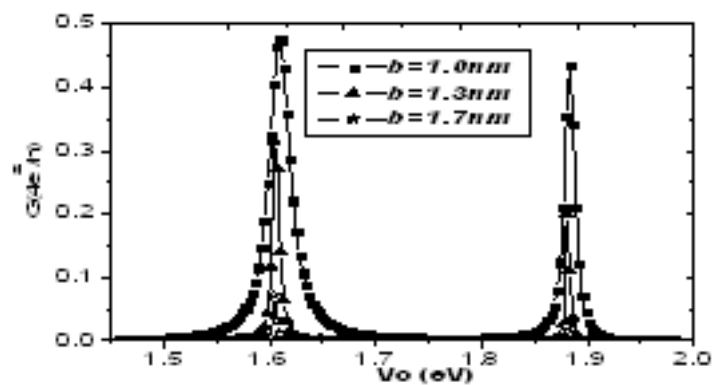


Fig. 3 The variation of the normalized conductance with the bias voltage, V_o (eV), at different values of the outer barrier width, b .

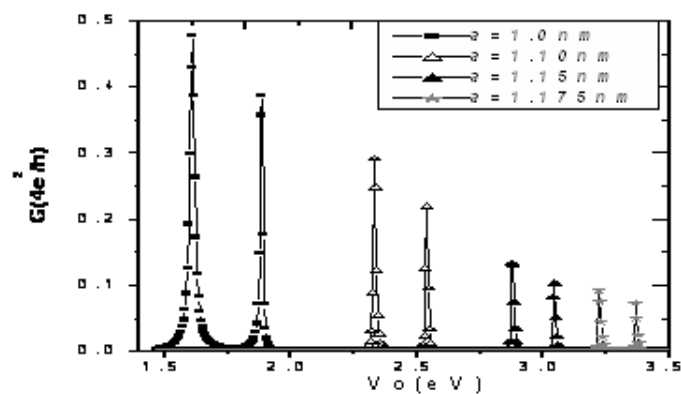


Fig. 4 The variation of the normalized conductance with the bias voltage calculated for different values of the diameter of the quantum, a .

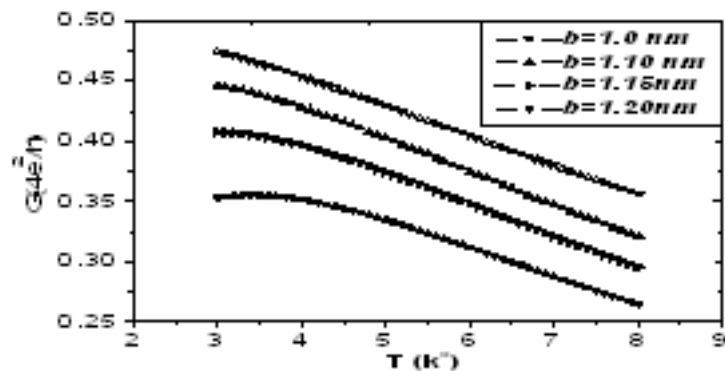


Fig. 5 The variation of the normalized conductance with the absolute temperature (K°) at different values of the outer barrier width, b .

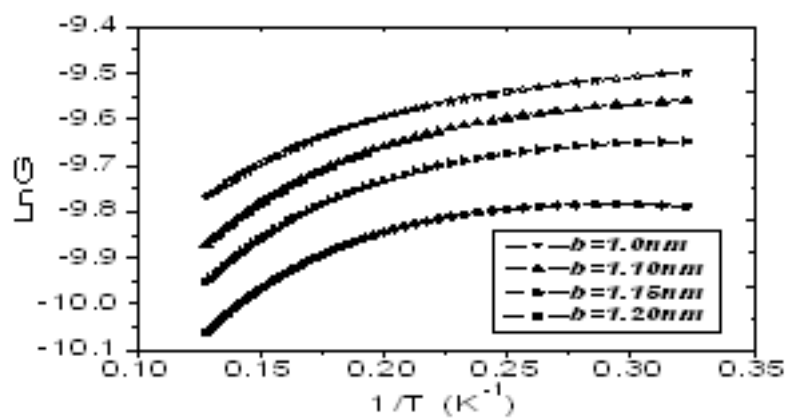


Fig. 6 The variation of the logarithm of the normalized conductance, $\text{Ln } G$, with the reciprocal of the absolute temperature, $1/T$, at different values of the outer barrier width, b .

Quantum Destructive Interference

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Abstract: An apparent paradox for unitarity non-conservation is investigated for the case of destructive quantum interference.

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Keywords: Quantum Mechanics; Quantum Interference; Wave-function

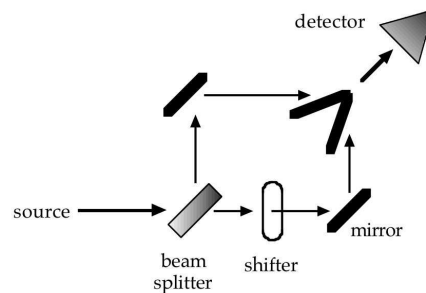
PACS (2008): 03.65.-w; 03.67.-a; 03.67.Hk; 42.25.Hz

1. Introduction

Destructive quantum interference might at first seem to potentially threaten unitarity since it seems to imply a reduction of the wave-function. Since this mechanism has been exploited for a variety of applications [1, 2, 3], this motivates a deeper investigation into the seeming paradox of unitary conservation.

2. Quantum Interference

An extreme example of destructive quantum interference is where two arms are brought into overlap, having first arranged for them to be in anti-phase



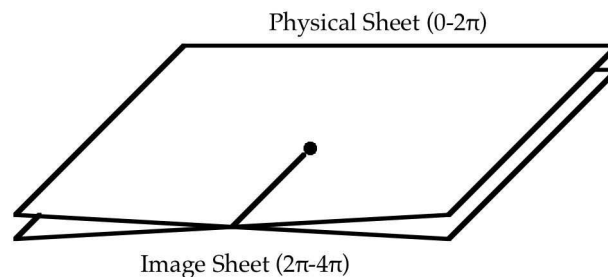
Destructive Interference

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In this case the destructive interference can be near complete in practice, and one might naturally wonder if this is in conflict with unitarity conservation.

2.1 Multi-valued wave-‘functions’

As is well known, expressions such as the square root are not functions as there is, in general, more than one result. In response to this dilemma one can either force the issue by defining the principle square root to be only one of the two valid outputs, or one can restore the single valuedness by defining it on a two sheeted Riemann surface.



Riemann double sheet

The interference situation being considered here is similar to the Aharonov-Bohm effect, and also lives on a Riemann surface due to the multi-valued nature of the situation; in this case the same two sheeted surface as for the square root.

Now the interfering arms on the physical (top) sheet combine destructively as

$$(|\psi\rangle - |\psi\rangle)/2$$

while on the lower (unphysical) sheet they combine constructively as

$$(|\psi\rangle + |\psi\rangle)/2$$

so preserving unitarity over the entire mathematical space; the normalization of $1/2$ is composed of $1/\sqrt{2}$ for each path, and $1/\sqrt{2}$ for each sheet.

More generally, for a phase shift of $2\pi \frac{n}{m}$ (n and m integers; m strictly positive), the sum of probabilities over the now m sheets becomes

$$\sum_{l=1}^m \left| \frac{1}{\sqrt{2m}} (1 + e^{2\pi i l \frac{n}{m}}) \right|^2$$

which when multiplied out yields

$$\frac{1}{2m} \sum_{l=1}^m (2 + e^{-2\pi i l \frac{n}{m}} + e^{2\pi i l \frac{n}{m}})$$

and is then seen to be identically equal to one since each exponent is the sum of vectors spread evenly around zero. This demonstrates that unitarity is indeed mathematically preserved over the whole Riemann surface.

Conclusion

The seeming paradox of lost unitarity is resolved by the realization that mathematically, over the whole Riemann surface, unitarity is actually preserved; while destructive interference can take place on the single physical sheet.

However, since the particle cannot actually be lost, the wave-function needs to be renormalized on the physical sheet, and this constitutes the mechanism that is being exploited for quantum computation and communication [1, 2, 3].

The use of Riemann surfaces to carry physical problems can be of great utility [4].

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Quantized Fields Around Field Defects

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Abstract: A heuristic exercise exploring analogies between different field theories. Similarities between the crystal defects and other various fields help to create a model to quantize these fields. The charge of the electromagnetic field, and the electromagnetic waves are used as examples.

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Keywords: Quantum Field Theory; Electrodynamics; Magnetic Monopoles

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1. Introduction

1.1 Physical Introduction

We can observe similarities between the phenomena appearing in solid states and in electrodynamics or particle physics. In the crystal lattice of the solid states various kinds of crystal defects can exist, for instance the line-defects such as the edge dislocations and the screw dislocations. [1][2] Around the crystal defects, the bulk material is intact all over, and inside these materials we can use such simple formulas which become unusable and invalid at the place of the figurable lattice defects. Drawing a parallel between these, we will consider a field in which in the "solid" domain we can use some rule, except of some "figurable" field defects. The "solid" domain works in a way, that the initial conditions and the boundary conditions determines the inside of the domain. On the other hand, the "figurable" domain can easily fit every kind of initial and boundary conditions, and the continuation in time and space is hardly predictable. Of course there are some rules existing in this domain too, but applying these rules, we get near chaos results. If the boundary of this kind of "solid" field is over determined, then it will inevitably contain "figurable" field defects. Since the "figurable" field defects have very

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unpredictable behavior, we will try to characterize them using the "solid" field encircling them.

1.2 Mathematical introduction

1.2.1 Quantized integrals

Let there be a field described by two physical properties: C , S . and let there be a soft constraint condition between them:

$$C^2 + S^2 = 1 \quad (1)$$

(I use the expression: "soft constraint", if the constraint criterion is not too stiff, and valid only mostly and approximately.) $C^2 + S^2$ is mainly equal to 1 but not allways. In some regions, the equation is not valid. We can close out the most invalid points of our region, so we will use multiply-connected regions, where the constraint-criterion is approximately valid. The C and S properties depends on each others. In a simply-connected region, we can describe our field with a phase of a vawe. Lets be $C = \cos(\phi)$ and $S = \sin(\phi)$. We know the two originally used properties, and we want to calculate the phase. We will create a vector from the following determinants :

$$v_i = Det \begin{bmatrix} C; \partial_i C \\ S; \partial_i S \end{bmatrix} \quad (2)$$

The determinants are created from the properties of the field and its partial derivatives with respect to space variables.

$$v_i = Det \begin{bmatrix} \cos(\phi); -\sin(\phi)\partial_i\phi \\ \sin(\phi); \cos(\phi)\partial_i\phi \end{bmatrix} = (\cos^2(\phi) + \sin^2(\phi))\partial_i\phi = \partial_i\phi \quad (3)$$

We can calculate the phase difference between two points:

$$\Delta\phi = \int v_i dx^i = \int \partial_i\phi dx^i \quad (4)$$

In a simply-connected region, if we calculate the integral on a closed curve, we will get zero, but in the case of multiply-connected regions we will get quantized results. The result can be $2\pi * n$, where n is a positive or negative or zero whole number.

1.2.2 Additions of fields with defects

We have defects, and we would like to have the field of the sum of the defects. If we are in the case of line surrounded defects, and our trajectory in the mapped field consists of

only a single loop, let's create two times two rotational operators for the mapped space. These operators are constitute closed group [3].

$$\underline{U} = \begin{bmatrix} C; & -S \\ S; & C \end{bmatrix} \quad (5)$$

The field with two defects can be generated from the single fields multiplying by the operators which represent the single fields. The result represents a field with multiple defects. We can repeat this procedure several times, so we can create fields with multiple defects. Of course we can multiply the result with the operators of the defectless field. In this two dimensional case the operators are commutative. In general: If we can project the subspace of the soft constraint condition to a grid, where the cell of the grid corresponds to the single encircling, then we can use vectors in the space of the grid. The defects are similar to the well known up and down counting operators [4].

2.

2.1 Line surrounded defects.

If we are on a surface, we can walk around a point-defect always staying inside the "solid" field. If we are in a three-dimensional space, we can walk around a line-defect on a curved line, while if we are in a time and space, we can walk around a surface or a moving curve. We will try to create properties similar to the Burgers-vectors [5] in this field. We would like any optional closed curve integrals which walk around inside the "solid" field to be quantized, and in a defectless case the result must be zero. Let there be a field described by a vector composed of two physical properties. It creates a two dimensional mapped space.

$$\underline{\Psi} = (A, B) \quad (6)$$

and let there be a soft constraint condition between them:

$$C(\underline{\Psi}) = 0 \quad (7)$$

$$v_i = Det \begin{bmatrix} F_A(A, B); & \partial_i A \\ F_B(A, B); & \partial_i B \end{bmatrix} \quad (8)$$

Where $F_A(A, B)$ and $F_B(A, B)$ are functions. The determinants are created from the properties of the field and its partial derivatives with respect to space variables. The curl

[6][7] of this vector is the following:

$$\begin{aligned}
 \text{curl} \underline{\mathbf{v}} &= \underline{\nabla} \times \underline{\mathbf{v}} \\
 &= (\underline{\mathbf{g}}^n \times \underline{\mathbf{g}}^m) \text{Det} \begin{bmatrix} \partial_n F_A(A, B); \partial_m A \\ \partial_n F_B(A, B); \partial_m B \end{bmatrix} \\
 &= \delta_{pq}^{nm} (\underline{\mathbf{g}}^p \otimes \underline{\mathbf{g}}^q) \text{Det} \begin{bmatrix} \partial_n F_A(A, B); \partial_m A \\ \partial_n F_B(A, B); \partial_m B \end{bmatrix}
 \end{aligned} \tag{9}$$

Where the $\underline{\mathbf{g}}^i$ is a contravariant base-vector, and δ is the Kronecker's symbol. We will see, that the result of a curve line integral around a defect is quantized:

$$\begin{aligned}
 \oint v_i dx^i &= \oint \text{Det} \begin{bmatrix} F_A(A, B); \partial_i A \\ F_B(A, B); \partial_i B \end{bmatrix} dx^i \\
 &= \oint \text{Det} \begin{bmatrix} F_A(A, B); \partial_i A dx^i \\ F_B(A, B); \partial_i B dx^i \end{bmatrix} = \oint \text{Det} \begin{bmatrix} F_A(A, B); dA \\ F_B(A, B); dB \end{bmatrix}
 \end{aligned} \tag{10}$$

Every point of our space is connected to a point of the mapped space, so while we walk around on a curve in our real space, it is a closed curve line at the same time in the mapped space too, which can walk around multiple times. The result of the closed line integral is equal with the integral made in the mapped space. For example the curve determined by the soft constraint condition in the mapped space can contain a loop. If it is without a loop, our integral has to be zero, but otherwise, it can be quantized. The size of the quantum step or steps depend on the loop curve determined by the soft constraint condition in the mapped space. The closed curve integral is the multiple of the single loop integral of the projected mapped space, in consequence of the fact, that the projected closed curve can be not only a single times encircled line, but a multiplied times encircled closed curve, so the result of the integral can be the result of the single times encircled case multiplied by a positive or negative or zero whole number.

2.2 Defect surrounded by surfaces.

2.2.1 Example: The Electric displacement field

In a three dimensional space a closed surface can encircle a point defect, or in a four dimensional spacetime it can encircle a trajectory. We will try to create quantized integrals. Let there be

$$\underline{\beta} = (\beta_1, \beta_2, \beta_3) \tag{11}$$

a three dimensional property mapped field. Let there be a soft constraint condition again:

$$C(\underline{\beta}) = 0 \tag{12}$$

Let's create the following electric displacement vectors in the spacetime:

$$\underline{\underline{D}} = \delta_{nm}^{jk} \frac{1}{2} (\underline{\mathbf{g}}^n \otimes \underline{\mathbf{g}}^m) \text{Det} \begin{bmatrix} F_1(\underline{\beta}); \partial_j \beta_1; \partial_k \beta_1 \\ F_2(\underline{\beta}); \partial_j \beta_2; \partial_k \beta_2 \\ F_3(\underline{\beta}); \partial_j \beta_3; \partial_k \beta_3 \end{bmatrix} \quad (13)$$

Where $F_1(\underline{\beta})$, $F_2(\underline{\beta})$ and $F_3(\underline{\beta})$ are functions.

$$D_{ij} = \text{Det}(F(\underline{\beta}), \partial_i \underline{\beta}, \partial_j \underline{\beta}) \quad (14)$$

In our model the electric and magnetic field transforms according to the general relativity. A relativistic four-tensor describes the electromagnetic field.

$$\underline{\underline{D}} = \begin{bmatrix} 0; & H_x; & H_y; & H_z \\ -H_x; & 0; & D_z; & -D_y \\ -H_y; & -D_z; & 0; & D_x \\ -H_z; & D_y; & -D_x; & 0 \end{bmatrix} \quad (15)$$

So the closed surface integral:

$$\oint D_{nm} da^{nm} = \oint \text{Det} \begin{bmatrix} F_1(\underline{\beta}); \partial_j \beta_1; \partial_k \beta_1 \\ F_2(\underline{\beta}); \partial_j \beta_2; \partial_k \beta_2 \\ F_3(\underline{\beta}); \partial_j \beta_3; \partial_k \beta_3 \end{bmatrix} da^{jk} \quad (16)$$

in this model is quantized. Every point of our real space is connected to a point of the mapped space, so while we integral around on a closed surface in our real space, it is a closed surface integral in the mapped space too. The result of the closed surface integral is equal with the integral made in the mapped space. If the surface of the mapped space is closed, the integral can differ from zero. In the case of certain kind of surface, it is possible to wrap around the defect in the mapped space multiple too. The size of the quantum step depends on the surface of property space determined by the soft constraint condition. In this case in a similar way as in the two dimensional case:

$$\underline{\underline{J}} = \underline{\nabla} \times \underline{\underline{D}} = \delta_{pnm}^{ijk} \frac{1}{2} (\underline{\mathbf{g}}^p \otimes \underline{\mathbf{g}}^n \otimes \underline{\mathbf{g}}^m) \text{Det} \begin{bmatrix} \partial_i F_1(\underline{\beta}); \partial_j \beta_1; \partial_k \beta_1 \\ \partial_i F_2(\underline{\beta}); \partial_j \beta_2; \partial_k \beta_2 \\ \partial_i F_3(\underline{\beta}); \partial_j \beta_3; \partial_k \beta_3 \end{bmatrix} \quad (17)$$

We can create a pseudo-vector for the four-vector electric current density, which is the following:

$$J^s = \frac{1}{6} \frac{1}{\sqrt{g}} \epsilon^{sijk} J_{ijk} = \frac{1}{2} \frac{1}{\sqrt{g}} \epsilon^{sijk} \nabla_i D_{jk} \quad (18)$$

Where ∇_j represents the covariant differential, and g is the determinant of the metric-tensor divided by $-c^2$.

In details:

$$\underline{J} = (\rho; J_x; J_y; J_z) \quad (19)$$

Of course this satisfies the electric charge continuity [8], a trivial way, as we can see it using the equation (17):

$$\underline{\nabla} \underline{J} = \nabla_s J^s = \partial_s J^s + J^s \Gamma_{ns}^n = 0 \quad (20)$$

2.3 Defect encircled by volume

2.3.1 The pure theory

In a four dimensional spacetime one closed volume can encircle one event defect. We will try to make quantized the event defect. Let there be

$$\underline{\gamma} = Det(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \quad (21)$$

a four-vector in the mapped space, where we project our field. Let there be a soft constraint condition again, which decreases the four dimensional freedom in the mapped space to three: $C(\underline{\gamma}) = 0$ Now, we create the following three-indexed tensor in the real spacetime:

$$\underline{\underline{v}} = \delta_{nml}^{ijk} (\underline{g}^n \otimes \underline{g}^m \otimes \underline{g}^l) Det \begin{bmatrix} F_0; \partial_j \gamma_0; \partial_k \gamma_0; \partial_l \gamma_0 \\ F_1; \partial_j \gamma_1; \partial_k \gamma_1; \partial_l \gamma_1 \\ F_2; \partial_j \gamma_2; \partial_k \gamma_2; \partial_l \gamma_2 \\ F_3; \partial_j \gamma_3; \partial_k \gamma_3; \partial_l \gamma_3 \end{bmatrix} \quad (22)$$

The quantized event count can be calculated based on the volume-boundary integral encircling the event The event count will be the particle count difference:

$$\Delta N = \oint v_{ijk} dV^{ijk} \quad (23)$$

2.3.2 The Magnetic induction field

Till now, in our model of the electromagnetic field, it was not possible to create a vectorpotential. We will create the electric intensity vector \underline{E} and the magnetic induction vector \underline{B} from the following vectorpotential and scalarpotential.

$$\underline{A} = \frac{1}{2} \underline{g}^i Det(\underline{G}(\underline{\alpha}); \partial_i \underline{\alpha}) \quad (24)$$

Where $\underline{G}(\underline{\alpha})$ is function of $\underline{\alpha}$. The vectorpotential in four-vector form is [9]:

$$\underline{A} = (\Phi, A_x, A_y, A_z) \quad (25)$$

In the usual way, if let there be: The magnetic induction field as a tensor derived from the vectorpotential:

$$\underline{\underline{B}} = \underline{\nabla} \times \underline{A} \tag{26}$$

$$B_{mn} = \frac{1}{2} Det(\partial_m \underline{G}(\underline{\alpha}); \partial_n \underline{\alpha}) - \frac{1}{2} Det(\partial_n \underline{G}(\underline{\alpha}); \partial_m \underline{\alpha}) \tag{27}$$

(There is a soft constraint condition too: $\underline{E} = \underline{D}$ and $\underline{B} = \underline{H}$.)

$$\underline{\underline{B}} = \begin{bmatrix} 0; & -E_x; & -E_y; & -E_z \\ E_x; & 0; & B_z; & -B_y \\ E_y; & -B_z; & 0; & B_x \\ E_z; & B_y; & -B_x; & 0 \end{bmatrix} \tag{28}$$

In this case, as a trivial result, the magnetic induction field is charge free.

$$\underline{\nabla} \times \underline{\underline{B}} = 0 \tag{29}$$

The magnetic flux:

$$\Phi = \int B_{ij} da^{ij} = \oint A_i ds^i \tag{30}$$

In parallel with the method used at the electric field due to the switched on soft constraint condition, we can move only on a loop in the mapped space, and if we suppose, that the magnetic field likes the magnetic induction vector and the electric intensity vector to be zero, then the magnetic flux is quantized. Then it is the same case as studied before, at the chapter "Line surrounded defects." We know, that the magnetic flux can really be quantized, for example in superconductors, where neither electric intensity vector nor magnetic induction vector is present. It is the Josephson-effect [10]. The flux quantum seems to be independent of the type of the material.

2.3.3 Boundary integral of the Event-current-density

To have quantized integrals, our boundary circle integrals of the real spacetime are needed to be boundary circle integrals in the mapped space too. For example the following expression complies this requirement.

$$L = \underline{ED} - \underline{BH} + \underline{JA} - \Phi\rho = div(\underline{H} \times \underline{A} - \underline{D}\Phi) - \partial_t(\underline{DA}) \tag{31}$$

(This is a soft constraint condition.) In this case:

$$\int (\partial_n L^n) dx^4 = \oint L^n dV_n \tag{32}$$

$$L \propto \epsilon^{nijk} \partial_n Det \begin{vmatrix} \underline{F}(\underline{\beta}); & 0; & \partial_i \underline{\beta}; & \partial_j \underline{\beta}; & \partial_k \underline{\beta} \\ 0; & \underline{G}(\underline{\alpha}); & \partial_i \underline{\alpha}; & \partial_j \underline{\alpha}; & \partial_k \underline{\alpha} \end{vmatrix} \tag{33}$$

This integral may be quantized.

$$\Delta N = \oint L^n dV_n \quad (34)$$

The event-current-density pseudo-vector:

$$L^s = -\frac{1}{2} \frac{1}{\sqrt{g}} \epsilon^{sijk} A_i D_{jk} \quad (35)$$

2.3.4 The Energy-impulse tensor

If we use (18) and (35) then the supposed Energy-Impulse tensor can be written in the following indexed form:

$$T_q^p = \nabla_q L^p - J^p A_q + \delta_q^p (J^s A_s - \nabla_s L^s) \quad (36)$$

where the upper index is pseudo. The energy-impulse tensor needn't to be symmetrical, because it is only a subsystem, and shouldn't be symmetrical, due to the photon count change. The force density of the electromagnetic field can be derived from the Energy-Impulse tensor:

$$f_i = \nabla_j T_i^j = B_{is} J^s + A_s \nabla_i J^s \quad (37)$$

The expression of the change of the impulse:

$$\Delta P_k = \oint T_k^j dV_j \quad (38)$$

So the impulse is on an infinite volume:

$$P_k = \int T_k^j dV_j = \int \nabla_k L^s - J^s A_k + \delta_k^s (J^j A_j - \nabla_j L^j) dV_s \quad (39)$$

Without the members which are proportional to the currents or the events, we will get the photon impulse:

$$P_k^{[photon]} = \int \nabla_k L^s dV_s \quad (40)$$

2.3.5 The moment of momentum

The torque causes the change of the moment of momentum. The torque density derived from (37) with an additional torque density is:

$$\begin{aligned} \underline{\underline{M}} &= \underline{\mathbf{x}} \times \underline{\mathbf{f}} + \underline{\mathbf{A}} \times \underline{\mathbf{J}} = \underline{\mathbf{g}}^j \partial_j (\cdot \underline{\mathbf{g}}_p T_i^p (\underline{\mathbf{x}} \times \underline{\mathbf{g}}^i)) + T_i^j (\underline{\mathbf{g}}^i \times \underline{\mathbf{g}}_j) + \underline{\mathbf{A}} \times \underline{\mathbf{J}} \\ &= \underline{\mathbf{g}}^j \partial_j (\cdot \underline{\mathbf{g}}_p T_i^p (\underline{\mathbf{x}} \times \underline{\mathbf{g}}^i)) + (\underline{\nabla} \times \underline{\mathbf{L}}) \end{aligned} \quad (41)$$

The first member is the change of the orbital moment of momentum. The second member is the change of the spin of the photon.

A corresponding operator was found for the spin. It was derived from a quantized property with the help of a rotational operator.

As we can see, in our model both the Energy-impulse and the Spin of the photon, can be derived from the quantized integral of the Event-current-density.

3. Conclusion

The soft constraint criterions in the mapped space can cause quantized integrals, where the quantum step depends on the subspaces which depend on the soft constraint criterions. The members of the Energy-Impulse tensors derived from the event-current-density are similar to the Bosons contrasted with the members derived from the current-densities. It was possible to use the method in the simple example of the electromagnetic field. As a result of our model, the electric charge, magnetic flux and the photon-properties are quantized, and magnetic monopoles are not exists.

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Path Integral Quantization of Brink-Schwarz Superparticle

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Abstract: The quantization of the Brink-Schwarz superparticle is performed by canonical phase-space path integral. The supersymmetric particle is treated as a constrained system using the Hamilton-Jacobi approach. Since the equations of motion are obtained as total differential equations in many variables, we obtained the canonical phase space coordinates and the phase space Hamiltonian with out introducing Lagrange multipliers and with out any additional gauge fixing condition.

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1. Introduction

Systems described by singular Lagrangians are called singular systems and this kind of systems contain inherent constraints [1, 2]. In a lot of physical domains, there extensively exist different singular systems, such as gauge field theories, gravitational field theory, supersymmetric theory, supergravity, superstring theory. A standard consistent way of dealing with singular systems was first formulated by Dirac [3]. In Dirac's method, when a singular Lagrangian in configuration space is transformed into a singular Lagrangian in phase space, the set of constraints would be generated, which are called primary constraints [4, 5]. Through the consistency conditions, using these primary constraints may generate more new constraints, which are called secondary constraints. Following Dirac, one classifies the constraints as being first or second class constraints. According to Dirac's conjecture each first class constraint generates a corresponding gauge symmetry,

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while second class constraints require for their implementation the replacement of Poisson brackets by Dirac brackets [6, 7, 8]. The quantization scheme for constrained systems is the path integral quantization. It is important because it serves as a basis to develop perturbation theory and to find out the Feynman rules. The path integral quantization of singular theories with first class constraints in canonical gauge was given by Faddeev and Popov [9, 10]. The generalization of the method to theories with second class constraints is given by Senjanovic [11]. Moreover, Fradkin and Vilkovisky [12, 13] considered quantization to bosonic theories with first class constraints and its extension to include fermions in the canonical gauge. When the constrained dynamical system possesses some second class constraints there exists another method given by Batalain and Fradkin [14]: the BFV- BRST operator quantization method. Which implies to extend the initial phase space by auxiliary variables to convert the original second class constraints into effective first class ones in the extended manifold. Recently, a new scheme of path integral quantization [19]-[22], depends on the Hamilton-Jacobi treatment of constrained systems [17]-[24]. According to Hamilton-Jacobi formalism the equations of motion are obtained as total differential equations in many variables which require to investigate the integrability conditions. The canonical path integral quantization is obtained directly as an integration over the canonical phase-space coordinates without any need to enlarge the initial phase-space by introducing extra-unphysical variables. The advantage of the Hamilton-Jacobi formalism is that we have no difference between first and second class constraints and we do not need gauge-fixing term to reduce or enlarge the physical phase-space. The better understanding of this feature arises by applying the Hamilton-Jacobi formalism for supersymmetric constraint systems [25], which are subject to mixed fermionic first and second class constraints in an arbitrary space-time dimension. The main aim of this paper is to apply the Hamilton-Jacobi technique to discuss the classical dynamics of the Brink-Schwarz superparticle, then we try to quantize it by using the canonical path integral method.

The material presented in this paper is divided as follows: In the next section the Hamilton-Jacobi formulation is presented. Section 3, is devoted to analyze the massive Brink-Schwarz superparticle model [26] by using Hamilton-Jacobi formalism. The conclusion is given in section 4.

2. Hamilton-Jacobi Formalism Of Constrained Systems

The system that is described by singular Lagrangian $L(q_i, \dot{q}_i, t)$ with $i = 1, \dots, N$, has a rank of Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, \dots, N, \quad (1)$$

equal to $(N - p)$, $p < N$. In this case we have p momenta which are dependent on each other. The generalized momenta P_i corresponding to the generalized coordinates q_i are

defined as,

$$P_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, \dots, N - p, \quad (2)$$

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = N - p + 1, \dots, N. \quad (3)$$

Since, the rank of the Hess matrix is $(N - p)$, one may solve (2) for \dot{q}_a as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, P_b) \equiv \omega_a. \quad (4)$$

Substituting (4) into (3), we obtain relations in q_i , P_a , \dot{q}_ν and t in the form

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu} \Big|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\nu, \dot{q}_a = \omega_a, P_a, t), \quad \nu = N - p + 1, \dots, N. \quad (5)$$

By mean of (4) and (5) the canonical Hamiltonian H_0 is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a = \omega_a, t) + P_a \omega_a + \dot{q}_\mu P_\mu \Big|_{P_\nu = -H_\nu}. \quad (6)$$

The set of Hamilton-Jacobi partial differential equations (HJPDE) is expressed as

$$H'_\alpha \left(q_\beta; q_a; P_a = \frac{\partial S}{\partial q_a}; P_\mu = \frac{\partial S}{\partial q_\mu} \right) = 0, \quad \alpha, \beta = 0, 1, \dots, p. \quad (7)$$

where

$$H'_0 = P_0 + H_0; \quad (8)$$

and

$$H'_\mu = P_\mu + H_\mu. \quad (9)$$

with $q_0 \equiv t$ and S being the action. The equations of motion are obtained as total differential equations in many variables such as,

$$dq_a = \frac{\partial H'_\alpha}{\partial P_a} dt_\alpha, \quad (10)$$

$$dP_r = -(-1)^{n_r n_\alpha} \frac{\partial H'_\alpha}{\partial q_r} dt_\alpha, \quad r = 0, 1, \dots, N, \quad (11)$$

$$dZ = \left(-H_\alpha + P_a \frac{\partial H'_\alpha}{\partial P_a} \right) dt_\alpha, \quad (12)$$

where $n_i = 0, 1$, ($i = r, \alpha$) define the Grassmann parity of the corresponding quantity, and $Z = S(t_\alpha, q_a)$. These equations are integrable if and only if [27, 28]

$$dH'_0 = 0, \quad (13)$$

and

$$dH'_\mu = 0, \quad \mu = N - p + 1, \dots, N. \quad (14)$$

If the conditions (13) and (14) are not satisfied identically, we consider them as new constraints and we examine their variations. Thus repeating this procedure, one may

obtain a set of constraints such that all the variations vanish, then we may solve the equations of motion (10) and (11) to get the canonical phase-space coordinates as

$$q_a \equiv q_a(t, t_\mu), \quad p_a \equiv p_a(t, t_\mu), \quad \mu = 1, \dots, p. \quad (15)$$

In this case the path integral representation may be written as

$$\langle Out | S | In \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp \left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right], \quad (16)$$

$$a = 1, \dots, n - p, \quad \alpha = 0, n - p + 1, \dots, n.$$

We should notice that the integral (16) is an integration over the canonical phase space coordinates (q_a, p_a) .

3. Hamilton-Jacobi Formulation of Brink-Schwarz Superparticle

One may write an action for a particle moving in a superspace; which is an extension of ordinary 4 D spacetime to include extra anticommuting coordinates in the form of N two-components Weyl spinors $\theta, \bar{\theta}$, where $\bar{\theta}$ is the conjugate of θ . Such action, firstly is written by Brink-Schwarz with simple supersymmetry $N = 1$ [26] by the Lagrangian

$$L = \frac{1}{2} [e^{-1} (\dot{x}^\mu - i\bar{\theta}\gamma^\mu\dot{\theta})^2 + em^2]. \quad (17)$$

The singularity of the the Lagrangian follows from the fact that the rank of the Hessian matrix A_{ij} is one.

The canonical momenta defined in (2) and (3) read as

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = e^{-1} \left(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} \right), \quad (18)$$

$$\pi_\theta = \frac{\partial_r L}{\partial \dot{\theta}} = -i\bar{\theta}P_\mu\gamma^\mu = -H_\theta, \quad (19)$$

$$\bar{\pi}_{\bar{\theta}} = \frac{\partial_r L}{\partial \dot{\bar{\theta}}} = 0 = -H_{\bar{\theta}}, \quad (20)$$

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_e. \quad (21)$$

Since the rank of the Hessian matrix is one, we can solve (18) for \dot{x}^μ in terms of P_μ and other coordinates, in the form

$$\dot{x}_\mu = eP_\mu + i\bar{\theta}\gamma_\mu\dot{\theta}. \quad (22)$$

The canonical Hamiltonian H_0 is

$$H_0 = \frac{1}{2} e [P^2 - m^2]. \quad (23)$$

The set of HJPDE's are

$$H'_0 = P_0 + \frac{1}{2}e[P^2 - m^2], \quad (24)$$

$$H'_\theta = P_\theta + i\bar{\theta}P_\mu\gamma^\mu, \quad (25)$$

$$H'_{\bar{\theta}} = P_{\bar{\theta}}, \quad (26)$$

$$H'_e = P_e. \quad (27)$$

Therefore, the total differential equations for the characteristics read as

$$dx_\mu = eP_\mu d\tau + i\bar{\theta}\gamma_\mu d\theta, \quad (28)$$

$$dP_\mu = 0, \quad (29)$$

$$dP_\theta = 0, \quad (30)$$

$$dP_{\bar{\theta}} = (-iP_\mu\gamma^\mu)d\theta, \quad (31)$$

$$dP_e = -\frac{1}{2}[P^2 - m^2]dt. \quad (32)$$

To check whether the set of (28) to (32) are integrable or not, let us consider the total variations of the set of (HJPDE)'s. The variation of

$$dH'_0 = 0, \quad (33)$$

$$dH'_\theta = 0, \quad (34)$$

$$dH'_{\bar{\theta}} = 0, \quad (35)$$

are identically zero, whereas

$$dH'_e = -\left(\frac{1}{2}[P^2 - m^2]\right)dt = H''_e dt. \quad (36)$$

where

$$H''_e = \frac{1}{2}[P^2 - m^2] = 0. \quad (37)$$

is a new constraint. We notice that the total differential of H''_e vanish identically, i.e.

$$dH''_e = 0. \quad (38)$$

Thus the set of equations (28)-(32) with (41) are integrable. According to (12) the action can be written as

$$\begin{aligned} dZ &= -H_0 d\tau - H_\theta d\theta - H_{\bar{\theta}} d\bar{\theta} - H_e de + P_\mu dx^\mu \\ &= \left\{ -\frac{1}{2}e\left(P^2 - m^2\right) + P_\mu\left(\dot{x} - i\bar{\theta}\gamma^\mu\dot{\theta}\right) \right\} d\tau, \end{aligned} \quad (39)$$

and the canonical action integral becomes

$$S = \int \left\{ \frac{1}{2}e\left(P^2 + m^2\right) \right\} d\tau. \quad (40)$$

By using (40) and (16) the canonical path integral quantization of Brink-Schwarz superparticle is expressed as

$$\langle x_\mu, \tau; x'_\mu, \tau' \rangle = \int dx_\mu dp_\mu \exp \left[i \int \left\{ \frac{1}{2} e (P^2 + m^2) \right\} d\tau \right] \quad (41)$$

This path integral representation as an integration over the canonical phase-space with no need to introduce any gauge fixing to reduce or enlarge the phase-space as in covariant quantization of Brink-Schwarz superparticle described in references [29, 30, 31].

Conclusion

In this work we presented Brink-Schwarz Superparticle as a singular system, and its Hamiltonian treatment contains all kinds of constraints (primary and secondary, first and second class ones). This model is very illustrative, since it allows a comparison between all features of Diracs and Hamilton-Jacobi formalisms. In Dirac's formalism, we must reduce any constrained singular system to one with first-class constraints only, we must call attention to the presence of arbitrary variables in some of the Hamiltonian equations of motion due to the fact that we have gauge dependent variables and we have made a gauge fixing. This does not occur in Hamilton-Jacobi formalism since it provides a gauge-independent description of the systems evolution due to the fact that the Hamilton-Jacobi function S contains all the solutions that are related by gauge transformations. The canonical path integral quantization of Brink-Schwarz Superparticle is done, since the system is integrable, and the integration is taken over the canonical phase space.

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Noncommutative Geometry and Modified Gravity

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Abstract: Using noncommutative deformed canonical commutation relations, a model of gravity is constructed and a schwarchild like static solutions are obtained. As a consequence, the Newtonian potential is modified and it is shown to have a form similar to the one postulated by Fishbach et al. to explain the proposed fifth force. More interesting is the form of the gravitational acceleration expression proposed in the modified Newtonian dynamics theories (MOND) which is obtained explicitly in our model without any ad hoc asymptions.

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1. Introduction

Some physicists think that the Einstein gravitation theory has to be modified at small as well as large scales. In fact, there are certain cosmological and astrophysical arguments suggesting the modification of general relativity. Studying the galaxies cluster mass, Zwicky has noticed that the resulted mass deduced from the measurement of the galaxies velocity is 10 to 100 times greater than the observed matter mass. Moreover, other cosmological problems such as galaxies curvature anomalies, gravitational mirage, galaxy formation, homogeneous and uniform structure of our universe, flatness, horizon problem, etc...[1] – [6] suggest that either the Newtonian gravitation theory is not applicable at the cosmological scale or 90% of the universe mass is not observable (about 25% dark matter and 71% dark energy which do not have electromagnetic interactions) and does manifest only by gravitational effects. Determining the nature of these dark matter and energy [17] – [19] is one of the problems and tasks of the modern cosmol-

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ogy and particle physics. Instead of admitting the existence of dark matter and energy some theoreticians believe that these cosmological anomalies are due essentially to the fact that the Newtonian gravitation is incomplete. To account for these observations at large scales, the gravitational force has to be much bigger than the one given by the Newton approximation. For this, some models such as the Modified Newtonian Dynamics (MOND)[20] – [22], Tensor-Vector-Scalar (TeVeS) [23] etc..., were proposed allowing the reproduction of spiral galaxies rotational curves without recourse to the dark matter scenario. It is important to mention that within Einstein Gravitational theory it was possible to develop relative astrophysics and cosmology in Riemann spacetime successfully describing the basic structures of observable Universe. However the difficulties of classical theoretical cosmology and up-to date state of art in observation cosmology result in new problems of fundamental physics. One way to solve these problems suggested by many authors is to generalize Gravitation theory. Moreover, Einstein's theory of relativity has not been tested on cosmological scales, and so one might contemplate if the observed acceleration could be the first direct indication of our lack of understanding of gravity. The goal of this paper is to study and understand some qualitative aspects of the space noncommutativity through a gravity model based on noncommutative deformed canonical commutation relations. In section2, we present the formalism, construct the noncommutative action and derive the noncommutative Schwarchild like static solutions. In section3, we discuss the modified Newtonian potential and finally in section4, we give some of qualitative results and draw our conclusions.

2. Formalism

In what follows, we take $\hbar = c = 1$ and consider a noncommutative geometry, characterized by the space-time coordinates \hat{x}^μ and momenta \hat{p}^μ which are non commuting operators satisfying the following matrices valued commutation relations:

$$[\hat{x}_\mu, \hat{x}_\nu] = 0 \quad (1)$$

$$[\hat{x}_\mu, \hat{p}_\nu] = i(\delta_{\mu\nu}I + \theta_{\mu\nu}) \quad (2)$$

and

$$[\hat{p}_\mu, \hat{p}_\nu] = 0 \quad (3)$$

(I is 4×4 identity matrix) where $\theta_{\mu\nu}$ are matrices valued tensor under ordinary general coordinates transformations and taken to be proportional to the Dirac matrices $\gamma_{\mu\nu}$ in a curved space-time such that:

$$\theta_{\mu\nu} = \frac{1}{4}\xi(x)\gamma_{\mu\nu} = \frac{1}{4}\xi(x) [\gamma_\mu, \gamma_\nu] \quad (4)$$

(here $\xi(x)$ is a scalar function of the space-time variable x_μ). Notice that although the above commutation relations do not fit into the case where the noncommutativity parameters are c-numbers, there is nothing fundamentally wrong with this choice.

2.1 Non commutative Gravity Model

The operators \hat{x}_ν and \hat{p}_ν have as representations:

$$\hat{x}_\nu = x_\nu, \quad \hat{p}_\nu = -i\hat{\partial}_\nu \quad (5)$$

where the noncommutative matrix derivative $\hat{\partial}_\nu$ has as expression

$$\hat{\partial}_\nu = \partial_\nu + i\theta_{\nu\alpha}\partial^\alpha \quad (6)$$

and

$$\partial^\alpha = \hat{g}^{\mu\alpha}\partial_\mu \quad (7)$$

(x_ν and ∂_ν are the ordinary coordinates and derivative respectively). $\hat{g}^{\mu\alpha}$ is the inverse of the noncommutative symmetric metric $\hat{g}_{\mu\alpha}$ (which is not a matrix) such that

$$\hat{g}_{\nu\mu}\hat{g}^{\mu\alpha} = \delta_\nu^\alpha \quad (8)$$

The modified affine connection (which is not Riemannian) denoted by $\hat{\Gamma}^\nu_{\mu\lambda}$ takes the form:

$$\hat{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2}\hat{g}^{\mu\nu} \left(\hat{\partial}_\beta\hat{g}_{\nu\alpha} + \hat{\partial}_\alpha\hat{g}_{\nu\beta} - \hat{\partial}_\nu\hat{g}_{\alpha\beta} \right) \quad (9)$$

which can be rewritten as:

$$\hat{\Gamma}^\mu_{\alpha\beta} = \bar{\Gamma}^\mu_{\alpha\beta} + \tilde{\Gamma}^\mu_{\alpha\beta} \quad (10)$$

where

$$\bar{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2}\hat{g}^{\mu\nu} \left(\partial_\beta\hat{g}_{\nu\alpha} + \partial_\alpha\hat{g}_{\nu\beta} - \partial_\nu\hat{g}_{\alpha\beta} \right) \quad (11)$$

and

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{i}{2}\hat{g}^{\mu\nu} \left(\theta_{\beta\sigma}\partial^\sigma\hat{g}_{\nu\alpha} + \theta_{\alpha\sigma}\partial^\sigma\hat{g}_{\nu\beta} - \theta_{\nu\sigma}\partial^\sigma\hat{g}_{\alpha\beta} \right) \quad (12)$$

Here $\tilde{\Gamma}^\mu_{\alpha\beta}$ represents a non metricity like a tensor. We remind the reader that in differential geometry, the affine connection on a differential manifold with a metric can be always decomposed into the sum of a Levi-Civita (metric) connection, a non metricity tensor and a torsion. This is the case of theories with more complicated geometrical structure like the Riemann-Cartan space with general metric-affine spaces (curvature, torsion and non-metricity) and the Weyl-Cartan space which is a connected differentiable manifold with a Lorenz metric obeying the Weyl non-metricity condition. In the Riemannian space

of general relativity the metric and the connection (which are considered respectively as a potential and strength of the gravitational field) are linked through the requirement of metric homogeneity (metricity condition). The latter assures that the length of a vector transported parallel in any direction remains invariant. Regarding the noncommutative matrices curvature and Ricci tensors $\widehat{R}^\sigma{}_{\lambda\mu\nu}$ and $\widehat{R}_{\mu\nu}$, they are given by:

$$\widehat{R}^\mu{}_{\alpha\beta\lambda} = \widehat{\partial}_\beta \widehat{\Gamma}^\mu_{\alpha\lambda} - \widehat{\partial}_\lambda \widehat{\Gamma}^\mu_{\alpha\beta} + \widehat{\Gamma}^\mu_{\sigma\beta} \widehat{\Gamma}^\sigma_{\alpha\lambda} - \widehat{\Gamma}^\mu_{\sigma\lambda} \widehat{\Gamma}^\sigma_{\alpha\beta} \quad (13)$$

and

$$\widehat{R}_{\mu\nu} = \widehat{R}^\lambda{}_{\mu\lambda\nu} = \widehat{\partial}_\nu \widehat{\Gamma}^\lambda_{\mu\lambda} - \widehat{\partial}_\lambda \widehat{\Gamma}^\lambda_{\mu\nu} + \widehat{\Gamma}^\lambda_{\mu\sigma} \widehat{\Gamma}^\sigma_{\lambda\nu} - \widehat{\Gamma}^\sigma_{\mu\nu} \widehat{\Gamma}^\lambda_{\sigma\lambda} \quad (14)$$

We can also define a noncommutative matrix Einstein tensor $\widehat{G}_{\mu\nu}$ as:

$$\widehat{G}_{\mu\nu} = \widehat{R}_{\mu\nu} - \frac{1}{2} \widehat{g}_{\mu\nu} \widehat{R} \quad (15)$$

and where the non commutative matrix scalar curvature \widehat{R} is defined as:

$$\widehat{R} = \widehat{g}^{\mu\nu} \widehat{R}_{\mu\nu} \quad (16)$$

Now, we define the noncommutative Hilbert-Einstein I_{NCG} by

$$I_{NCG} = \frac{1}{64\pi\kappa} \int d^4x \sqrt{\widehat{g}} \widehat{g}^{\mu\nu} Tr \widehat{R}_{\mu\nu} \quad (17)$$

Where κ is the gravitational constant, \widehat{g} stands for $|\det(\widehat{g}_{\mu\nu})|$ and Tr is the trace over the gamma Dirac matrices. Since $Tr\theta^{\beta\sigma} = 0$, thus the terms linear in $\theta^{\beta\sigma}$ do not contribute in the expression of $Tr\widehat{R}^\mu{}_{\alpha\beta\lambda}$ and therefore:

$$\mathcal{R}_{\mu\nu} \equiv Tr \widehat{R}_{\alpha\lambda} = \overline{R}_{\alpha\lambda} + Tr \widetilde{R}_{\alpha\lambda} \quad (18)$$

where $\widetilde{R}_{\mu\nu}$ and $\overline{R}_{\mu\nu}$ are given by:

$$\widetilde{R}_{\mu\nu} = \left(i\theta_{\beta\sigma} \partial^\sigma \widetilde{\Gamma}^\beta_{\mu\nu} - i\theta_{\nu\sigma} \partial^\sigma \widetilde{\Gamma}^\beta_{\mu\beta} + \widetilde{\Gamma}^\beta_{\sigma\beta} \widetilde{\Gamma}^\sigma_{\mu\nu} - \widetilde{\Gamma}^\beta_{\sigma\nu} \widetilde{\Gamma}^\sigma_{\mu\beta} \right) \quad (19)$$

and

$$\overline{R}_{\mu\nu} = 4(\partial_\nu \overline{\Gamma}^\lambda_{\mu\lambda} - \partial_\lambda \overline{\Gamma}^\lambda_{\mu\nu} + \overline{\Gamma}^\lambda_{\mu\sigma} \overline{\Gamma}^\sigma_{\lambda\nu} - \overline{\Gamma}^\sigma_{\mu\nu} \overline{\Gamma}^\lambda_{\sigma\lambda}) \quad (20)$$

It is worth to mention that the principle of a least action leads to the following noncommutative Einstein field equation in the vacuum:

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \mathcal{R}^\alpha{}_\alpha = 0 \quad (21)$$

which is equivalent to:

$$G_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu} \quad (22)$$

where $G_{\mu\nu}$ has the form:

$$G_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\bar{R}^\alpha{}_\alpha \quad (23)$$

and $\mathcal{T}_{\mu\nu}$ is an effective matter energy-momentum tensor induced by the non commutativity of the space and has as expression:

$$\mathcal{T}_{\mu\nu} = \frac{-1}{\kappa}(Tr\tilde{R}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}Tr\tilde{R}^\alpha{}_\alpha) \quad (24)$$

This means that the non commutativity deform the space and generate a more complex structure with a non metricity contributing to the field equations and induce an effective macroscopic matter energy-momentum tensor as an additional source of gravity. This is not surprising about the role of the deformed canonical commutation relations in quantum mechanics where it was shown in ref.[27], that there exist an intimate connection to the curved space. Moreover, a suitable choice of the position-momentum commutator can elegantly describe many features of gravity, including the IR/UV correspondence and dimensional reduction (holography)[28]. In what follows, we set:

$$\xi^2(x) = 4\eta^2\sigma(x) \quad (25)$$

($\eta \ll 1$ is a constant parameter which characterizes the noncommutativity). Since $\hat{g}_{\nu\alpha}$ is a solution of the field equations, we assume to have the form:

$$\hat{g}_{\nu\alpha} = g_{\nu\alpha} + \eta^2 g_{\nu\alpha}^{(1)} \quad (26)$$

where $g_{\nu\alpha}$ is the classical Einstein Riemanian metric and $g_{\nu\alpha}^{(1)}$ is a non commutative correction to be determined later. Furthermore, to simplify our calculation, we assume that

the only non vanishing matrix valued parameters are θ_{01} (of course our qualitative results remain valid in the general case). Then, it is easy to show that at the $O(\eta^2)$, one has:

$$\frac{1}{4}Tr\theta_{01}\theta_{01} \approx \eta^2\sigma(x)(g_{01}g_{01} - g_{00}g_{11}) \quad (27)$$

It is important to mention that the noncommutative Hilbert-Einstein action given in eq.(17) is invariant under general coordinate transformations.

2.2 Nocommutative Schwarzschild metric

In the case of Schwarzschild metric, one has in spherical coordinates system (where $0 \rightarrow t$, $1 \rightarrow r$, etc.): $g_{01} = 0$ and $g_{00}.g_{11} = -1$. Therefore, eq. (27) takes the simple form:

$$\frac{1}{4}Tr\theta_{01}\theta_{01} \approx \eta^2\sigma(x) \quad (28)$$

For static solutions, tedious but straightforward calculations lead to the following non vanishing components of $Tr\tilde{R}_{\alpha\lambda}$:

$$Tr\tilde{R}_{00} = -\frac{1}{2}\eta^2 e^{-2\lambda}\sigma \left\{ -\lambda'' + \frac{1}{2}\lambda'^2 + \frac{4}{r}\lambda' + \frac{1}{2}\lambda'\nu' + \frac{2}{r}\nu' \right\} + \frac{1}{4}\eta^2 e^{-2\lambda}\sigma' \left(\lambda' + \frac{4}{r} \right) \quad (29)$$

$$Tr\tilde{R}_{11} = -\frac{1}{2}\eta^2 e^{-(\lambda+\nu)}\sigma \left\{ \lambda'' - \frac{1}{2}\lambda'^2 + \frac{2}{r}\lambda' - \frac{1}{2}\lambda'\nu' \right\} - \frac{1}{4}\eta^2 e^{-(\lambda+\nu)}\sigma'\lambda' \quad (30)$$

$$Tr\tilde{R}_{22} = -\frac{1}{2}\eta^2 e^{-2\lambda}e^{-\nu}\sigma \{2 - r\lambda' - r\nu'\} - \frac{1}{2}\eta^2 e^{-2\lambda}e^{-\nu}\sigma'r \quad (31)$$

$$Tr\tilde{R}_{33} = Tr\tilde{R}_{22} \sin^2\theta \quad (32)$$

Using a perturbative expansion around the classical solutions λ_0 and ν_0 at the $O(\eta^2)$:

$$\lambda \approx \lambda_0 + \eta^2\lambda_1 \quad \nu \approx \nu_0 + \eta_1^2\nu_1 \quad (33)$$

Direct calculations give the following noncommutative Einstein equations in the vacuum:

$$-\frac{1}{2}\nu_1'' - \frac{1}{r}\nu_1' - \frac{1}{4}(3\nu_1' - \lambda_1')\nu_0' - \sigma \left(\frac{1}{2}\nu_0'' - \frac{1}{r}\nu_0' \right) + \frac{1}{4} \left(\frac{4}{r} - \nu_0' \right) \sigma' = 0 \quad (34)$$

$$\frac{1}{2}\nu_1'' - \frac{1}{r}\lambda_1' + \frac{1}{4}(3\nu_1' - \lambda_1')\nu_0' + \sigma \left(\frac{1}{2}\nu_0'' + \frac{1}{r}\nu_0' \right) + \frac{1}{4}\nu_0'\sigma' = 0 \quad (35)$$

and

$$\lambda_1' - \nu_1' + \frac{2}{r-b}\lambda_1 + \frac{2}{r}\sigma + \sigma' = 0 \quad (36)$$

where ν_0 and λ_0 are given by the classical Schwarzschild like static solutions that is:

$$e^{\nu_0} = e^{-\lambda_0} = 1 - \frac{b}{r} \quad (37)$$

from eqs.(34) and (35) we get:

$$\lambda_1' + \nu_1' - 2\nu_0'\sigma - \sigma' = 0 \quad (38)$$

Now, eqs.(36) and (37) lead to:

$$\lambda_1' + \frac{1}{r-b}\lambda_1 + \left(\frac{2}{r} - \frac{1}{r-b} \right) \sigma = 0 \quad (39)$$

and its solution has the form:

$$\lambda_1 = \frac{C}{r-b} - \frac{1}{r-b} \int \left(1 - \frac{2b}{r} \right) \sigma dr \quad (40)$$

from eq.(38) we deduce that:

$$\nu_1 = -\lambda_1 + 2 \int \nu_0'\sigma dr + \sigma + D \quad (41)$$

Using eq.(40) we obtain the following second order differential equation:

$$[(r-b)\nu_1]'' = (r-b)\sigma' + 3\sigma'' + 2\nu_0'\sigma \quad (42)$$

(the notation $''''$ means space coordinates second derivative). Eq.(34) together with the differential equation of the classical solution ν_0 :

$$\frac{1}{2}\nu_0'' + \frac{1}{2}\nu_0'^2 + \frac{1}{r}\nu_0' = 0 \quad (43)$$

imply that:

$$[(r-b)\nu_1]'' = \frac{2}{r}(r-b)\sigma' - 2(r-b)\nu_0''\sigma \quad (44)$$

Moreover, as the classical solutions give:

$$2(r-b)\nu_0'' = \frac{2}{r^2}(r-b) - \frac{2}{(r-b)} \quad (45)$$

eqs.(40), (44) and (45) lead to the following second order differential equation:

$$(r-b)\sigma'' + \left(1 + \frac{2b}{r}\right)\sigma' - \frac{2b}{r^2}\sigma = 0 \quad (46)$$

where the general solution has the form:

$$\sigma(r) = A\frac{r^2}{(r-b)^2} + B\frac{r}{(r-b)^2}(r \ln r + b) \quad (47)$$

If we set the integration constant $B = 0$ (for the sake of simplicity), eqs.(40) and (41) read:

$$\lambda_1 = -A + \frac{C - Ab}{r - b} - \frac{Ab^2}{(r - b)^2} \quad (48)$$

and

$$\nu_1 = 2A + D - \frac{C - Ab}{r - b} + \frac{Ab^2}{(r - b)^2} \quad (49)$$

where A, C, D are integration constants and:

$$b = 2\kappa M. \quad (50)$$

(κ and M are the Newton constant and the body mass producing the gravitational field). Thus, in the noncommutative space, the Schwarzschild metric takes the final form:

$$ds^2 = -\left(1 - \frac{b}{r}\right)e^{\eta^2\nu_1}dt^2 + \left(1 - \frac{b}{r}\right)^{-1}e^{\eta^2\lambda_1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (51)$$

Notice that in the classical limit where $\eta \rightarrow 0$, eq.(51) is reduced to the classical Schwarzschild metric.

3. Newtonian Potential in the Curved Noncommutative Space

Recently, some experiments were undertaken to measure the gravitational constant (Australian mine experiment by S.Franks, Greenland ice bring experiment by Fischbach et al...). It was found that the measured value is smaller by about 2% than the predicted one. Then, it was concluded that there is a possibility of the existence of a fifth force. The latter is repulsive with a small action range λ ($\sim 1\text{cm} - 100\text{m}$) and depends on the substances nature of the bodies. Fischbach et al. have proposed to modify the Newtonian potential $V(r)$ to have the form:

$$V(r) = -\frac{\kappa M}{r} \left\{ 1 + \alpha \exp\left(-\frac{r}{\lambda}\right) \right\} \quad (52)$$

where the second term gives rise to the fifth force with strength α (~ 0.01). Moreover, to reproduce the spiral galaxies rotational curves, the MOND (Modified Newtonian Dynamics) theories, postulate that at the limit of very weak accelerations, the gravitational acceleration g_M takes the form:

$$g_M \approx \sqrt{\alpha_0 g_N} \quad (53)$$

where the Newtonian gravitational acceleration g_N has the form:

$$g_N = \frac{\kappa M}{r^2} \quad (54)$$

and $\alpha_0 \approx 1.2 \cdot 10^{-10} \text{m/s}^2$. In 1983 Milgrom has proposed other modifications to this law namely [20] – [22]:

$$g_M \approx g_N \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4\alpha_0}{g_N}} \right) \quad (55)$$

or

$$g_M \approx g_N \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4\alpha_0}{g_N}} \right)^{\frac{1}{2}} \quad (56)$$

In our case, from the noncommutative Schwarzschild metric of eq(51) we deduce that :

$$g_{00} = -(1 + 2V(r)) = -\left(1 - \frac{r_g}{r}\right) e^{\eta^2 \nu_1} \approx -\left(1 - \frac{r_g}{r}\right) (1 + \eta^2 \nu_1) \quad (57)$$

with

$$\nu_1 = 2A + D - \frac{C - Ar_g}{r - r_g} + \frac{Ar_g^2}{(r - r_g)^2} \quad (58)$$

Thus, the modified Newtonian potential produced by a body of mass M and radius R at a distance r ($r > R$) takes the form:

$$V(r) = -\frac{\kappa M}{r} + \eta^2 \frac{1}{r} \left(Er + \frac{\bar{A}}{r - r_g} + \bar{E} \right) \quad (59)$$

where the first and the second terms (proportional to η^2) are the classical Newtonian potential and the contribution of the space noncommutativity respectively. Here E , \bar{E} and \bar{A} , are integration constants and r_g is the Schwarzschild radius ($r_g \approx 3km$ for the sun and $r_g \approx 0.88cm$ for the earth etc..). Notice that there is a singularity at $r = r_g$ for very massive bodies (black holes) where $r_g > R$, whereas for small masses where $r_g < R$ the singularity is absent. The modified Newtonian potential of eq.(59), can be rewritten in the form:

$$V(r) = -\frac{\kappa M}{r} \left\{ 1 + \eta^2 \alpha \left(1 + \frac{r}{\lambda} + \frac{\gamma}{r - r_g} \right) \right\} \quad (60)$$

with α , λ and γ are new constants. Now, for small values of α and γ and the noncommutative parameter η , the modified Newtonian potential takes the following expression:

$$V(r) = -\frac{\kappa M}{r} \left\{ \frac{1}{2} + \frac{1}{2} \tilde{\alpha} \exp \left(-\eta^2 \frac{r}{\tilde{\lambda}} \right) \right\} \quad (61)$$

with

$$\tilde{\alpha} = \exp 2\eta^2 \alpha \left(1 + \frac{\gamma}{r - r_g} \right) \quad (62)$$

and

$$\tilde{\lambda} = -\frac{\lambda}{2\alpha} \quad (63)$$

(for $\gamma \approx 0$ we have $\tilde{\alpha} \approx 1 + \eta^2 \alpha$). Notice that the expression in eq.(61) of the potential is similar to the one postulated by Fishbach et al.[23] – [25] to explain the origin of the fifth force. In our case the value of $\tilde{\alpha}$ is a function of r , except for $\gamma \approx 0$. Moreover, the sign of the second term which gives rise to the fifth force depends on the parameters α , λ , γ , and the variable r . The resulted force can be repulsive as well as attractive. Moreover, eq.(61) can be rewritten in the form:

$$V(r) = -\frac{\tilde{\kappa}(r)M}{r} \quad (64)$$

where the factor $\tilde{\kappa}(r)$ behaves like a running Newton constant.

$$\tilde{\kappa}(r) = \left[1 + \eta^2 \alpha \left(1 + \frac{r}{\lambda} + \frac{\gamma}{r - r_g} \right) \right] \kappa \quad (65)$$

Thus, eq.(61) can be interpreted as a Newton potential with a running coupling due to the noncommutativity of the space [26].

4. Results and Conclusions

In what follows, to get qualitative results, study the behavior of the modified Newtonian force according to the possible cases and make our analysis clear and simplified, we take for illustration (in arbitrary units) $r_g = 1$, $\bar{E} = -1$, $E = 0, \pm 1$ and $\bar{A} = 0, \pm 1$:

4.1 $\eta^2 = 0$, or $E = 0, \bar{A} = 0$:

In this case, the noncommutative Newtonian potential is reduced to the classical one. Figure(1) displays the behavior of the Newtonian force $F(r)$ acting on a body of mass unity ($m = 1$) as a function of the distance r ($F(r) = -\frac{1}{r^2}$).

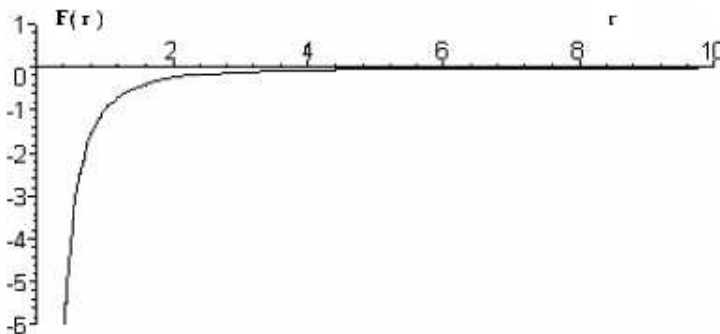


Fig. 1

4.2 $E > 0, \bar{A} > 0$:

In this case and in arbitrary units, the behavior of the noncommutative Newtonian potential $V(r)$ is presented in fig.2 and the corresponding Newtonian force $F(r)$ acting on a body of mass $m = 1$ is displayed in fig.3. We distinguish eight main regions:

a) For $0 \leq r \leq \tilde{r}_1$ ($\tilde{r}_1 \approx 0.9$), the non commutative Newtonian potential (NCNP) is negative. It is an increasing function of r . It varies from $V(0) \approx -\infty$ to $V(\tilde{r}_1) \approx -1.223$. The related non commutative force (NCF) is attractive and its intensity decreases from $|F(0)| = +\infty$ to $|F(\tilde{r}_1)| \approx 0$.

b) For $\tilde{r}_1 \leq r \leq r_1$ ($r_1 \approx 0.92$), the NCNP is negative. It is an increasing function of r until a maximum value $V(r_1) \approx -1.222$. The NCF is repulsive and its intensity increases from $|F(\tilde{r}_1)| = 0$ to $|F(r_1)| \approx 0.16$.

c) For $r_1 \leq r \leq r_g$, the NCNP remains negative but a decreasing function of r and gives a repulsive NCF where the intensity increases form $|F(r_1)| \approx 3.3$ to $|F(r_g)| = +\infty$,

d) For $r_g \leq r \leq r_2$ ($r_2 \approx 1.01$), the NCNP is positive. It is a decreasing function of r and it vanishes at a distance $r_2 \approx 1.01$. The NCF is a repulsive force where the intensity increases from $|F(r_g)| = +\infty$ to $|F(r_2)| \approx 97.03$.

e) For $r_2 \leq r \leq r_3$ ($r_3 \approx 1.09$), the NCNP is negative. It is a decreasing function of r . It varies from $V(r_2) = 0$ to a minimal value $V(r_3) \approx -1.02$. The corresponding NCF is repulsive where the intensity decreases from $|F(r_2)| \approx 97.03$. to $|F(r_3)| \approx 0.19$.

f) For $r_3 \leq r \leq r_4$ ($r_4 \approx 1.1$), the NCNP is negative. It is an increasing function of r . It varies from $V(r_3) \approx -1.02$ to value $V(r_4) \approx -0.99$. The NCF is a repulsive force with an intensity decreasing from $|F(r_3)| \approx 0.19$ to $|F(r_4)| \approx 0$.

g) For $r_4 \leq r \leq r_5$ ($r_5 \approx 1.25$), the NCNP is negative. It is an increasing function of r . It varies from $V(r_4) \approx -0.99$ to $V(r_5) \approx -0.83$. The NCF is an attractive force and

its intensity increases from $|F(r_4)| \approx 0$ to $|F(r_5)| \approx 0.46$.

h) For $r_5 \leq r \leq +\infty$, the NCNP has a behavior which looks like the classical Newtonian one. Thus, it is negative and increasing function of r . It gives an attractive NCF with an intensity decreasing from $|F(r_5)| \approx 0.46$ to $|F(+\infty)| = 0$.

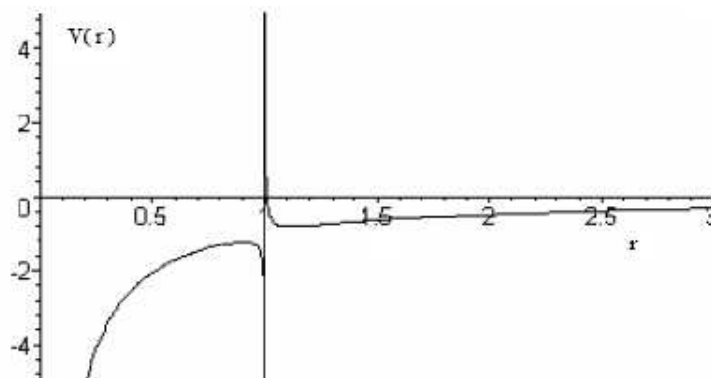


Fig. 2

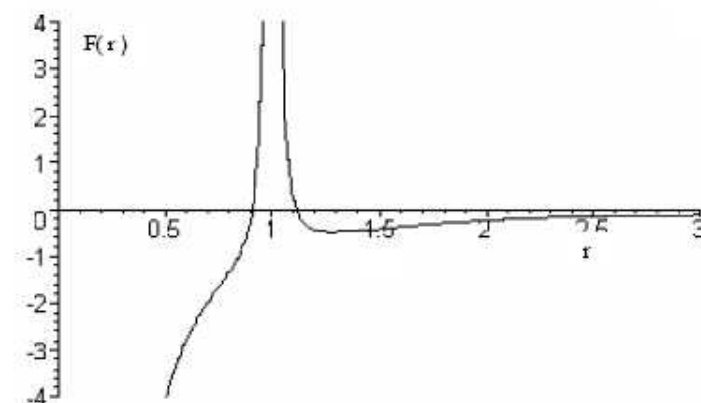


Fig. 3

4.3 $E > 0, \bar{A} < 0$:

In this case and in arbitrary units, the behavior of the NCNP is presented in fig.4 and the corresponding Newtonian force $F(r)$ acting on a body of $m = 1$ is displayed in fig.5. We distinguish four main regions:

a) for $0 \leq r \leq r_6$ ($r_6 \approx 0.82$), the NCNP is negative and an increasing function of r . It varies from $V(0) = -\infty$ to $V(r_6) \approx -1.15$. This gives an attractive NCF where the intensity decreases from $|F(0)| = +\infty$ to $|F(r_6)| \approx 1.79$.

b) for $r_6 \leq r \leq r_7$ ($r_7 \approx 0.99$), the NCNP is negative and an increasing function of r . It varies from $V(r_6) \approx -1.15$ to $V(r_7) \approx 0$. This will give an attractive NCF where the intensity increases from $|F(r_6)| \approx 1.79$ to $|F(r_7)| \approx 101.02$.

c) for $r_7 \leq r \leq r_g$, the NCNP is positive and an increasing function of r . It varies from

$V(r_7) \approx 0$ to $V(r_g) \approx +\infty$. The related NCF is an attractive force where the intensity increases from $|F(r_7)| \approx 101.02$ to $F(r_g) = +\infty$.

d) for $r_g \leq r \leq +\infty$, the NCNP has a behavior which looks like the classical one. Thus, it is negative and increasing function of r . This gives an attractive NCF with an intensity decreasing from $|F(r_g)| = +\infty$ to $F(+\infty) = 0$.

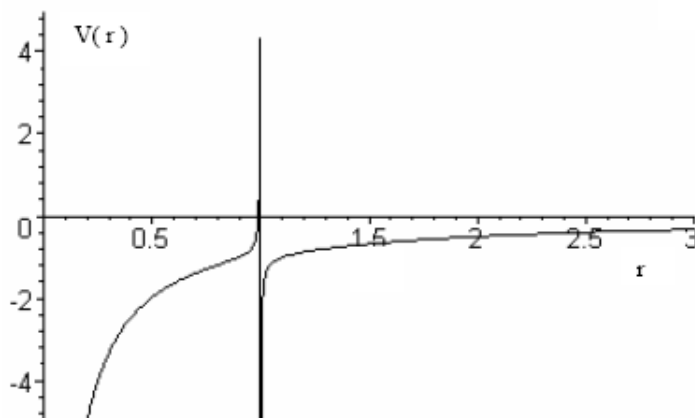


Fig. 4

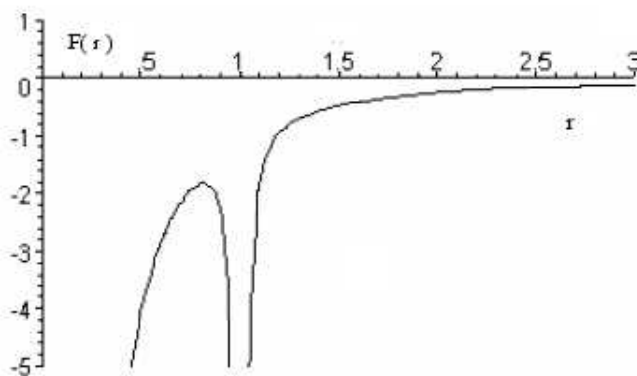


Fig. 5

Notice that, if we add the logarithmic term in the expression of $\sigma(r)$ see eq.(47) (which diverges for $r \rightarrow \infty$), we find the following expression for the noncommutative potential:

$$\begin{aligned}
2V(r) = & -\frac{b}{r} - \eta^2 \frac{1}{r} \left(C - (A - B)r - Bb \ln r - \frac{1}{r-b} (Ab^2 + Bb^2) \right) \\
& + \eta^2 \left(1 - \frac{b}{r} \right) \left(\frac{\ln r}{(r-b)^2} (Bb^2 - Bbr + Br^2) + B \ln(r-b) - B \ln r \right) \\
& + \eta^2 \left(1 - \frac{b}{r} \right) \left(2(\ln r) \frac{Bb^2 - 2Bbr}{2b^2 - 4br + 2r^2} + 2 \frac{Ab^2 - Bbr - 2Abr}{2b^2 - 4br + 2r^2} \right) \\
& + \eta^2 \left(1 - \frac{b}{r} \right) \left(A \frac{r^2}{(r-b)^2} + B \frac{r}{(r-b)^2} (r \ln r + b) + D \right)
\end{aligned} \tag{66}$$

For large values of r , it reduces to :

$$2V(r) \approx -\frac{r_g}{r} + B\eta^2 \left(\ln(r - r_g) + \frac{r^2}{(r - r_g)^2} \ln r \right) \approx -\frac{r_g}{r} + 2B\eta^2 \ln r \tag{67}$$

Thus, the modified Newtonian acceleration behaves like:

$$g_M \approx \frac{GM}{r^2} (1 + B_1 \eta^2 r) \approx g_N \left(1 + \eta^2 \frac{\tilde{B}}{g_N} \right) \tag{68}$$

or equivalently:

$$g_M \approx g_N \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4\alpha_0}{g_N}} \right) \tag{69}$$

Therefore, we obtain the same form for the gravitational acceleration as in the modified Newtonian dynamics theories (MOND) at low energies (large scales).

As a conclusion, using noncommutative deformed canonical commutation relations, we have constructed a non Riemannian model of gravity with non metricity like tensor and complex geometric structure. As a consequence, the non commutative Schwarchild like static solutions yield to a modification of the Newtonian potential. The latter is shown to have a form similar to the one postulated by Fishbach et al. to explain the fifth force. One can also interpret the resulted potential as the classical Newtonian potential with a running Newton coupling constant. More interesting, is the form of the gravitational acceleration (obtained in our noncommutative space approach without any ad hoc assumption) which looks like the one proposed in the modified Newtonian dynamics theories (MOND).

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Classification of Electromagnetic Fields in non-Relativist Mechanics

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Abstract: We study the classification of electromagnetic fields using the equivalence relation on the set of all 4-potentials of the Schrödinger equation. In the general case we find the relations among the equivalent fields, currents, and charge densities. Particularly, we study the fields equivalent to the null field. We show that the non-stationary state function for a particle in arbitrary uniform time-dependent magnetic field is equivalent to a plane wave. We present that the known coherent states of a free particle are equivalent to the stationary states of an isotropic oscillator. We reveal that the only constant magnetic field is not equivalent to the null field (contrary to a constant electrical field) and we find other fields that are equivalent to the constant magnetic field. We establish that one particular transformation of the free Schrödinger equation puts a plane wave and Green's function in a equivalence relation.

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Keywords: Schrödinger Equation, Intrinsic Characteristics, Equivalent Fields, the Shapovalov Group

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1. Exposition of the Problem

It is known that the Schrödinger equation for a particle in the electromagnetic field in Cartesian coordinates

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \quad \text{with} \quad \hat{H} = \frac{1}{2m} \left[-i\hbar\nabla - e\vec{A}(t, \vec{r}) \right]^2 + e\varphi(t, \vec{r}). \quad (1)$$

must be completed with the following known characteristics: a physical interpretation of the wave function and the fact that the 4-potential must be real so that the Hamilton

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operator is hermitian (if not the normalization integrals could not be constant). These characteristics are intrinsic of the physics problem.

The Schrödinger equation completed by their intrinsic characteristics we call a physical problem in non-Relativist Mechanics. In this paper we realize the classification on the set of all physical problems using the Shapovalov's approach [1]. It is evident that for different intrinsic characteristics we obtain different equivalence relations and, consequently, different classifications.

Traditional studies add to the list of known intrinsic characteristics the structure of the electromagnetic field: \vec{E} and \vec{B} must be fixed. This imposition reduces the group of equivalence to the Cartesian product of the gauge invariance group and the Galilean transformations group (see, for example, [2]).

On the contrary, Shapovalov's approach does not include the structure of the electromagnetic field in the intrinsic characteristics. This allows us to find a broader equivalence group, and gives the new exact solutions of different physical problems.

The following theorem defines the equivalence group of the Schrödinger equation (we call it *The Shapovalov group*):

Shapovalov's Theorem 1. The Schrödinger equation in the Euclidian space completed by mentioned intrinsic characteristics admits following equivalence group: $G_{sh} = R \otimes T \otimes V \otimes \Gamma \otimes Y$. Where Γ is the gauge invariance group and $R(\theta_1(t), \theta_2(t), \theta_3(t))$; $T(c_1(t), c_2(t), c_3(t))$ and $V(s(t))$ are the rotations, translations, and scale change groups. The Shapovalov group contains 7 arbitrary time-dependent functions: $s(t), \theta_i(t), c_i(t)$ and $\frac{dt'}{dt} := s^2 > 0$, and Y is the discrete invariance group of the Schrödinger equation from Section 4.5.

The Shapovalov group is very effective for finding new solutions. We note that the traditional equivalence group is a subgroup of the $G_{sh} : \theta_i, c_i, s = const$.

For a specification of the Shapovalov group, we compare two Schrödinger equations such as (1) with different 4-potentials: $\{\vec{A}(t, \vec{r}), \varphi(t, \vec{r})\}$ and $\{\vec{A}'(t', \vec{r}'), \varphi'(t', \vec{r}')\}$. Additionally, we designate $\{t', \vec{r}'\}$ to be the variables of the second equation, this allows us to distinguish equivalent fields. The equivalence relations must conserve the structure (1) of the Schrödinger equation. When using only this imposition we found [1] that the Shapovalov group of the Schrödinger equation is the following :

$$t' = t'(t) \quad \left[\frac{dt'}{dt} := s^2 > 0 \right] \quad (2)$$

$$\vec{r}' = s(t) a(t) \vec{r} + c(t) \quad (3)$$

$$\Psi'(t', \vec{r}') = \Psi(t, \vec{r}) s^{-3/2}. \quad (4)$$

The seven arbitrary time-dependent functions of the Shapovalov group mean the following: $s(t)$ represents a freedom of time-scale and the coordinate-scale choice. Three functions $c_i(t)$ define three independent displacements with arbitrary time-dependent linear velocities. The orthogonal matrix $a(t)$ describes the rotations with three arbitrary time-dependent angular velocities. In Section 4 we study the subclasses of the null field equivalence class generated by each one of these three types of equivalence relations.

If a reference system were inertial, the Shapovalov group puts it in equivalence with the non-inertial systems. This fact is used frequently, for example, to describe the circular movement of a classic particle with a constant linear velocity. After being transferred to a rotating reference system, this physical problem is reduced to a free particle (see, for instance, [3], [4]). We emphasize that in the traditional approach such obvious equivalence does not exist. At least, this fact shows a need for extension of the traditional equivalence group.

2. Equivalent Potentials and Fields

Using the fact that the form (1) of the Schrödinger equation must be invariant and the equivalence relations of the Shapovalov group (2), (3) and (4), we obtain the equivalent potentials:

$$A'_k = s^{-1} \sum_j a_{kj} A_j - \frac{m}{e} s^{-2} x_k, \quad (5)$$

$$\varphi' = \frac{1}{s^2} \varphi + \sum_{k,j} \frac{1}{s^3} \dot{x}'_k a_{kj} A_j - \sum_k \frac{m}{2e} \frac{1}{s^4} \dot{x}'_k \dot{x}'_k, \quad (6)$$

where $\dot{x}'_k := \frac{\partial x'_k}{\partial t}$. Rigorously, the Shapovalov group is specified through formulas (2), (3), (4), (5), (6), and (39). Now, we can easily calculate the relation among equivalent electromagnetic fields:

$$s^2 B'_i(t', \vec{r}') = \sum_k a_{ik} B_k(t, \vec{r}) + \frac{m}{e} \sum_{l,k,j} \varepsilon_{ilk} \dot{a}_{lj}(t) a_{kj}(t), \quad (7)$$

$$s^2 \vec{E}'(t', \vec{r}') = \frac{1}{s} a \vec{E}(t, \vec{r}) - \dot{\vec{r}}' \times \vec{B}' + \frac{m}{e} \left(\frac{\dot{\vec{r}}'}{s^2} \right). \quad (8)$$

For a compact presentation, we introduce Larmor's notation: $\omega := \frac{eB}{2m}$ and the magnetic field's anti-symmetrical 3×3 matrix: $F_{ij} := \sum_k \varepsilon_{ijk} \omega_k$. The relation (7) between equivalent magnetic fields in matrix form appears as:

$$s^2 F'(t', \vec{r}') = a F(t, \vec{r}) a^T + \dot{a} a^T. \quad (9)$$

If, for example, we have only time-dependent magnetic fields: $B' = B'(t')$ and $B = B(t)$ they both are equivalent (a is an orthogonal matrix):

$$\dot{a}(t) = s^2 F'(t') a - a F(t). \quad (10)$$

Particularly, if both magnetic fields are constant, the following orthogonal matrix provides the equivalent relations:

$$a(t) = \exp \{ F' t'(t) - F t \}; \quad t'(0) = 0. \quad (11)$$

Then we observe that naturally the equivalent fields (7) and (8) simultaneously verify the first pair of the Maxwell equations. It is easy to find that:

$$s^3 \nabla' \cdot \vec{B}' = \nabla \cdot \vec{B}, \quad s^5 \left[\nabla' \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t'} \right] = sa \left[\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right] - (\operatorname{div} \vec{B}) \dot{\vec{r}}.$$

If one referential system does not have a “magnetic charge”, than no other equivalent system will have it. The second pair of the Maxwell equations gives the relations between the equivalent densities of charge and current:

$$s^4 \left(\frac{\rho'(t', \vec{r}')}{\epsilon_0} - \frac{2m}{e} \dot{\omega}^2 \right) = \frac{\rho(t, \vec{r})}{\epsilon_0} - \frac{2m}{e} \dot{\omega}^2 + s^2 \dot{\vec{r}} \operatorname{rot}' \vec{B}' - \frac{3m}{e} f(t), \quad (12)$$

$$\vec{J}'(t', \vec{r}') + \epsilon_0 \frac{\partial \vec{E}'}{\partial t'} = \frac{1}{s^3} a \left(\vec{J}(t, \vec{r}) + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right). \quad (13)$$

Where $\omega(t)$ is the Larmor frequency, \vec{E} and \vec{E}' verify (8) and

$$f(t) := s \left(\frac{1}{s} \right)'' . \quad (14)$$

3. Fields Equivalent to the Null Field

Here we study the null field equivalence class. If $\vec{E}' = \vec{B}' = 0$, then relations (7), (8), and (10) allow us to describe all fields of this equivalence class:

$$\vec{B} = \vec{B}(t)$$

defined by (7)

$$\dot{a} = -aF(t); \quad (15)$$

$$\vec{E}(t, \vec{r}) = \frac{m}{e} \left[b(t) \vec{r} + \vec{d}(t) \right], \quad (16)$$

where we denote the matrix:

$$b(t) := -sa^T \left[\frac{(sa)'}{s^2} \right]' = f(t)I + \dot{F} - F^2, \quad (17)$$

with $f(t)$ from (14), I unity matrix, and the vector

$$\vec{d}(t) := -sa^T \left[\frac{\dot{\vec{c}}}{s^2} \right]'. \quad (18)$$

We conclude from (15) and (16) that the widest field equivalent to the null field can contain an arbitrary time-dependent magnetic field, whereas the electric field is linear related to \vec{r} (with the matrix of time-dependent coefficients) :

$$\vec{B} = \vec{B}(t) \quad (19)$$

$$\vec{E}(t, \vec{r}) = \left(\frac{m}{e}\right) \left[\vec{r} \times \dot{\vec{\omega}} + (\vec{\omega} \times \vec{r}) \times \vec{\omega} + f(t)\vec{r} + d(t) \right], \quad (20)$$

where the second term of \vec{E} has the form of the centrifugal force. We observe that the class has non-orthogonal fields $\vec{E} \cdot \vec{B} \neq 0$, but all are equivalent to one orthogonal. From (12) and (13) we find the null equivalent densities of charge and current:

$$e\rho(t) = 2m\epsilon_0\omega^2(t) + 3m\epsilon_0f(t), \quad (21)$$

$$\vec{J}(t, \vec{r}) = -\epsilon_0 \frac{\partial \vec{E}}{\partial t} = -\frac{\epsilon_0 m}{e} \left[\dot{b}\vec{r} + \dot{d} \right]. \quad (22)$$

4. Particular Cases

4.1 Exclusively Rotation Equivalence ($s = 1$, $\vec{c} = 0$)

Now we treat the subclass of the fields which are equivalent to the null field using only the rotating equivalent reference systems. From (19) and (20) we have:

$$\vec{B} = \vec{B}(t), \quad (23)$$

$$\vec{E}(t, \vec{r}) = \left(\frac{m}{e}\right) \vec{r} \times \dot{\vec{\omega}} + \left(\frac{m}{e}\right) (\vec{\omega} \times \vec{r}) \times \vec{\omega}. \quad (24)$$

The following 4-potential creates such a field:

$$\vec{A} = \left(\frac{m}{e}\right) (\vec{\omega} \times \vec{r}), \quad \varphi = -\left(\frac{m}{2e}\right) (\vec{\omega} \times \vec{r})^2,$$

where the scalar potential represents the centrifugal energy. We observe that the electric and magnetic fields are not orthogonal in the general case: $\vec{E} \cdot \vec{B} = \left(\frac{1}{2}\right) \left(\dot{\vec{B}} \times \vec{B}\right) \cdot \vec{r}$, but they are if, for example, $\dot{\vec{B}} \parallel \vec{B}$. Hamilton's operator of this system has the form:

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta - \vec{\omega}(t) \cdot \hat{\vec{L}}, \quad (25)$$

where Δ is the Laplacian and $\hat{\vec{L}} := \vec{r} \times \hat{\vec{p}}$ is the operator of the angular momentum. The formula (25) corresponds exactly to the classic mechanics theorem of transformation of energy after passing from an inertial reference system to a system in rotation: $E_{rotation} = E_{inertial} - \vec{\omega} \cdot \vec{L}$ (see for instance [4]). If the field (23) and (24) is equivalent to the null field, then the system with Hamilton's operator (25) has a symmetry that corresponds to one of a free particle. The first integrals of motion of (25) are:

$$\hat{X}_1 = a(t)\hat{p}, \quad \hat{X}_2 = a(t) \left[-\frac{t}{m}\hat{p} + \vec{r} \right], \quad \hat{X}_3 = a(t)\hat{L}. \quad (26)$$

These first order operators of symmetry are unique for the system and constitute the base of all superior-order symmetry operators. The corresponding integrals of motion of

a free particle are: $\hat{X}'_i = a^T \hat{X}_i$ ($i = 1, 2, 3$). It is known that the eigenvectors of the linear momentum and the initial coordinate operators

$$\hat{X}'_1 = \hat{p}', \quad \hat{X}'_2 = - \left(\frac{t'}{m} \right) \hat{p}' + \hat{r}', \quad (27)$$

are respectively a plane wave and Green's function (we remove the primes on top of wave functions and variables):

$$\Psi_{plane\ wave} = C \exp \left[-i \frac{E}{\hbar} + i \frac{\vec{p}}{\hbar} \cdot \vec{r} \right]; \quad \Psi_{Green} = \frac{C}{t^{3/2}} \exp \left[i \frac{m(\vec{r} - \vec{\lambda})^2}{2\hbar t} \right]. \quad (28)$$

In Section 4.5 we show that an equivalence exists among these functions and among the operators (27). Now, using the known solutions (28), we can construct new solutions to the Schrödinger equation for the field (23) and (24) for an arbitrary $\vec{B}(t)$. The first operator of (26) \hat{X}'_1 has the following eigenfunction:

$$\Psi(t, \vec{r}) = C \exp \left[i \frac{\vec{\lambda}^T a(t) \vec{r}}{\hbar} - i \frac{\lambda^2}{2m\hbar} t \right]. \quad (29)$$

This solution describes a non-stationary state, it is equivalent to a plane wave for a free particle (stationary state function). The constant $\frac{\lambda^2}{2m}$ is quasi-energy, the expression $\vec{p}_0 := a^T(t)\lambda$ is understood to be a quasi-momentum. We did not need to solve the Schrödinger equation for a particle in an arbitrary time-dependent field (23), (24). But we directly wrote its solution (29) without solving this equation, only using an equivalence relation of the Shapovalov group. We observe that contrary to the null field, for a particle in the field (23) and (24) the vector momentum is not a first integral of motion.

The Hamilton's operator (25) and the vector operator \hat{X}'_2 of (26) provide the solution of the Schrödinger equation that corresponds to the Green's function:

$$\Psi(t, \vec{r}) = \frac{C}{t^{3/2}} \exp \left[i \frac{m(\vec{r} - \vec{r}_0(t))^2}{2\hbar t} \right], \quad (30)$$

where $\vec{r}_0(t) := a^T(t)\vec{\lambda}$, $\vec{\lambda}$ is the initial coordinate in the inertial reference system; $\vec{r}_0(t)$ is the same initial coordinate in a rotating reference system. The vector λ is an arbitrary constant vector (eigenvalue of \hat{X}'_2). An orthogonal matrix $a(t)$ verifies (15) with the arbitrary vector function $\vec{\omega}(t)$. It is known that the solution (30) can be orthonormalized to the Dirac delta function, but its interpretation is difficult. It is a case of equivalence of a non-stationary solution to another non-stationary one.

The third vector operator of symmetry (26) \hat{X}'_3 does not allow a direct construction of a solution because as its components do not commute, it is necessary to use the set equivalent to: \hat{H}' , \hat{L}'^2 , \hat{L}'_z .

4.2 Exclusively Euclidean Translational Equivalence (the Unity Matrix $a = I$ and $s = 1$)

The subclass of the Euclidean translational displacements ($\vec{c} \neq 0$) relates to the equivalence between reference systems moving with arbitrary linear velocities. For example, the only time-dependent uniform electrical field (16) defined by vector $\vec{d}(t)$ from (18), with the uniform corresponding current (22) is equivalent to the null field without a current. Naturally, Galileo's equivalence belongs to the Shapovalov group (see Section 4.5).

4.3 Isotropic Oscillator (Exclusively Time Scale Equivalence: the Unity Matrix $a = I$ and $\vec{c} = 0$)

Let be $\vec{B} = 0 \implies b(t)$ is a constant matrix. Using formulas (19 and 20), for $\vec{c} = 0$ and $\vec{\omega} = 0$, we find the field of the isotropic oscillator: $E = \left(\frac{m}{e}\right) f(t)r$. In case of $f(t) = const$, it corresponds to the potential energy $U = \frac{m\omega_0^2 r^2}{2}$, with $\omega_0 = const$. We note that constant ω_0 does not have any relation to the Larmor frequency ω (which is null here). From this form of potential energy and definition (14) we obtain the well known relation:

$$\left(\frac{1}{s}\right)'' + \omega_0^2 \left(\frac{1}{s}\right) = 0.$$

Solving it, we find the time-scale equivalence (scale change group of Shapovalov's theorem):

$$\frac{dt'}{dt} := s^2 = \sec^2(\omega_0 t) \implies \omega_0 t' = \tan(\omega_0 t), \quad \vec{r}' = s(t)\vec{r}. \quad (31)$$

This equivalence relation provides an equivalence between the free Schrödinger equation

$$i\hbar \frac{\partial \Psi'}{\partial t'} = \hat{H}' \Psi' = -\frac{\hbar^2}{2m} \Delta' \Psi', \quad (32)$$

and the equation for the isotropic oscillator:

$$i\hbar \frac{\partial \psi_{nlm}(t, \vec{r})}{\partial t} = \hat{H} \psi_{nlm}(t, \vec{r}) := \left[-\frac{\hbar^2}{2m} \Delta + \frac{m\omega_0^2 r^2}{2} \right] \psi_{nlm}(t, \vec{r}).$$

The solutions of the last equation are known (for instance, [5], [7]). We observe that these states are defined by the following set of integrals of movement: \hat{H} , \hat{L}^2 , \hat{L}_z . In addition, in the equivalence relation (31), it is necessary to use gauge invariance:

$$\Psi'_{nlm}(t', \vec{r}') = \frac{\psi_{nlm}(t, \vec{r})}{s^{3/2}} \exp \left[i \frac{m \dot{s}}{2\hbar s} r^2 \right]; \quad (33)$$

with the wave function:

$$\psi_{nlm}(t, \vec{r}) = \varphi_{nlm}(\vec{r}) \exp \left[-i \frac{E_{nl}}{\hbar} t \right]; \quad (34)$$

$$E_{nl} = \hbar\omega_0 \left[2n + l + \frac{3}{2} \right]; \quad n, l = 0, 2, \dots; \quad m = 0, \pm 1, \dots, \pm l;$$

$$\varphi_{nlm}(r, \theta, \phi) = \frac{1}{\xi} R_{nl}(\xi) Y_{lm}(\theta, \phi); \quad \xi := r \sqrt{\frac{m\omega_0}{\hbar}}.$$

Here the functions $Y_{lm}(\theta, \phi)$ are the spherical harmonics and $R_{nl}(\xi)$ are the radial functions expressed by confluent hypergeometric functions. The non-stationary solutions Ψ' (33) of the free Schrödinger equation (32) are known as coherent states and are equivalent to the stationary states of the isotropic oscillator. For example, the function of the fundamental state that verifies the equation (32) can be written (we abandon the primes on top of the variables and the wave function):

$$\Psi_{000}(t, \vec{r}) = \frac{C}{\sqrt{4\pi} (1 + \omega_0^2 t^2)^{3/4}} \exp \left[-\frac{m\omega_0}{2\hbar} \frac{1}{1 + i\omega_0 t} \vec{r}^2 - i\frac{3}{2} \arctan(\omega_0 t) \right]. \quad (35)$$

We observe that to establish the equivalence of the free particle's coherent states to the stationary states of an isotropic oscillator, the time scale equivalence relation (31) is indispensable. For another equivalent solution to (35) of the free Schrödinger equation see (40). The equivalence between a free particle and an isotropic oscillator of one dimension is discussed in [8].

If $f(t)$ from (14) is not a constant, we have an arbitrary time-dependent radial dilation. In this case, new solutions of other physical problems can be constructed.

4.4 Constant Magnetic Field

In the null field equivalent class (15) and (16), we find a field with a constant magnetic component:

$$\vec{B} = (0, 0, B) = \text{const}; \quad \vec{E}(t, \vec{r}) = \frac{m}{e} [f(t)\vec{r} + \omega^2 (x_1, x_2, 0)] + \vec{\varepsilon}(t). \quad (36)$$

Here $f(t)$ is defined by (14), $\vec{\varepsilon}(t)$ is an arbitrary time-dependent vector function. The first term of the electrical component corresponds to an arbitrary time-dependent dilation; particularly to an isotropic oscillator from Section 4.3 with an arbitrary time-dependent frequency. The second term corresponds to the constant centrifugal force field and the third term results from Euclidean translational freedom ($c(t) \neq 0$). Here are two particular cases of interest:

$$\text{if } f = 0 \quad (s = 1) \quad \Longrightarrow \quad \vec{B} = (0, 0, B); \quad \vec{E}(t, \vec{r}) = \frac{m\omega^2}{e} (x_1, x_2, 0); \quad (37)$$

$$\text{if } f := s \left(\frac{1}{s} \right)'' = -\omega_0^2 \quad \Longrightarrow \quad \vec{B} = (0, 0, B); \quad \vec{E}(t, \vec{r}) = -\frac{m\omega^2}{e} (0, 0, x_3).$$

We can see that only constant magnetic field does not belong to the null field equivalent class, but always some electrical component of the field is present. By using formulas (7)

and (8), we find that only constant magnetic field is equivalent to the constant centripetal force field:

$$\vec{B}' = (0, 0, B') = \text{const}; \quad \vec{E}' = 0 \quad \Leftrightarrow \quad \vec{B} = 0; \quad \vec{E}(t, \vec{r}) = -\frac{m\omega'^2}{e}(x_1, x_2, 0). \quad (38)$$

This result explains the formula (37). If $f = \text{const}$, $\vec{\varepsilon} = \text{const}$, the field (36) defines the widest stationary field equivalent to the null field.

4.5 Free Schrödinger Equation's Nucleus

It is interesting to find the set of equivalence relations that put the null field in equivalence to itself (see also [6]). By choosing $\vec{E}' = \vec{B}' = \vec{E} = \vec{B} = 0$, we easily find that such a nucleus is the Galilean group:

$$t' = \alpha^2 t + \beta; \quad \vec{r}' = \alpha a \vec{r} + \vec{v}_0 t + \vec{r}_0; \quad \Psi' = \alpha^{-3/2} \Psi$$

with arbitrary constants α , β , \vec{v}_0 , \vec{r}_0 , and a constant orthogonal matrix a . In addition, one isolated equivalence operation exists and leaves the free Schrödinger equation invariant:

$$t' = -\frac{1}{t}; \quad \vec{r}' = \frac{1}{t} \vec{r}; \quad \Psi'(t', \vec{r}') = \Psi(t, \vec{r}) t^{3/2} \exp\left[-i \frac{m r^2}{2\hbar t}\right], \quad (39)$$

$$\frac{t}{\Psi} \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \Delta \right] \Psi = -\frac{t'}{\Psi'} \left[i\hbar \frac{\partial}{\partial t'} + \frac{\hbar^2}{2m} \Delta' \right] \Psi'.$$

Let us denote this transformation e . It is easy to show that $\{e, e^2, e^3, e^4\}$ constitutes the discrete invariance group of the Schrödinger equation Y from the Shapovalov's theorem. This transformation has a very interesting property: it transforms Green's function (28) to the plane wave propagating in the direction $-\vec{\lambda}$:

$$\Psi'(t', \vec{r}') = C' \exp\left[-i\omega' t' - i\vec{k}' \vec{r}'\right], \quad \hbar\omega' := \frac{m\lambda^2}{2}; \quad \hbar\vec{k}' := m\vec{\lambda}.$$

And inversely, the equivalence relation (39) transforms a plane wave into Green's function. Of course, the relation (39) also puts in equivalence the operator of linear momentum $\hat{\vec{p}}'$ and the operator of initial coordinate: $-(\frac{t}{m}) \hat{\vec{p}} + \vec{r}$ (see formula (27)). This operation is similar to passing to momentum space. This transformation does not make sense if $\vec{\lambda} = 0$.

We easily find that relation (39) does not change the angular momentum: $\vec{L} = \hat{\vec{L}}'$. Consequently, the well-known states of a free particle with a constant angular momentum are invariant in respect to relation (39).

Another example, relation (39) puts the coherent state (35) in equivalence with the following function:

$$\Psi'_{000}(t', \vec{r}') = \frac{C}{\sqrt{4\pi} (\omega_0^2 + t'^2)^{3/4}} \exp\left[-\frac{m}{2\hbar} \frac{1}{\omega_0 + it'} \vec{r}'^2 - i\frac{3}{2} \arctan\left(\frac{\omega_0}{t'}\right)\right]. \quad (40)$$

It is a new solution of the free Schrödinger equation (32).

Also, we can note that formula (33) with the relations $\frac{dt'}{dt} := s^2(t)$ and $\vec{r}' = s(t)\vec{r}$ contains the operation of equivalence (39) if we just impose the transformation of the Green's function to a plane wave or inverse.

4.6 Classic Particle

The Shapovalov group was found by Shapovalov and Sukhomlin (1974) [1] who also enumerated all cases of separation of variables in a parabolic equation and proved that all of these cases recur in quantum mechanics and in classical Hamilton-Jacobi approach.

In 1980, S. Benenti and M. Francaviglia [9] applied the Shapovalov group to the Hamilton-Jacobi equation and G. Reid [10] extended it in 1986 to the space of n -dimensions.

A study of the Hamilton-Jacobi equation gives the same equivalence group that the Schrödinger equation (2), (3), (4) does. In fact, the equivalent potentials and the fields are the same as (5), (6) and (7), (8).

In particular, we refer to the results to the equivalence study in some cases. Using (31) we establish the relation between equivalent actions as in Section 4.3:

$$W'(t', \vec{r}') = W(t, \vec{r}) + \frac{m \dot{s}}{2s} \vec{r}^2, \quad (41)$$

where $W(t, \vec{r})$ corresponds to the isotropic oscillator and $W'(t', \vec{r}')$ to coherent states of a free particle.

Finally:

$$\begin{aligned} W'(t', r') = & -\frac{\alpha}{\omega_0} \cos^{-1} \left(\sqrt{\frac{1}{2\alpha} \frac{m\omega_0^2}{1 + \omega_0^2 t'^2} r'} \right) + \frac{m\omega_0}{2} r' \sqrt{\frac{1}{1 + \omega_0^2 t'^2} \left(\frac{2\alpha}{m\omega_0^2} - \frac{r'^2}{1 + \omega_0^2 t'^2} \right)} \\ & + \frac{m\omega_0}{2} \frac{t'}{1 + \omega_0^2 t'^2} r'^2 - \frac{\alpha}{\omega_0} \arctan(\omega_0 t'). \end{aligned} \quad (42)$$

Here α is energy. As in Section 4.5 we can verify that the nucleus of the Hamilton-Jacobi equation is specified by the same relation (39) as gauge invariance:

$$W'(t', \vec{r}') = W(t, \vec{r}) - \frac{m}{2} \frac{r^2}{t}, \quad (43)$$

where $W(t, \vec{r})$ is (42) without the primes. The solution of Hamilton-Jacobi that corresponds to (40), presents:

$$\begin{aligned} W'(t', r') = & -\frac{\alpha}{\omega_0} \cos^{-1} \left(\sqrt{\frac{1}{2\alpha} \frac{m\omega_0^2}{\omega_0^2 + t'^2} r'} \right) + \frac{m\omega_0}{2} r' \sqrt{\frac{1}{\omega_0^2 + t'^2} \left(\frac{2\alpha}{m\omega_0^2} - \frac{r'^2}{\omega_0^2 + t'^2} \right)} \\ & + \frac{m\omega_0}{2t'} \frac{r'^2}{\omega_0^2 + t'^2} - \frac{\alpha}{\omega_0} \arctan\left(\frac{\omega_0}{t'}\right). \end{aligned} \quad (44)$$

Conclusions

- (1) The null field is equivalent to the uniform magnetic field with arbitrary time dependence. The corresponding electric component (linear related to \vec{r}) is defined by (16) or (20). The equivalence relation between such a magnetic field and the null field is similar to the passing from an initial reference system to one which is in rotation defined by an orthogonal matrix from (15).
- (2) The null field equivalence class has several non-orthogonal fields.
- (3) The arbitrary-uniform charge distribution (time-dependent or not) is equivalent to the null charge density according to (21). Its time dependence comes from two types of equivalence: one from rotation of a reference system and the other from time scale equivalence. It is possible to have a null charge distribution simultaneously for both fields: one of type (15), (16) and the null field. Its current densities can be null also. The widest current density equivalent to the null current is linear related to \vec{r} (22).
- (4) The time scale equivalence ($\dot{s} \neq 0$) has an important role in the equivalence relations on all physical problem sets (see the isotropic oscillator, Section 4.3). Particularly it is known that the non-stationary coherent states of a free particle (33) are equivalent to the stationary states of a harmonic oscillator (34). Another example is when concrete relations establish the equivalence between a non-dissipative wave packet and one with dissipation [5].
- (5) Only Euclidean translational equivalence ($\vec{c}(t) \neq 0$) establishes a correspondence between arbitrary time-dependent uniform electrical fields and the null field (see Section 4.2).
- (6) Since the free Schrödinger equation has both stationary and non-stationary solutions, in the null field equivalence class we find four types of equivalence relations: the stationary wave functions equivalent to stationary or non-stationary states. Also non-stationary states could be equivalent to one other stationary or non-stationary wave functions. For example, the non-stationary wave function (29) is equivalent to a stationary one (plane wave (28)); the non-stationary function (30) is equivalent to another non-stationary one (Green's function of the free Schrödinger equation (28)).
- (7) The widest stationary field equivalent to the null field is defined by formula (36) with conditions $f = const$, $\vec{\varepsilon} = const$.
- (8) The formulas (36) and (38) show that only a constant magnetic field does not belong to the null field equivalent class (contrary to a constant electrical field).
- (9) The known isolated equivalence relation (39) puts the free Schrödinger equation in equivalence to itself. The plane wave is equivalent by means of this relation to Green's function. It is similar to the rotation in a phase space.

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Magnetized Bianchi Type VI_0 Barotropic Massive String Universe with Decaying Vacuum Energy Density Λ

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Abstract: Bianchi type VI_0 massive string cosmological models using the technique given by Letelier (1983) with magnetic field are investigated. To get the deterministic models, we assume that the expansion (θ) in the model is proportional to the shear (σ) and also the fluid obeys the barotropic equation of state. It was found that vacuum energy density $\Lambda \propto \frac{1}{t^2}$ which matches with natural units. The behaviour of the models from physical and geometrical aspects in presence and absence of magnetic field is also discussed.

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1. Introduction

The problem of the cosmological constant is one of the most salient and unsettled problems in cosmology. The smallness of the effective cosmological constant recently observed ($\Lambda_0 \leq 10^{-56} \text{cm}^{-2}$) constitutes the most difficult problems involving cosmology and elementary particle physics theory. To explain the striking cancelation between the “bare” cosmological constant and the ordinary vacuum energy contributions of the quantum fields, many mechanisms have been proposed during last few years [1]. The “cosmological constant problem” can be expressed as the discrepancy between the negligible value of Λ has for the present universe (as can be seen by the successes of Newton’s theory of gravitation [2]) and the values 10^{50} larger expected by the Glashow-Salam-Weinberg

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model [3] or by grand unified theory (GUT) where it should be 10^{107} larger [4]. The cosmological term Λ is then small at the present epoch. The problem in this approach is to determine the right dependence of Λ upon S or t . Recent observations of Type Ia supernovae (Perlmutter et al. [5], Riess et al. [6]) and measurements of the cosmic microwave background [7] suggest that the universe is an accelerating expansion phase [8].

Several ansätze have been proposed in which the Λ term decays with time (see Refs. Gasperini [9], Berman [10]–[12], Berman et al. [13]–[15], Freese et al. [16], Özer and Taha [17], Ratra and Peebles [18], Chen and Hu [19], Abdussattar and Vishwakarma [20], Gariel and Le Denmat [21], Pradhan et al. [22]). Of the special interest is the ansatz $\Lambda \propto S^{-2}$ (where S is the scale factor of the Robertson-Walker metric) by Chen and Wu [19], which has been considered/modified by several authors (Abdel-Rahaman [23], Carvalho et al. [24], Silveira and Waga [25], Vishwakarma [26]).

One of the outstanding problems in cosmology today is developing a more precise understanding of structure formation in the universe, that is, the origin of galaxies and other large-scale structures. Existing theories for the structure formation of the Universe fall into two categories, based either upon the amplification of quantum fluctuations in a scalar field during *inflation*, or upon symmetry breaking phase transition in the early Universe which leads to the formation of *topological defects* such as domain walls, cosmic strings, monopoles, textures and other 'hybrid' creatures. Cosmic strings play an important role in the study of the early universe. These arise during the phase transition after the big bang explosion as the temperature goes down below some critical temperature as predicted by grand unified theories (see Refs. Zel'dovich et al. [27], Kibble [28, 29], Everett [30], Vilenkin [31]). It is believed that cosmic strings give rise to density perturbations which lead to formation of galaxies (Zel'dovich [32]). These cosmic strings have stress energy and couple to the gravitational field. Therefore, it is interesting to study the gravitational effect which arises from strings. The general treatment of strings was initiated by Letelier [33, 34] and Stachel [35].

The occurrence of magnetic fields on galactic scale is well-established fact today, and their importance for a variety of astrophysical phenomena is generally acknowledged. Several authors (Zeldovich [36], Harrison [37], Misner, Thorne and Wheeler [38], Asseo and Sol [39], Pudritz and Silk [40], Kim, Tribble, and Kronberg [41], Perley, and Taylor [42], Kronberg, Perry, and Zukowski [43], Wolfe, Lanzetta and Oren [44], Kulsrud, Cen, Ostriker and Ryu [45] and Barrow [46]) have pointed out the importance of magnetic field in different context. As a natural consequences, we should include magnetic fields in the energy-momentum tensor of the early universe. The string cosmological models with a magnetic field are also discussed by Benerjee et al. [47], Chakraborty [48], Tikekar and Patel ([49, 50], Patel and Maharaj [51] Singh and Singh [52].

Recently, Bali et al. [53]–[57], Pradhan et al. [58] – [60], Yadav et al. [61] and Pradhan [62] have investigated Bianchi type I, II, III, V, IX and cylindrically symmetric magnetized string cosmological models in presence and absence of magnetic field. Tikekar and Patel [50] have investigated some solutions for Bianchi type VI_0 cosmology in presence and absence of magnetic field. In this paper we have derived some Bianchi type VI_0 string cosmological models for perfect fluid distribution in presence and absence of magnetic field and discussed the variation of Λ with time. This paper is organized as follows: The metric and field equations are presented in Section 2. In Section 3, we deal with the solution of the field equations in presence of magnetic field. In Section 4, we have described some geometric and physical behavior of the model. Section 5 includes the solution in absence of magnetic field. In Section 6, we have discussed the variation of cosmological constant Λ with time in presence and absence of magnetic field. In the last Section 7, concluding remarks are given.

2. The Metric and Field Equations

We consider the Bianchi Type VI_0 metric in the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)e^{2x}dy^2 + C^2(t)e^{-2x}dz^2. \quad (1)$$

The energy-momentum tensor for a cloud of strings in presence of magnetic field is taken into the form

$$T_{ik} = (\rho + p)v_iv_k + pg_{ik} - \lambda x_ix_k + [g^{lm}F_{il}F_{km} - \frac{1}{4}g_{ik}F_{lm}F^{lm}], \quad (2)$$

where v_i and x_i satisfy conditions

$$v^i v_i = -x^i x_i = -1, \quad v^i x_i = 0. \quad (3)$$

In equations (2), p is isotropic pressure, ρ is rest energy density for a cloud strings, λ is the string tension density, F_{ij} is the electromagnetic field tensor, x^i is a unit space-like vector representing the direction of string, and v^i is the four velocity vector satisfying the relation

$$g_{ij}v^i v^j = -1. \quad (4)$$

Here, the co-moving coordinates are taken to be $v^1 = 0 = v^2 = v^3$ and $v^4 = 1$ and $x^i = (\frac{1}{A}, 0, 0, 0)$. The Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0, \quad (5)$$

$$F_{;k}^{ik} = 0, \quad (6)$$

are satisfied by

$$F_{23} = K(\text{say}) = \text{constant}, \quad (7)$$

where a semicolon (;) stands for covariant differentiation.

The Einstein's field equations (with $\frac{8\pi G}{c^4} = 1$)

$$R_i^j - \frac{1}{2}Rg_i^j = -T_i^j - \Lambda g_i^j, \quad (8)$$

for the line-element(1) lead to the following system of equations:

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} + \frac{1}{A^2} = - \left[p - \lambda - \frac{K^2}{2B^2C^2} \right] - \Lambda, \quad (9)$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} - \frac{1}{A^2} = - \left[p + \frac{K^2}{2B^2C^2} \right] - \Lambda, \quad (10)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} - \frac{1}{A^2} = - \left[p + \frac{K^2}{2B^2C^2} \right] - \Lambda, \quad (11)$$

$$\frac{A_4B_4}{AB} + \frac{B_4C_4}{BC} + \frac{C_4A_4}{CA} - \frac{1}{A^2} = \left[\rho + \frac{K^2}{2B^2C^2} \right] - \Lambda, \quad (12)$$

$$\frac{1}{A} \left[\frac{C_4}{C} - \frac{B_4}{B} \right] = 0, \quad (13)$$

where the sub indice 4 in A, B, C denotes ordinary differentiation with respect to t . The velocity field v^i is irrotational. The scalar expansion θ and components of shear σ_{ij} are given by

$$\theta = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}, \quad (14)$$

$$\sigma_{11} = \frac{A^2}{3} \left[\frac{2A_4}{A} - \frac{B_4}{B} - \frac{C_4}{C} \right], \quad (15)$$

$$\sigma_{22} = \frac{B^2}{3} \left[\frac{2B_4}{B} - \frac{A_4}{A} - \frac{C_4}{C} \right], \quad (16)$$

$$\sigma_{33} = \frac{C^2}{3} \left[\frac{2C_4}{C} - \frac{A_4}{A} - \frac{B_4}{B} \right], \quad (17)$$

$$\sigma_{44} = 0. \quad (18)$$

Therefore

$$\sigma^2 = \frac{1}{2} \left[(\sigma^1_1)^2 + (\sigma^2_2)^2 + (\sigma^3_3)^2 + (\sigma^4_4)^2 \right],$$

which leads to

$$\sigma^2 = \frac{1}{3} \left[\frac{A_4^2}{A^2} + \frac{B_4^2}{B^2} + \frac{C_4^2}{C^2} - \frac{A_4B_4}{AB} - \frac{B_4C_4}{BC} - \frac{C_4A_4}{CA} \right].$$

Above relation after using (13) reduces to

$$\sigma = \frac{1}{\sqrt{3}} \left(\frac{A_4}{A} - \frac{B_4}{B} \right). \quad (19)$$

3. Solutions of the Field Equations

The field equations (9)-(13) are a system of five equations with seven unknown parameters A , B , C , ρ , p , λ and Λ . We need two additional conditions to obtain explicit solutions of the system.

Equation (13) leads to

$$C = mB, \quad (20)$$

where m is an integrating constant.

We first assume that the expansion (θ) in the model is proportional to shear (σ). The motive behind assuming this condition is explained with reference to Thorne [63], the observations of the velocity-red-shift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic today within ≈ 30 per cent [64, 65]. To put more precisely, red-shift studies place the limit

$$\frac{\sigma}{H} \leq 0.3$$

on the ratio of shear, σ , to Hubble constant, H , in the neighborhood of our Galaxy today. Collins et al. [66] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfies that the condition $\frac{\sigma}{\theta}$ is constant. This condition and Eq. (20) lead to

$$\frac{1}{\sqrt{3}} \left(\frac{A_4}{A} - \frac{B_4}{B} \right) = l \left(\frac{A_4}{A} + \frac{2B_4}{B} \right) \quad (21)$$

which yields to

$$\frac{A_4}{A} = n \frac{B_4}{B}, \quad (22)$$

where $n = \frac{(2l\sqrt{3}+1)}{(1-l\sqrt{3})}$ and l are constants. Eq. (22), after integration, reduces to

$$A = \beta B^n, \quad (23)$$

where β is a constant of integration. Eqs. (10) and (12) lead to

$$p = -\frac{K^2}{2B^2C^2} - \left(\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} - \frac{1}{A^2} \right) - \Lambda, \quad (24)$$

and

$$\rho = \frac{A_4B_4}{AB} + \frac{B_4C_4}{BC} + \frac{C_4A_4}{CA} - \frac{1}{A^2} - \frac{K^2}{2B^2C^2} + \Lambda, \quad (25)$$

respectively. Now let us consider that the fluid obeys the barotropic equation of state

$$p = \gamma\rho, \quad (26)$$

where $\gamma(\gamma \leq 0 \leq 1)$ is a constant. Eqs. (24) to (26) lead to

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + (1 + \gamma) \frac{A_4 C_4}{AC} + \gamma \left(\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} \right) - (1 + \gamma) \frac{1}{A^2} + (1 - \gamma) \frac{K^2}{2B^2 C^2} + (1 + \gamma) \Lambda = 0. \quad (27)$$

Eq. (27) with the help of (20) and (23) reduces to

$$2B_{44} + \frac{2(n^2 + 2\gamma n + \gamma) B_4^2}{(n + 1) B^2} = \frac{2(1 + \gamma)}{\beta^2 B^{2n-1}} + \frac{(1 - \gamma) K^2}{m^2 B^3} + 2l_0 B, \quad (28)$$

where $l_0 = (1 + \gamma)\Lambda$.

Let us consider $B_4 = f(B)$ and $f' = \frac{df}{dB}$. Hence Eq. (28) takes the form

$$\frac{d}{df}(f^2) + \frac{2\alpha}{B} f^2 = \frac{2(1 + \gamma)}{\beta^2 B^{2n-1}} + \frac{(1 - \gamma) K^2}{m^2 B^3} + 2l_0 B, \quad (29)$$

where $\alpha = \frac{(n^2 + 2n\gamma + \gamma)}{(n+1)}$. Eq. (29) after integrating reduces to

$$f^2 = \frac{2(1 + \gamma) B^{-2n+2}}{\beta^2(2\alpha - 2n + 2)} + \frac{(1 - \gamma) K^2}{2m^2(\alpha - 1)} + \frac{l_0 B^2}{(\alpha + 1)} + M B^{-2\alpha}, \quad \gamma \neq 1, \quad (30)$$

where M is an integrating constant. To get deterministic solution in terms of cosmic string t , we suppose $M = 0$ without any loss of generality. In this case Eq. (30) takes the form

$$f^2 = aB^{-2(n-1)} + bB^{-2} + kB^2, \quad (31)$$

where

$$a = \frac{2(1 + \gamma)}{\beta^2(2\alpha - 2n + 2)}, \quad b = \frac{(1 - \gamma) K^2}{2m^2(\alpha - 1)}, \quad k = \frac{(1 + \gamma)\Lambda}{(\alpha + 1)}.$$

Therefore, we have

$$\frac{dB}{\sqrt{aB^{-2(n-1)} + bB^{-2} + kB^2}} = dt. \quad (32)$$

To get deterministic solution, we assume $n = 2$. In this case integrating Eq. (32), we obtain

$$B^2 = \sqrt{(a + b)} \frac{\sinh(2\sqrt{k}t)}{\sqrt{k}}. \quad (33)$$

Hence, we have

$$C^2 = m^2 \sqrt{(a + b)} \frac{\sinh(2\sqrt{k}t)}{\sqrt{k}}, \quad (34)$$

$$A^2 = \beta^2(a + b) \frac{\sinh^2(2\sqrt{k}t)}{k}, \quad (35)$$

where $k > 0$ without any loss of generality.

Therefore, the metric (1), in presence of magnetic field, reduces to the form

$$ds^2 = -dt^2 + \beta^2(a + b) \frac{\sinh^2(2\sqrt{k}t)}{k} dx^2 + \sqrt{(a + b)} \frac{\sinh(2\sqrt{k}t)}{\sqrt{k}} e^{2x} dy^2 + m^2 \sqrt{(a + b)} \frac{\sinh(2\sqrt{k}t)}{\sqrt{k}} e^{-2x} dz^2. \quad (36)$$

4. The Geometric and Physical Significance of Model

The pressure (p), energy density (ρ), the string tension density (λ), the particle density (ρ_p), the scalar of expansion (θ), the shear tensor (σ) and the proper volume (V^3) for the model (36) are given by

$$p = \left[\frac{k}{\beta^2(a+b)} - \frac{K^2 k}{2m^2(a+b)} \right] \coth^2(2\sqrt{kt}) + \left[\frac{K^2}{2m(a+b)} - \frac{1}{\beta^2(a+b)} - 8 \right] k - \Lambda, \quad (37)$$

$$\rho = \left[5k - \frac{k}{(a+b)} \left(\frac{K^2}{2m^2} + \frac{1}{\beta^2} \right) \right] \coth^2(2\sqrt{kt}) + \frac{k}{(a+b)} \left(\frac{K^2}{2m^2} + \frac{1}{\beta^2} \right) + \Lambda, \quad (38)$$

where $p = \gamma\rho$ is satisfied by (27).

$$\lambda = \left[\frac{2k}{\beta^2(a+b)} - \frac{K^2 k}{m^2(a+b)} - k \right] \coth^2(2\sqrt{kt}) + \left\{ \frac{K^2 k}{m^2(a+b)} - \frac{2k}{\beta^2(a+b)} - 4k \right\}, \quad (39)$$

$$\rho_p = \rho - \lambda = \left[\frac{K^2 k}{2m^2(a+b)} - \frac{3k}{\beta^2(a+b)} + k \right] \coth^2(2\sqrt{kt}) + 9k + \left\{ \frac{3k}{\beta^2(a+b)} - \frac{K^2}{2m^2(a+b)} \right\}, \quad (40)$$

$$\theta = 4\sqrt{k} \coth(2\sqrt{kt}), \quad (41)$$

$$\sigma = \sqrt{\frac{k}{3}} \coth(2\sqrt{kt}), \quad (42)$$

$$V^3 = \frac{\beta m(a+b)}{k} \sinh^2(2\sqrt{kt}). \quad (43)$$

From Eqs. (30) and (31), we obtain

$$\frac{\sigma}{\theta} = \text{constant}. \quad (44)$$

The deceleration parameter is given by

$$q = -\frac{\ddot{R}/R}{\dot{R}^2/R^2} = -\left[\frac{\frac{8k}{3} - \frac{8k}{9} \coth^2(2\sqrt{kt})}{\frac{16k}{9} \coth^2(2\sqrt{kt})} \right]. \quad (45)$$

From (45), we observe that

$$q < 0 \text{ if } \coth^2(2\sqrt{kt}) < 3$$

and

$$q > 0 \text{ if } \coth^2(2\sqrt{kt}) > 3.$$

From (38), $\rho \geq 0$ implies that

$$\coth^2(2\sqrt{kt}) \leq \left[\frac{\frac{k}{(a+b)} \left(\frac{K^2}{2m^2} + \frac{1}{\beta^2} \right) + \Lambda}{\frac{k}{(a+b)} \left(\frac{K^2}{2m^2} + \frac{1}{\beta^2} \right) - 5k} \right]. \quad (46)$$

Also from (40), $\rho_p \geq 0$ implies that

$$\coth^2(2\sqrt{kt}) \leq \left[\frac{\frac{3k}{\beta^2(a+b)} - \frac{K^2}{2m^2(a+b)} + 9k}{\frac{3k}{\beta^2(a+b)} - \frac{K^2k}{2m^2(a+b)} - k} \right]. \quad (47)$$

Thus the energy conditions $\rho \geq 0$, $\rho_p \geq 0$ are satisfied under conditions given by (46) and (47).

The model (36) starts with a big bang at $t = 0$. The expansion in the model decreases as time increases. The proper volume of the model increases as time increases. Since $\frac{\sigma}{\theta} = \text{constant}$, hence the model does not approach isotropy. Since ρ , λ , θ , σ tend to infinity and $V^3 \rightarrow 0$ at initial epoch $t = 0$, therefore, the model (36) for massive string in presence of magnetic field has Line-singularity (Banerjee et al. [47]). For the condition $\coth^2(2\sqrt{kt}) < 3$, the solution gives accelerating model of the universe. It can be easily seen that when $\coth^2(2\sqrt{kt}) > 3$, our solution represents decelerating model of the universe.

5. Solutions in Absence of Magnetic Field

In absence of magnetic field, i.e. when $b \rightarrow 0$ i.e. $K \rightarrow 0$, we obtain

$$\begin{aligned} B^2 &= 2\sqrt{2} \frac{\sinh(2\sqrt{kt})}{2\sqrt{k}}, \\ C^2 &= 2m^2\sqrt{a} \frac{\sinh(2\sqrt{kt})}{2\sqrt{k}}, \\ A^2 &= 4a\beta^2 \frac{\sinh^2(2\sqrt{kt})}{4k}. \end{aligned} \quad (48)$$

Hence, in this case, the geometry of the universe (36) reduces to

$$\begin{aligned} ds^2 &= -dt^2 + 4\beta^2 a \frac{\sinh^2(2\sqrt{kt})}{4k} dx^2 + \\ &2\sqrt{2} \frac{\sinh(2\sqrt{kt})}{2\sqrt{k}} e^{2x} dy^2 + 2m^2\sqrt{a} \frac{\sinh(2\sqrt{kt})}{2\sqrt{k}} e^{-2x} dz^2. \end{aligned} \quad (49)$$

The pressure (p), energy density (ρ), the string tension density (λ), the particle density (ρ_p), the scalar of expansion (θ), the shear tensor (σ) and the proper volume (V^3) for the model (49) are given by

$$p = \frac{k}{a\beta^2} \coth^2(2\sqrt{kt}) - \left(\frac{1}{a\beta^2} + 8\right)k - \Lambda, \quad (50)$$

$$\rho = \left(5k - \frac{k}{a\beta^2}\right) \coth^2(2\sqrt{kt}) + \frac{k}{a\beta^2} + \Lambda, \quad (51)$$

$$\lambda = \left[\frac{2k}{a\beta^2} - k\right] \coth^2(2\sqrt{kt}) - \left\{\frac{2k}{a\beta^2} + 4k\right\}, \quad (52)$$

$$\rho_p = \rho - \lambda = \left[k - \frac{3k}{a\beta^2}\right] \coth^2(2\sqrt{kt}) + 9k + \frac{3k}{\beta^2 a}, \quad (53)$$

$$\theta = 4\sqrt{k} \coth(2\sqrt{kt}), \quad (54)$$

$$\sigma = \sqrt{\frac{k}{3}} \coth(2\sqrt{kt}), \quad (55)$$

$$V^3 = \frac{\beta ma}{k} \sinh^2(2\sqrt{kt}). \quad (56)$$

From Eqs. (54) and (55), we obtain

$$\frac{\sigma}{\theta} = \text{constant}. \quad (57)$$

From (51), $\rho \geq 0$ implies that

$$\coth^2(2\sqrt{kt}) \leq \left[\frac{\frac{k}{a\beta^2} + \Lambda}{\frac{k}{a\beta^2} - 5k}\right]. \quad (58)$$

Also from (53), $\rho_p \geq 0$ implies that

$$\coth^2(2\sqrt{kt}) \leq \left[\frac{\frac{3k}{a\beta^2} + ak}{\frac{3k}{a\beta^2} - k}\right]. \quad (59)$$

Thus the energy conditions $\rho \geq 0$, $\rho_p \geq 0$ are satisfied under conditions given by (58) and (59).

The model (49) starts with a big bang at $t = 0$ and the expansion in the model decreases as time increases. The spatial volume of the model increases as time increases. Since $\frac{\sigma}{\theta} = \text{constant}$, hence the anisotropy is maintained throughout. Since ρ , λ , θ , σ tend to infinity and $V^3 \rightarrow 0$ at initial epoch $t = 0$, therefore, the model (49) for massive string in absence of magnetic field has Line-singularity [47].

6. Variation of Λ with time

Equations (37) and (38) with the use of (26) reduce to

$$\coth^2(2\sqrt{kt}) = \left[\frac{\ell + \alpha + 1}{\ell - 5\gamma} \right], \quad (60)$$

where

$$\ell = \frac{(1 + \gamma)}{\beta^2(a + b)} - \frac{K^2(1 - \gamma)}{2m^2(a + b)}. \quad (61)$$

Thus ℓ decreases as magnetic field increases. From equation (60), we obtain

$$2\sqrt{kt} = \coth^{-1} \left[\frac{\ell + \alpha + 1}{\ell - 5\gamma} \right]^{\frac{1}{2}}, \quad (62)$$

where $\ell + \alpha + 1 > \ell - 5\gamma$ implies that $\alpha + 1 + 5\gamma > 0$ which is true. Putting the value of k in (62), we obtain

$$\sqrt{\Lambda t} = \frac{\sqrt{1 + \alpha}}{2\sqrt{1 + \gamma}} \coth^{-1} \left[\frac{\ell + \alpha + 1}{\ell - 5\gamma} \right]^{\frac{1}{2}} = \text{constant}, \quad (63)$$

which implies that

$$\Lambda = \frac{L}{t^2}, \quad (64)$$

where L is constant. Here we observe that when $t \rightarrow 0$ then $\Lambda \rightarrow \infty$ and when $t \rightarrow \infty$ then $\Lambda \rightarrow 0$. Here $\Lambda \propto \frac{1}{t^2}$ which gives fundamental condition supported by observations.

In absence of magnetic field i.e. when $K \rightarrow 0$ then

$$\ell \rightarrow \frac{(1 + \gamma)}{\beta^2(a + b)} = s \text{ (say)}. \quad (65)$$

In this case equations (37) and (38) with the use of (26) reduce to

$$\coth^2(2\sqrt{kt}) = \left[\frac{s + \alpha + 1}{s - 5\gamma} \right], \quad (66)$$

Putting the value of k in (66), we obtain

$$\sqrt{\Lambda t} = \frac{\sqrt{1 + \alpha}}{2\sqrt{1 + \gamma}} \coth^{-1} \left[\frac{s + \alpha + 1}{s - 5\gamma} \right]^{\frac{1}{2}} = \text{constant}, \quad (67)$$

which implies that

$$\Lambda = \frac{Q}{t^2}, \quad (68)$$

where Q is constant. Here we observe that when $t \rightarrow 0$ then $\Lambda \rightarrow \infty$ and when $t \rightarrow \infty$ then $\Lambda \rightarrow 0$. Here $\Lambda \propto \frac{1}{t^2}$ which gives fundamental condition supported by observations. A number of authors have argued in favor of the dependence $\Lambda \rightarrow t^{-2}$ in different context. It has also be found, by several authors, that when one supposes variable gravitational and cosmological “constant” in Brans-Dicke theories one finds the relation like equations

(64) and (68). Berman and Som [13] pointed out that the relation $\Lambda \rightarrow t^{-2}$ seems to play a major role in cosmology. In fact, Berman, Som, and Gomide [14] found this relation in Brans-Dicke static models; Berman [10] found it in a static universe with Endo-Fukui modified Brans-Dicke cosmology; Berman and Som [13] found it again in general Brans-Dicke models which obey the perfect gas equation of state [11, 12]. Berman [15] also found this relation in general relativity. We have derived the same variation of Λ with time in massive string cosmology in this article.

7. Concluding Remarks

Some Bianchi type VI_0 massive string cosmological models with a perfect fluid as the source of matter are obtained in presence and absence of magnetic field. Generally, the models are expanding, shearing and non-rotating. In presence of perfect fluid it represents an accelerating universe during the span of time mentioned below equation (45) as decelerating factor $q < 0$ and it represents decelerating universe as $q > 0$. All the two massive string cosmological models obtained in the present study have Line-singularity (Banerjee et al. [47]) at the initial epoch $t = 0$. The variation of cosmological term in presence and absence of magnetic field is consistent with recent observations. To solve the age parameter and density parameter, one requires the cosmological constant to be positive or equivalently the deceleration parameter to be negative. The nature of the cosmological constant Λ and the energy density ρ have been examined. We have found that the cosmological parameter Λ varies inversely with the square of time, which matches its natural units. This supports the views in favour of the dependence $\Lambda \rightarrow t^{-2}$ first expressed by Bertolami [67, 68] and later on observed by several authors [9]–[22]. The density is easily adjustable to what we observe today, so that there is no need to have recourse to any critical density, and the $\Lambda \rightarrow t^{-2}$ law guarantees that we may explain why the present value of Λ is negligible in comparison with the early universe values as required by particle physics.

We have also observed that the magnetic field gives positive contribution to expansion, shear and the free gravitational field which die out for large value of t at a slower rate than the corresponding quantities in the absence of magnetic field.

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Bianchi Type V Magnetized String Dust Universe with Variable Magnetic Permeability

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Abstract: Bianchi Type V magnetized string dust universe with variable magnetic permeability is investigated. The magnetic field is due to an electric current produced along x-axis. Thus F_{23} is the only non-vanishing component of electro-magnetic field tensor F_{ij} . Maxwell's equations $F_{[ij;k]} = 0$, $F_{;j}^{ij} = 0$ are satisfied by $F_{23} = \text{constant}$. The physical and geometrical aspects of the model with singularity in the model are discussed. The physical implications of the model are also explained.

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1. Introduction

Bianchi Type V universes are the natural generalization of FRW (Friedmann-Robertson-Walker) models with negative curvature. These open models are favoured by the available evidences for low density universes (Gott et al [1]). Bianchi Type V cosmological model where matter moves orthogonally to the hyper surface of homogeneity, has been studied by Heckmann and Schucking[2]. Exact tilted solutions for the Bianchi Type V space-time are obtained by Hawking[3], Grishchuk et al. [4]. Ftaclas and Cohen[5] have investigated LRS (Locally Rotationally Symmetric) Bianchi Type V universes containing stiff matter with electromagnetic field. Lorentz[6] has investigated LRS Bianchi Type V tilted models with stiff perfect fluid and electromagnetic field. Roy and Singh [7] have investigated a Bianchi Type V universe with stiff fluid and a source free electromagnetic field. Banerjee and Sanyal [8] have investigated Bianchi Type V cosmological models with viscosity and heat flow. Coley [9] has investigated Bianchi type V imperfect fluid cosmological models in General Relativity. Nayak and Sahoo [10] have investigated Bianchi type V models with

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matter distribution admitting anisotropic pressure and heat flow. Bali and Meena[11] have investigated Bianchi Type V tilted cosmological model for stiff perfect fluid distribution.

Cosmic string play a significant role in the study of the early universe. These strings arise during the phase transitions after the big-bang explosion. Linde [12] conjectured that universe might have experienced a number of phase transitions after the big-bang explosions. The phase transitions produce vacuum domain structure such as domain walls, strings and monopoles(Kibble[13], Zel'dovich[14]). Cosmic strings create a considerable interest as these act as a gravitational lenses and give rise to density perturbations leading to the formation of galaxies (Vilekin[15]). These strings have stress energy and they can be classified as massive and geometrical strings. Each massive string is formed by geometric string with particles attached along its extension. This is the interesting situations wherein we have particles and strings together. The pioneering work in the formulation of the energy-momentum tensor for classical massive strings is due to Letelier[16] Who explained that the massive strings are formed by geometric string(Stachel[17]) with particles attached along its extension. Letelier[18] first used this idea in finding some cosmological solutions for massive strings for Bianchi Type I and Kantowski-Sachs space-time. Melvin[19] in his cosmological solution for dust and electromagnetic field suggested that during the evolution of the universe, the matter was in a highly ionized state and is smoothly coupled with the field. Hence the presence of magnetic field in string dust universe is not unrealistic. Banerjee et al.[20] have investigated an axially symmetric Bianchi Type I string dust cosmological model in presence and absence of magnetic field. A class of cosmological solutions of massive strings has been derived by Chakraborty[21] for Bianchi Type VI_0 space-time. Tikekar and Patel [22,23]have investigated cosmological models in Bianchi Type III and VI_0 space-times in presence and absence of magnetic field. Patel and Maharaj[24] investigated stationary rotating world model with magnetic field. Singh and Singh [25] have investigated string cosmological models with magnetic field in the context of space-time with G_3 symmetry. Wang [26] has investigated massive string cosmological model in presence of magnetic field in the context of Bianchi Type III space-time. Bali and Upadhaya [27]have investigated LRS(Locally Rotationally Symmetric)Bianchi Type I string dust magnetized cosmological models using the condition that σ (shear) is proportional to the expansion (θ). Bali and Anjali [28]have investigated Bianchi Type I magnetized string dust cosmological model using supplementary condition between metric potentials A, B, C as $A = (BC)^n$, n being a constant. Recently Bali et al.[29]have investigated Bianchi Type I massive string cosmological model with magnetic field for Barotropic perfect fluid distribution. In the above mentioned studies, the magnetic permeability where it is considered, is assumed as constant quantity. In this paper, we have investigated Bianchi Type V string dust universe in the presence of magnetic field with variable magnetic permeability. To get the deterministic model, we have assumed that F_{23} is the only non-vanishing component of F_{ij} . The physical implications of the model are also discussed.

2. Formation of Field Equations

We consider Bianchi Type V space-time in the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{2x} dy^2 + C^2 e^{2x} dz^2 \quad (1)$$

where A, B, C are functions of t. The energy-momentum tensor (T_i^j) for a cloud of string is given by Letelier[16]

$$T_i^j = \rho v_i v^j - \lambda x_i x^j + E_i^j \quad (2)$$

where v^i and x^i satisfy the conditions

$$\begin{aligned} v_i v^i &= -x_i x^i = -1, v^i x_i = 0, \\ x_1 &\neq 0, x_2 = x_3 = x_4 \end{aligned} \quad (3)$$

ρ being the proper energy density for a cloud of string with particles attached to them, λ the string tension density, v^i the four-velocity of the particles and x^i is a unit space-like vector representing the direction of string. If the particle density of the configuration is denoted by ρ_p then we have

$$\rho = \rho_p + \lambda \quad (4)$$

In a comoving coordinate system, we have

$$v^i = (0, 0, 0, 1), x^i = (1/A, 0, 0, 0) \quad (5)$$

E is electromagnetic field given by Lichnerowicz [30] as

$$E_i^j = \bar{\mu} [|h|^2 (v_i v^j + 1/2 g_i^j) - h_i h^j] \quad (6)$$

with

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}} \epsilon_{ijkl} F^{kl} v^j \quad (7)$$

Where h_i is the magnetic flux vector, ϵ_{ijkl} the Levi-Civita tensor, F^{kl} the electromagnetic field tensor, $\bar{\mu}$ the magnetic permeability and $|h|^2 = h_l h^l$, g_{ij} the metric tensor. We assume that magnetic field is due to an electric current produced along x-axis. Thus F_{23} is the only non-vanishing component of electromagnetic field tensor F_{ij} and $h_1 \neq 0, h_2 = 0 = h_3 = h_4$. Maxwell's equations $F_{ij;k} + F_{jk;i} + F_{ki;j} = 0$ and $F_{;j}^{ij} = 0$ are satisfied by

$$F_{23} = H(\text{constant}) \quad (8)$$

We also find that $F_{14} = 0 = F_{24} = F_{34}$ due to the assumption of infinite electrical conductivity (Roy Maartens[31]). From equation (7), we find that

$$h_1 = \frac{AH e^{-2x}}{\bar{\mu} BC} \quad (9)$$

Using equation (9) in (6), we have

$$E_1^1 = -\frac{H^2 e^{-4x}}{2\bar{\mu}B^2C^2} = -E_2^2 = -E_3^3 = E_4^4 \quad (10)$$

The Einstein's field equations

$$R_i^j - 1/2Rg_i^j = -8\pi T_i^j \quad (11)$$

for the line-element (1) with equations (2), (5) and (10) lead to the following system of equations

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} - \frac{1}{A^2} = 8\pi \left(\frac{H^2}{2B^2C^2} + \lambda \right) \quad (12)$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} - \frac{1}{A^2} = -8\pi \left(\frac{H^2}{2B^2C^2} \right) \quad (13)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} - \frac{1}{A^2} = -8\pi \left(\frac{H^2}{2B^2C^2} \right) \quad (14)$$

$$\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC} - \frac{3}{A^2} = 8\pi \left(\rho + \frac{H^2}{2B^2C^2} \right) \quad (15)$$

$$\frac{2A_4}{A} - \frac{B_4}{B} - \frac{C_4}{C} = 0 \quad (16)$$

where we have assumed that magnetic permeability is a variable quantity and assumed as

$$\bar{\mu} = e^{-4x} \quad (17)$$

Thus $\bar{\mu} \rightarrow 0$ as $x \rightarrow \infty$ and $\bar{\mu} = 1$ when $x \rightarrow 0$, Zel'dovich[14] in his investigation has explained that $\rho_s/\rho_c \sim 2.5 \times 10^{-3}$ where ρ_s is the mass density and ρ_c the critical density then strings frozen in plasma would change their density like a^{-2} i.e. like t^{-1} in the radiation dominated universe where a is the radius of the universe. In this approximation, the strings would soon be dominant and the tension along the string (λ) is equal to its energy density (ρ) per unit length and the particle density (ρ_p) of the configuration is zero. Thus from equation (4) we have string dust condition as

$$\rho = \lambda$$

3. Solution of Field Equations

Equations (12) and (15) after using string dust condition $\rho = \lambda$ lead to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{A_4}{A} \left(\frac{B_4}{B} + \frac{C_4}{C} \right) + \frac{2}{A^2} = 0 \quad (18)$$

Equation (16) leads to

$$\frac{A_4}{A} = \frac{1}{2} \left(\frac{B_4}{B} + \frac{C_4}{C} \right) \quad (19)$$

which on integration leads to

$$A = L\sqrt{BC} \quad (20)$$

where L is the constant of integration. Equation (19) leads to

$$\frac{2A_{44}}{A} = \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{1}{2} \frac{B_4^2}{B^2} - \frac{1}{2} \frac{C_4^2}{C^2} + \frac{B_4 C_4}{BC} \quad (21)$$

From equations (13) and (14), we have

$$\frac{2A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{A_4}{A} \left(\frac{B_4}{B} + \frac{C_4}{C} \right) - \frac{2}{A^2} = -\frac{K}{B^2 C^2} \quad (22)$$

where

$$K = 8\pi H^2 \quad (23)$$

Using equations (20) and (21) in (22), we have

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} - \frac{1}{L^2 BC} = -\frac{K}{B^2 C^2} \quad (24)$$

Let us assume

$$BC = \mu, \frac{B}{C} = \nu \quad (25)$$

Using equation (25) in (18) and (24), we have

$$\frac{\mu_{44}}{2\mu} - \frac{\mu_4^2}{2\mu^2} + \frac{\nu_4^2}{4\nu^2} + \frac{1}{L^2 \mu} = 0 \quad (26)$$

and

$$\frac{\mu_{44}}{\mu} - \frac{\mu_4^2}{4\mu^2} + \frac{\nu_4^2}{4\nu^2} - \frac{1}{L^2 \mu} = -\frac{K}{2\mu^2} \quad (27)$$

Equations (26) and (27) lead to

$$2\mu_{44} + 1/4\mu_4^2 = \frac{8}{L^2} - \frac{2K}{\mu} \quad (28)$$

which leads to

$$f^2 = \left(\frac{d\mu}{dt} \right)^2 = \frac{4}{L^2} \mu - 2K + \frac{N}{\mu} \quad (29)$$

where $\mu_4 = f(\mu)$, $\mu_{44} = f f'$, $f' = \frac{df}{d\mu}$
and N is the constant of integration.

From equations (13), (14) and (19), we have

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{1}{2} \left(\frac{B_4}{B} + \frac{C_4}{C} \right) \left(\frac{B_4}{B} - \frac{C_4}{C} \right) = 0 \quad (30)$$

which after using the condition (25) leads to

$$\frac{\nu_4}{\nu} = \frac{l}{L\mu^{3/2}} \quad (31)$$

which again leads to

$$\frac{d\nu}{\nu} = \frac{l}{L\mu^{3/2}} \frac{dt}{d\mu} d\mu \quad (32)$$

where l is the constant of integration. Equation (32) after using (29) leads to

$$\nu = M \left[\frac{\alpha + \tan \theta/2 - \frac{4\gamma}{L^2 K}}{\alpha - \tan \theta/2 + \frac{4\gamma}{L^2 K}} \right]^{2l/L^2 \alpha K} \quad (33)$$

where $\tan \theta/2$ is determined by

$$\tan \theta/2 = \frac{\sqrt{(4T - L^2 K)^2 + (L^4 K^2 + 4NL^2)} - 1}{(4T - L^2 K)} \quad (34)$$

and

$$K = \frac{1}{L^2} \quad (35)$$

Hence the metric (1) reduces to the form

$$\begin{aligned} ds^2 &= - \left(\frac{dt}{d\mu} \right)^2 d\mu^2 + L^2 \mu dx^2 + \mu \nu e^{2x} dy^2 + \frac{\mu}{\nu} e^{2x} dz^2 \\ &= - \frac{dT^2}{\frac{4T}{L^2} - 2K + \frac{N}{T}} + T dX^2 + T \nu e^{\frac{2X}{L}} dY^2 + \frac{T}{\nu} e^{\frac{2X}{L}} dZ^2 \end{aligned} \quad (36)$$

where $Lx = X, y = Y, z = Z, \mu = T$ and ν is given by (33).

In the absence of magnetic field i.e. when $K \rightarrow 0$ then the metric (36) reduces to the form

$$ds^2 = - \frac{dT^2}{\frac{4T}{L^2} + \frac{N}{T}} + T dX^2 + T \nu e^{\frac{2X}{L}} dY^2 + \frac{T}{\nu} e^{\frac{2X}{L}} dZ^2 \quad (37)$$

where ν is determined by (33) in absence of magnetic field as

$$\nu = M \left[\frac{\alpha + \tan \theta/2 - 4\gamma}{\alpha - \tan \theta/2 + 4\gamma} \right]^{2l/\alpha} \quad (38)$$

and $\tan \theta/2$ in the absence of magnetic field is given by (34) as

$$\tan \theta/2 = \frac{\sqrt{16T^2 + 4NL^2} - 1}{4T} \quad (39)$$

The energy density (ρ), the string tension density (λ), the scalar of expansion (θ) and the shear (σ) for the model (36) in the presence of magnetic field, are given by

$$\begin{aligned} 8\pi\rho &= \left(\frac{3N}{4} - \frac{M^2}{4} \right) \frac{1}{T^3} - \frac{2K}{T^2} \\ &= 8\pi\lambda \end{aligned} \quad (40)$$

$$\begin{aligned} \theta &= \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \\ &= 3\sqrt{\frac{4}{L^2 T} + \frac{N}{T^3} - \frac{2K}{T^2}} \end{aligned} \quad (41)$$

$$\sigma = \frac{l}{2LT^{3/2}} \quad (42)$$

The energy condition $\rho \geq 0$ leads to

$$0 < T \leq \frac{3N-M^2}{64\pi K}$$

Conclusions

The model (36) starts with a big-bang at $T = 0$ and the expansion in the model decreases as time increases. When $T \rightarrow 0$ then $\rho \rightarrow \infty$ and when $T \rightarrow \infty$ then $\rho \rightarrow 0$. Since $\sigma \rightarrow 0$ when $T \rightarrow \infty$ then the model isotropizes for large values of T . There is a Point type singularity in the model (36) at $T = 0$ (MacCallum[32]). The scale factor R is given by

$$R^3 = ABCe^{2x} = Le^{2x}T^{3/2}$$

Thus R increases as T increases. The deceleration parameter (q) is given by

$$\begin{aligned} q &= -\frac{\ddot{R}/R}{\dot{R}^2/R^2} \\ &= -\frac{(2K - \frac{4N}{T})}{(\frac{4T}{L^2} - 2K + \frac{N}{T})} \end{aligned} \quad (43)$$

The deceleration parameter approaches the value -1 as in de-Sitter universe when

$$\frac{5N}{T} + \frac{4T}{L^2} = 4K$$

In the absence of magnetic field i.e. when $K \rightarrow 0$, The energy density (ρ), the string tension density (λ), the scalar of expansion (θ) and the shear (σ) for the model (37), are given by

$$\begin{aligned} 8\pi\rho &= \left(\frac{3N}{4} - \frac{M^2}{4}\right) \frac{1}{T^3} \\ &= 8\pi\lambda \end{aligned} \quad (44)$$

$$\theta = 3\sqrt{\frac{4}{L^2T} + \frac{N}{T^3}} \quad (45)$$

$$\sigma = \frac{l}{2LT^{3/2}} \quad (46)$$

The energy condition $\rho \geq 0$ leads to $3N \geq M^2$.

The model (37) in the absence of magnetic field, starts with a big-bang at $T = 0$ and the expansion in the model decreases as time increases. Since $\sigma \rightarrow 0$ when $T \rightarrow \infty$. Therefore the model isotropizes for large values of T . The scale factor R is given by

$$R^3 = Le^{2x}T^{3/2}$$

Thus R increases as T increases. The deceleration parameter (q) in the absence of magnetic field is given by

$$q = -\frac{\ddot{R}/R}{\dot{R}^2/R^2} = \frac{4N/T}{\frac{4T}{L^2} + \frac{N}{T}}$$

Thus the deceleration parameter approaches the value -1 as in de-Sitter universe if $5NL^2 + 4T^2 = 0$.

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Dynamics of Shell With a Cosmological Constant

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Abstract: Spherically symmetric thin shell in the presence of a cosmological constant are constructed, applying the Darmois-Israel formalism. An equation governing the behavior of the radial pressure across the junction surface is deduced. The cosmological constant term slows down the collapse of matter. The spherical N-shell model with an appropriate initial condition imitates the FRW universe with $\Lambda \neq 0$, quite well.

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1. Introduction

The possible existence of a cosmological constant is one of the most important challenges in high energy physics today [1]. However, a surprising recent result coming from the analysis of high redshift supernovae, indicating that the universe may be accelerating now [2]. This suggests that there is in fact a cosmological constant, that dominates the content of energy of the universe today. The cosmological implication of the existence of a cosmological constant today are enormous, concerning not only the evolution of the universe but also the structure formation and age problems. The gravitational collapse is one example of these extreme physical conditions where black holes seem to be formed.

The general relativistic treatment of an infinitely thin shell has been given by Israel [3]. The motion of a shell is described as a timelike hypersurface between two different given space- times. This metric junction method was generalized to include a non vacuum metric. The compact stellar objects such as white dwarf and neutron star are formed by a period of gravitational collapse. It is interesting to consider the appropriate geometry of interior and exterior regions and determine proper junction conditions which allow the

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matching of these regions. Most of the problems related to gravitational collapse have been discussed by considering spherically symmetric system. The gravitational collapse of dust was first shown by Oppenheimer and Snyder [4], the evolution of bubbles and domain walls in cosmological settings [5], and shells around black hole solutions [6].

An interesting application to the motion of dust shell with a cosmological constant was done in [7, 8]. The effect of a positive cosmological constant on spherically symmetric collapse with perfect fluid has been studied in [9].

The goal of this work is to extend the study of the gravitational collapse in the presence of a cosmological constant. This paper is organized as follows. In Section 2 the Darmois-Israel thin shell formalism is briefly reviewed. Match an interior Schwarzschild de-Sitter solution to an exterior Schwarzschild de-Sitter solution and the expression governing the behavior of the radial pressure across the junction boundary are given in Section 3. The equations of motion of thin shell and the general form of this equations in N-shell are deduced in Section 4. Finally, some concluding remarks are made in Section 5. Also adopt the units such that $c = G = 1$.

2. The Darmois – Israel Formalism

Consider two distinct spacetime manifolds M_+ and M_- with metrics given by $g_{\mu\nu}^+(x_+^\mu)$ and $S^{ij}\bar{K}_{ij} = [-T_{\mu\nu}n^\mu n^\nu - \frac{\Lambda}{8\pi}]_+^+$, in terms of independently defined coordinate systems x_\pm^μ . The manifolds are bounded by hypersurfaces Σ_+ and Σ_- , respectively, with induced metrics g_{ij}^\pm . The hypersurfaces are isometric, i.e. $g_{ij}^+(\xi) = g_{ij}^-(\xi) = g_{ij}(\xi)$, in terms of the intrinsic coordinates, invariant under the isometry. A single manifold M is obtained by gluing together M_+ and M_- at their boundaries, i.e. $M = M_+ \cup M_-$, with the natural identification of the boundaries $\Sigma = \Sigma_+ = \Sigma_-$. The basic vectors $e^\mu = \frac{\partial}{\partial \xi^a}$ tangent to Σ have the components $e_{a\pm}^\mu = \frac{\partial x_\pm^\mu}{\partial \xi^a}$, with respect to the two four dimensional coordinate systems in M^\pm . The second fundamental forms (extrinsic curvature) associated with the two sides of the shell are:

$$K_{ij}^\pm = -n_\gamma^\pm \left(\frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^\gamma \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right);_\Sigma \quad (1)$$

where n_γ^\pm are the unit normal 4-vector to Σ in M , with $n_\mu n^\mu = 1$ and $n_\mu e_i^\mu = 0$. The Israel formalism requires that the normal point from M_- to M_+ . For the case of a thin shell K_{ij} is not continuous across Σ , so that, the discontinuity in the second fundamental form is defined as $[K_{ij}] = K_{ij}^+ - K_{ij}^-$. The Einstein equation determines the relations between the extrinsic curvature and the three dimensional intrinsic energy momentum tensor are given by The Lanczos equations,

$$S_{ij} = \frac{-1}{8\pi} ([K_{ij}] - [K] g_{ij}) \quad (2)$$

where $[K]$ is the trace of $[K_{ij}]$ and S_{ij} is the surface stress-energy tensor on Σ . The first contracted Gauss- Kodazzi equation or the ‘‘Hamiltonian’’ constraint

$$G_{\mu\nu}n^\mu n^\nu = \frac{1}{2}(K^2 - K_{ij}K^{ij} - {}^3R), \quad (3)$$

with the Einstein equations provide the evolution identity

$$S^{ij} \bar{K}_{ij} = \left[-T_{\mu\nu} n^\mu n^\nu - \frac{\Lambda}{8\pi} \right]_+^+ \quad (4)$$

The convention, $[X] = X^+ - X^-$, and $\bar{X} = \frac{1}{2}(X^+ + X^-)$, is used. The second contracted Gauss- Kodazzi equation or the ‘‘ADM’’ constraint,

$$G_{\mu\nu} e_i^\mu n^\nu = K_{i;j}^j - K_{,i} \quad (5)$$

With the Lanczos equations gives the conservation identity

$$S_{j;i}^i = [T_{\mu\nu} e_i^\mu n^\nu]_+^+ \quad (6)$$

The surface stress-energy tensor may be written in terms of the surface energy density σ , and surface pressure p : $S_j^i = \text{diag} \cdot (-\sigma, p, p)$.

For spherically symmetric thin shell, the Lanczos equations reduce to

$$\sigma = \frac{-1}{4\pi} [K_\theta^\theta] \quad (7)$$

$$p = \frac{1}{8\pi} ([K_\tau^\tau] + [K_\theta^\theta]) \quad (8)$$

If the surface stress-energy terms are zero, the junction is denoted as a boundary surface. If surface stress terms are present, the junction is called a thin shell.

3. Generic Dynamic Spherically Symmetric Thin Shell

The matching of two Schwarzschild de-Sitter space-times of M^\pm , given by the following line elements:

$$ds_\pm^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (9)$$

with

$$F_\pm = 1 - \frac{2m_\pm}{R} - \frac{1}{3}\Lambda_\pm R^2 \quad (10)$$

where m_\pm and Λ_\pm are the gravitational mass and the cosmological constant outside and inside the shell. The suffix ‘+’ denotes a quantity evaluated just outside the shell and ‘-’ just inside the shell. Let r be the area radius, i.e. the radial coordinate such that $A = 4\pi r^2$ is the area of the spheres of symmetry at constant r . The area radius is continuous across Σ , which is not true for the time coordinates. Let τ be the proper time parameter along the lines element of constant angular coordinates in Σ . Let the equation of the shell be $r_\pm = R_\pm(\tau)$, the history of the shell is described by the hypersurface $x_\pm^\alpha = x_\pm^\alpha(\tau, \theta, \phi)$, in the regions M^\pm , respectively; the function $R(\tau)$ describes the time evolution of the shell.

Using the Einstein field equation in an orthogonal reference frame, the stress-energy tensor components are given by

$$\rho(R) = \frac{-1}{8\pi} \left[-\frac{1}{R^2} + \frac{F}{R^2} + \frac{F'}{R} \right]_+^+ \quad (11)$$

$$P_r(R) = \frac{1}{8\pi} \left[\frac{1}{R^2} - \frac{F}{R^2} - \frac{F'}{R} \right]_+ \quad (12)$$

$$P_t(R) = \frac{1}{8\pi} \left[-\frac{F''}{2} - \frac{F'}{R} \right]_+ \quad (13)$$

where $\rho(R)$ is the energy density, $P_r(R)$ is the radial pressure, and $P_t(R)$ is the lateral pressure measured in the orthogonal direction to the radial direction; the prim denotes a derivative with respect to R . The four velocity is given by

$$U_{\pm}^{\mu} = (F_{\pm}^{-1} \sqrt{F_{\pm} + \dot{R}^2}, \dot{R}, 0, 0) \quad (14)$$

where the overdot denotes a derivative with respect to τ . The unit normal to the junction surface is

$$n_{\pm}^{\mu} = (\dot{R}F_{\pm}^{-1}, \sqrt{F_{\pm} + \dot{R}^2}, 0, 0). \quad (15)$$

Using equation (1), the non-trivial components of the extrinsic curvature are given by:

$$K_{\theta}^{\theta\pm} = K_{\phi}^{\phi\pm} = \frac{1}{R} \sqrt{F_{\pm} + \dot{R}^2} \quad (16)$$

$$K_{\tau}^{\tau\pm} = \frac{1}{\sqrt{F_{\pm} + \dot{R}^2}} \left(\frac{m_{\pm}}{R^2} - \frac{1}{3} \Lambda_{\pm} R + \ddot{R} \right) \quad (17)$$

Therefore, the Lanczos equations (7)-(8), with the extrinsic curvature equations (16)-(17), are given by

$$\sigma = \frac{-1}{4\pi R} \left[\sqrt{F + \dot{R}^2} \right]_+ \quad (18)$$

$$p = \frac{1}{8\pi R} \left[\frac{1 - \frac{m}{R} - \frac{2}{3} \Lambda R^2 + \dot{R}^2 + R\ddot{R}}{\sqrt{F + \dot{R}^2}} \right]_+ \quad (19)$$

Taking into account the transparency condition, $[G_{\mu\nu} U^{\mu} n^{\nu}]_{\pm}^{\pm} = 0$, the conservation identity, equation (6), provides the simple relationship:

$$\dot{\sigma} = \frac{-2\dot{R}}{R} (\sigma + P) \quad (20)$$

where the first term represents the variation of the internal energy of the shell, and the second term is the work done by the shell's internal force.

In general case, the conservation identity provides the following relationship:

$$\sigma' = \frac{-2}{R} (\sigma + P) + H \quad (21)$$

where H is the momentum flux given by

$$H = \frac{1}{4\pi R^2} \left[\sqrt{F + \dot{R}^2} \right]_+ \quad (22)$$

This flux term vanishes in the particular case of $P = -\rho$. Taking into account these relationship

$$\sigma + p = \frac{1}{8\pi R} \left[\frac{-1 + \frac{3m}{R} - \dot{R}^2 + R\ddot{R}}{\sqrt{F + \dot{R}^2}} \right]_+^- , \quad (23)$$

$$\dot{\sigma} = \frac{-\dot{R}}{4\pi R^2} \left[\frac{-1 + \frac{3m}{R} - \dot{R}^2 + R\ddot{R}}{\sqrt{F + \dot{R}^2}} \right]_+^- , \quad (24)$$

$$\sigma' = \frac{1}{4\pi R^2} \left[\frac{1 - \frac{3m}{R} + \dot{R}^2 - R\ddot{R}}{\sqrt{F + \dot{R}^2}} \right]_+^- . \quad (25)$$

For the static solution R_0 , with $\dot{R} = \ddot{R} = 0$, equation (25), reduced to

$$\sigma'(R_0) = \frac{1}{4\pi R_0^2} \left[\frac{1 - \frac{3m}{R_0}}{\sqrt{F(R_0)}} \right]_+^- . \quad (26)$$

Using the surface mass of the shell, $M = 4\pi R^2\sigma$, and taking into account the radial derivative of σ' , equation (21) can be rearranged to provide the following relationship

$$\left(\frac{M}{2R} \right)' = \frac{4\pi}{R} (\sigma + P) + 2\pi R H' - 4\pi \sigma' \eta \quad (27)$$

with the parameter η defined as $\eta = \frac{P'}{\sigma'}$.

One may obtain an equation governing the behavior of the radial pressure in terms of the surface stresses at the junction boundary from the following identity:

$$[T_{\mu\nu} n^\mu n^\nu] = \frac{1}{2} (K_j^{i+} + K_j^{i-}) S_j^i. \quad (28)$$

The tension acting on the shell is, by definition, the normal component of the stress-energy tensor, $-\Xi = T_{\mu\nu} n^\mu n^\nu \equiv P_r(R)$, then the pressure balance equation is

$$-\Xi^+ + \Xi^- = -\frac{\sigma}{2} \left(\frac{\frac{m_+}{R^2} - \frac{1}{3}\Lambda^+ R + \ddot{R}}{\sqrt{F_+ + \dot{R}^2}} + \frac{\frac{m_-}{R^2} - \frac{1}{3}\Lambda^- R + \ddot{R}}{\sqrt{F_- + \dot{R}^2}} \right) + \frac{P}{R} N \quad (29)$$

where , $N = \left(\sqrt{F_+ + \dot{R}^2} + \sqrt{F_- + \dot{R}^2} \right)$.

This equation relates the difference of the radial tension across the shell in terms of a combination of the surface stresses, σ and P , given by equations (18)-(19), respectively, and the geometrical quantities. For the exterior vacuum solution $\Xi^+ = 0$. For the case of a null surface energy density $\sigma = 0$, and considering that the interior and exterior cosmological constants are equal $\Lambda^- = \Lambda^+$, equation (29) reduces to

$$\Xi^-(R) = \frac{2P}{R} \sqrt{1 - \frac{2m_\pm}{R} - \frac{1}{3}\Lambda R^2}. \quad (30)$$

For a radial tension $\Xi^-(R) > 0$, acting on the shell from the interior, a tangential surface pressure, $P > 0$, is needed to hold the thin shell form collapsing. For a radial interior pressure $\Xi^-(R) < 0$, then a tangential surface tension, $P < 0$, is needed to hold the structure form expansion.

4. The Motion of Dust Shell

Rearranging equation (18) into the form

$$4\pi\sigma R = \sqrt{F_- + \dot{R}^2} - \sqrt{F_+ + \dot{R}^2} \equiv \frac{M}{R} \quad (31)$$

where $M = 4\pi\sigma R^2$ is the rest mass of the shell. Equation (31) can be written in the form

$$\dot{R}^2 = -1 + \frac{M^2}{4R^2} + \left(\frac{m_+ - m_-}{M}\right)^2 + \left(\frac{m_- + m_+}{R}\right) + \frac{R^2}{4M^2}\Gamma \quad (32)$$

where

$$\Gamma = \frac{R^4}{9}(\Lambda_+ - \Lambda_-)^2 + \frac{2}{3}M^2(\Lambda_+ + \Lambda_-) + \frac{4R^2}{3}\left(\frac{m_+ - m_-}{R}\right)(\Lambda_+ - \Lambda_-)$$

From equation (31), the total gravitational mass of the shell is

$$m \equiv m_+ - m_- = \frac{R^3}{6}(\Lambda_- - \Lambda_+) - \frac{M^2}{2R} + M\sqrt{1 - \frac{2m_-}{R} - \frac{1}{3}\Lambda_-R^2 + \dot{R}^2}$$

If the discontinuity of Λ across the thin shell, $[\Lambda] = 0$, then

$$\dot{R}^2 = -1 + \frac{M^2}{4R^2} + \left(\frac{m_+ - m_-}{M}\right)^2 + \left(\frac{m_- + m_+}{R}\right) + \frac{\Lambda R^2}{3} \quad (33)$$

It represents the energy equation of the shell, and can be called the expansion law of the dust shell.

Generalize this equation of motion for a sequence of shells, by defining a number of spherically symmetric shells with a common center at $R = 0$. The innermost shell is called the first shell, the next one is called the second shell, and so on. The region enclosed by the $(i + 1)^{th}$ shell and i^{th} shell is called the i^{th} region. The general form of the energy equation for i^{th} shell is

$$\dot{R}_i^2 = -1 + \frac{M_i^2}{4R_i^2} + E_i^2 + \frac{2\bar{m}_i}{R_i} + \frac{\Lambda R_i^2}{3} \quad (34)$$

where

$$E_i = \frac{m_{i+1} - m_i}{M_i},$$

$$\bar{m}_i = \frac{1}{2}(m_{i+1} + m_i),$$

$$M_i = 4\pi\sigma_i R_i^2.$$

For $\Lambda = 0$, equation (34), reduced to the energy equation of N-shell in the Schwarzschild space-time. Therefore, the thin shell's equation of motion will be written in the form

$$\dot{R}_i^2 + V(R) = 0 \quad (35)$$

where

$$V(R) = 1 - \frac{M_i^2}{4R_i^2} - E_i^2 - \frac{2\bar{m}_i}{R_i} - \frac{\Lambda R_i^2}{3} \quad (36)$$

is an effective potential that determines the motion of the shells. From (34) the expansion law of the dust shells can be written as

$$\left(\frac{\dot{R}_i}{R_i}\right)^2 = \frac{M_i^2}{4R_i^4} + \frac{E_i^2 - 1}{R_i^2} + \frac{2\bar{m}_i}{R_i^3} + \frac{\Lambda}{3} \quad (37)$$

To compare this equation with the Friedmann equation in the Friedmann universe, let y_i be the initial circumferential radius of the i^{th} shell, and the difference between these shells is defined by the interval Δy_i . Let the radius of the i^{th} shell be

$$R_i = y_i a_i(\tau), \quad (38)$$

where $a(\tau)$ is a dimensionless scale (expansion) factor depending on the time τ , called the radius of the universe. Inserting (38) into (37) to get

$$\left(\frac{\dot{a}_i}{a_i}\right)^2 = \frac{E_i^2 - 1}{y_i^2 a_i^2} + \frac{2\bar{m}_i}{y_i^3 a_i^3} + \frac{M_i^2}{4y_i^4 a_i^4} + \frac{\Lambda}{3} \quad (39)$$

Define the density ρ_i of the i^{th} shell by

$$\frac{4\pi}{3} \rho_i = \frac{\bar{m}_i}{y_i^3 a_i^3}$$

and the curvature k_i by

$$k_i = \frac{1}{y_i^2} (1 - E_i^2)$$

Therefore, equation (39) will be

$$\dot{a}_i^2 = -k_i + \frac{8\pi}{3} \rho_i a_i^2 + \frac{M_i^2}{4y_i^4 a_i^2} + \frac{\Lambda a_i^2}{3} \quad (40)$$

This represents the motion of the i^{th} shell in the Friedmann universe with $\Lambda \neq 0$, where the first term corresponds to the curvature, the second term behaves like a non-relativistic matter term in the Friedmann universe, and the third term behaves like a radiation source, and the last term corresponds to the effect of cosmological constant. For great a , the third term is neglected and the expansion law is equivalent to the Friedmann equation with $\Lambda \neq 0$,

$$\dot{a}_i^2 = -k_i + \frac{8\pi}{3} \rho_i a_i^2 + \frac{\Lambda a_i^2}{3} \quad (41)$$

Therefore, the spherical N-shell model with an appropriate initial condition imitates the Friedmann- Robertson-Walker (FRW) universe with $\Lambda \neq 0$, quite well.

Conclusion

By using the Darmois-Israel formalism technique, thin shell in the presence of a cosmological constant is constructed. General solution by matching an interior Schwarzschild de-Sitter space-time to a Schwarzschild de-Sitter space-time exterior solution at a junction surface was constructed.

An equation governing the behavior of the radial pressure across the shell was determined. The pressure term creates a bound for the cosmological constant to act as a repulsive force. It is mentioned here that if $P = 0$, the results reduce to the dust case. The cosmological constant can slow down the collapse initially, but at later times the sphere's Self gravity dominates entirely and eventually pulls the sphere into the final singularity.

An FRW model with $\Lambda \neq 0$ can be considered as a limiting case of dust shell model. Therefore, the non-homogeneities are more important in earlier phases of the expansion because they are represented by a term similar to the addition term in the expansion law. The spherical N-shell model with an appropriate initial condition imitates the FRW universe with $\Lambda \neq 0$, quite well.

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Discrete Cosmological Self-Similarity and Delta Scuti Variable Stars

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Abstract: Within the context of a fractal paradigm that emphasizes nature's well-stratified hierarchical organization, the δ Scuti class of variable stars is investigated for evidence of discrete cosmological self-similarity. Methods that were successfully applied to the RR Lyrae class of variable stars are used to identify Atomic Scale analogues of δ Scuti stars and their relevant range of energy levels. The mass, pulsation mode and fundamental oscillation period of a well-studied δ Scuti star are then shown to be quantitatively self-similar to the counterpart parameters of a uniquely identified Atomic Scale analogue. Several additional tests confirm the specificity of the discrete fractal relationship.

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1. Introduction

In two previous papers (Oldershaw, 2008a, b) evidence was presented for discrete fractal phenomena associated with RR Lyrae variable stars. The masses, radii and oscillation periods of RR Lyrae stars were shown to have a discrete self-similar relationship to the masses, radii and transition periods of their Atomic Scale counterparts: helium atoms in moderately excited Rydberg states undergoing single-level transitions. The techniques that were used to achieve these unique results are applied here to a distinctly different class of variable stars: δ Scuti stars. The new results contain some interesting surprises, but they are in general agreement with the discrete fractal paradigm and offer additional evidence for the principle of discrete cosmological self-similarity.

Briefly, the Self-Similar Cosmological Paradigm (SSCP) proposes that nature is ordered in a transfinite hierarchy of discrete cosmological Scales, of which we can currently

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observe the Atomic, Stellar and Galactic Scales (Oldershaw, 1989a,b). Spatial lengths (R), temporal periods (T) and masses (M) of analogue systems on neighboring Scales Ψ and $\Psi-1$ are related by the following set of discrete self-similar transformation equations:

$$R_{\Psi} = \Lambda R_{\Psi-1} \quad (1)$$

$$T_{\Psi} = \Lambda T_{\Psi-1} \quad (2)$$

$$M_{\Psi} = \Lambda^D M_{\Psi-1} \quad (3)$$

where Λ and D are dimensionless scaling constants with values of $\approx 5.2 \times 10^{17}$ and ≈ 3.174 , respectively, and $\Lambda^D \approx 1.70 \times 10^{56}$. The most readily available resource for a detailed presentation of this discrete fractal cosmology is the author's website (Oldershaw, 2001), where a full list of publications on the SSCP and downloadable copies of relevant papers are available. A familiarity with the previous fractal analysis (Oldershaw, 2008a) of the RR Lyrae class would be beneficial to a full appreciation of the present work on δ Scuti stars, but it is not mandatory.

2. Delta Scuti Variable Stars

Compared with RR Lyrae stars, δ Scutis are a somewhat erratic and heterogeneous class of stars. For example, the amplitudes of their oscillation periods can vary radically. As Breger and Pamyatnykh (2005) point out: "A star may change its pulsation spectrum to such an extent as to appear as a different star at different times", although "modes do not completely disappear, but are still present at small amplitudes." Here we will work exclusively with high amplitude δ Scuti stars (HADS) because their pulsation behavior is simpler (less multi-periodic) and more regular than low amplitude δ Scuti stars (LADS), and because they are thought to pulsate mainly in radial modes (Pigulski *et al.*, 2005) which is useful for identifying specific energy level transitions. Delta Scuti stars have spectral classifications of A to F and can be designated dwarf or subgiant stars (Alcock *et al.*, 2000). *Most importantly*, their masses typically range from $1.5 M_{\odot}$ to $2.5 M_{\odot}$ (Fox Machado *et al.*, 2005). This is a major change from the situation with the RR Lyrae class, which has a lower, narrower and better-defined mass range of $0.4 M_{\odot}$ to $0.6 M_{\odot}$.

As of 2004, roughly 400 δ Scuti stars were known. A typical oscillation period would be 0.1 day, and period cutoffs for this class occur at roughly 0.04 day and 0.2 day. Figure 1 shows a representative histogram of oscillation periods for 193 HADS (Pigulski *et al.*, 2005). Superimposed upon this distribution are lines corresponding to typical oscillation periods for Stellar Scale n values, which will be derived and explained below.

In this particular sample of 193 HADS stars, there were 40 double-mode pulsators and twelve stars that simultaneously pulsed at 3 or more periods. Because data on the radii of HADS stars appear to be very limited, there is a minor problem with regard to identifying relevant energy levels, but the problem can be circumvented.

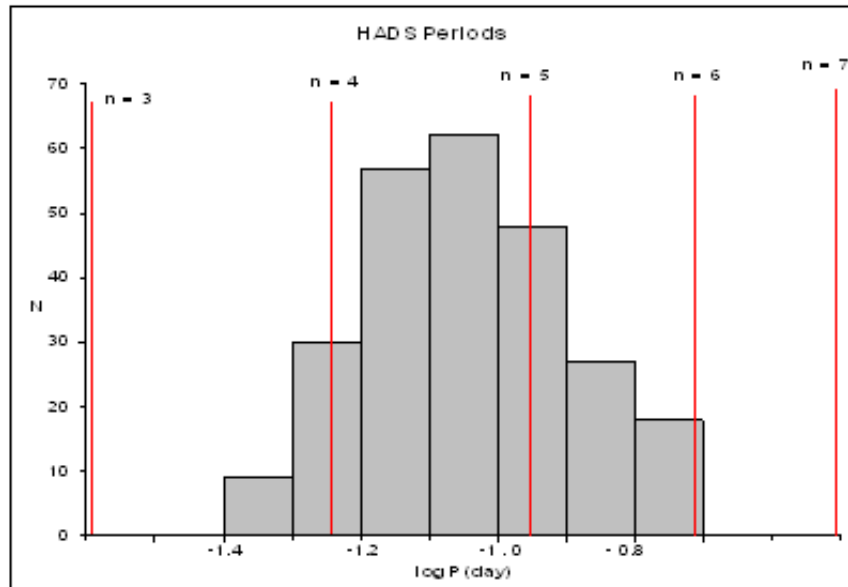


Fig. 1 Histogram of periods for 193 high amplitude δ Scuti stars (Pigulski et al., 2005). Superimposed upon the period distribution are lines representing the *scaled* oscillation periods for Rydberg atoms excited to various n values.

3. Atomic Scale Counterparts of Delta Scuti Stars

Given Eq. (3) and the mass of the proton (m_p), we can calculate that the Stellar Scale proton analogue will have a mass of about $0.145 M_\odot$. The mass range ($1.5 M_\odot$ to $2.5 M_\odot$) for δ Scuti stars would then correspond to an Atomic Scale mass range of approximately $10 m_p$ to $17 m_p$, or about 10 to 17 atomic mass units (amu). The atoms that dominate this mass range are Boron (11 amu), Carbon (12 amu), Nitrogen (14 amu) and Oxygen (16 amu). Therefore the majority of δ Scuti stars are hypothesized to be analogues of these atoms. Because reliable radius data are not available for δ Scuti stars, an alternative method that does not require $R_{\Psi=0}$ data must be used for determining the approximate energy levels of their relevant Atomic Scale analogue systems. There is a general relationship between the principal quantum numbers (n) for Rydberg atoms and their oscillation periods of the form:

$$p_n \approx n^3 p_0 \quad (4)$$

where p_0 is the classical orbital period of the ground state H atom: $\approx 1.5 \times 10^{-16}$ sec. The Stellar Scale equivalent to this relationship would be:

$$p_n \approx n^3 P_0 \quad (5)$$

where $P_0 \approx \Lambda p_0 \approx (5.2 \times 10^{17})(1.5 \times 10^{-16} \text{ sec}) \approx 78 \text{ sec}$. Using Eq. (5) we can generate the lines plotted in Fig. 1 for the different oscillation periods associated with Stellar Scale n values. These results tell us that if the discrete fractal paradigm is correct, and if we are dealing with analogues to Rydberg atoms undergoing transitions with $\Delta n \approx 1$, then the relevant range of n values for δ Scuti variables is predominantly $3 \leq n \leq 6$, with $n = 5$ being the most probable radial quantum number for this class of stars.

The above considerations lead us to conclude that δ Scuti variables correspond to a very heterogeneous set of B, C, N and O atoms excited to Rydberg states with n varying primarily between 3 and 6. The next step is to test this hypothesis by comparing the oscillation periods of δ Scuti stars with specific oscillation periods of their predicted Atomic Scale counterparts. However, there is a serious problem with a straightforward comparison between Stellar Scale and Atomic Scale frequency/period spectra, as was achieved in the case of the RR Lyrae class. The main source of this problem is the heterogeneity of the δ Scuti class of stars. We are dealing with analogues of *at least four different atoms*, each of which can be in several different isotopic configurations. Moreover, each of those four species of atom can be in several different ionization states: -1, neutral, +1, +2, etc. A further complication is that each of the atoms can have an entirely different set of energy levels for each of the singlet, doublet, triplet, etc., spin-related designations that apply. Also, we are less confident about using the $\Delta n \approx 1$ and $l \leq 1$ restrictions that were so helpful in the RR Lyrae case. Therefore, we are faced with an extremely large number of potential Atomic Scale transition periods to compare with the δ Scuti period distributions. A meaningful test requires that both the observed Stellar Scale period spectrum and the experimental Atomic Scale period spectrum have a limited number of uniquely identifiable peaks. In the case of the δ Scuti class of stars, the very large number of potential Atomic Scale comparison periods precludes a meaningful test of this type. Compounding this general lack of Atomic Scale specificity are the usual uncontrollable physical factors that can result in additional shifting of Stellar Scale energy levels away from unperturbed values: ambient Galactic Scale pressures, temperatures, electric fields and magnetic fields.

Fortunately, there is a way to circumvent the alarming specificity problems discussed above. If we have enough accurate information for an individual δ Scuti star, then we can use the data to identify the specific Atomic Scale analogue of the star, to restrict the number of possible energy level transitions for that specific atom, and to construct a valid test between a uniquely predicted oscillation period and a limited number of Atomic Scale comparison periods. Very recently, the δ Scuti star GSC 00144–03031 was analyzed in detail by Poretti *et al* (2005) and this system has certain characteristics that make it an excellent test star for our purposes. First and foremost, its mass has been determined with reasonable accuracy and is approximately $1.75 M_{\odot}$. Using our knowledge that $0.145 M_{\odot} \approx 1$ smu (stellar mass unit, $\approx \Lambda^D$ amu), we can determine that $1.75 M_{\odot}$ corresponds to 12 smu and therefore GSC 00144–03031 can be identified with a high degree of confidence as an analogue of a ^{12}C atom. Other advantages of using this star as a test system are that it is a classic HADS system (regular, high amplitude pulsations), that it has a dominant fundamental mode that is highly radial in character (which helps in narrowing energy level possibilities), and finally that it is a pure double-mode pulsator (providing us with an second test period). The fundamental radial mode has a period of about 0.058 day (≈ 5017.42 sec) and its amplitude is 4 times greater than the secondary pulsation which has a period of ≈ 3872.70 sec. The basic characteristics of GSC 00144–03031 are summarized in Table 1.

Table 1. Physical Properties (Poretti *et al.*, 2005) of GSC 00144–03031

Class	high amplitude δ Scuti (HADS)
Mass	$\approx 1.75 M_{\odot}$
Mode	“pure double-mode pulsator” with radial fundamental mode
Fundamental Period	5017.42 sec (radial, amplitude = 0.1383 mag)
Secondary Period	3872.70 sec (amplitude = 0.0331mag)

4. Test of the Discrete Self-Similarity Principle

If the principle of discrete cosmological self-similarity is correct, then we should find a self-similar relationship between the oscillation periods of GSC 00144-03031 and the empirical oscillation periods of its Atomic Scale analogue undergoing corresponding transitions, in accordance with Eq. 2. We have identified the Atomic Scale analogue as a ^{12}C atom and the transition has a strong radial mode ($l = 0$) character. From Figure 1, we can determine that the position of the dominant period for GSC 0144–03031 falls between the $n \approx 5$ and the $n \approx 4$ lines and so we anticipate a correlation with a $n = 5 \rightarrow 4$, low l , transition. Using Eq. 2 we can calculate a predicted Atomic Scale period for the counterpart to the δ Scuti fundamental mode:

$$\begin{aligned} P_{\Psi-1} &\approx P_{\Psi} \div \Lambda \approx 5017.42 \text{ sec} \div 5.2 \times 10^{17} \\ &\approx 9.65 \times 10^{-15} \text{ sec.} \end{aligned}$$

We assume that the ^{12}C atom is most likely to be uncharged, rather than being in an ionized state. Therefore the quantitative test is whether a neutral ^{12}C atom has a radial mode transition between the $n = 5$ and $n = 4$ levels that involves an oscillation period of about 9.65×10^{-15} sec. We use a standard source for atomic energy level data (Bashkin and Stoner, Jr, 1975) and find that the $n = 5$ energy level with the least non-radial character is the $1s^2 2s^2 2p^5 p(J=0) \ ^1S$ singlet level with an energy of 85625.18 cm^{-1} . The $n = 4$ energy level with the least non-radial character is the $1s^2 2s^2 2p^4 p(J=0) \ ^1S$ singlet level with an energy of 82251.71 cm^{-1} . Subtracting the energies for these neighboring energy levels gives 3373.47 cm^{-1} as the transition energy (ΔE) for the $n = 5 \rightarrow 4(J=0)$ transition. We can calculate the oscillation frequency for the transition by using the relation $\nu = \Delta E c$ for electromagnetic radiation, and we find that $\nu = (3373.47 \text{ cm}^{-1})(2.99 \times 10^{10} \text{ cm/sec}) = 1.01 \times 10^{14} \text{ sec}^{-1}$. Since $p = 1/\nu$, the period for the transition is 9.88×10^{-15} sec. This value is higher than the predicted value of 9.65×10^{-15} sec by a factor of 0.024, but considering the numerous sources of small uncertainties that are involved in this test, and the uncontrollable physical factors that can shift the Stellar Scale oscillation period, the agreement between the predicted and experimental values is quite good. Table 2 summarizes the discrete self-similarity between GSC 00144–03031 and $^{12}\text{C} [1s^2 2s^2 2p^5 p \rightarrow 4p, (J=0), \ ^1S]$.

Table 2. Comparison of the Fundamental Mode Properties of GSC 00144–03031 and Its ^{12}C Analogue Undergoing a $[1s^2 2s^2 2p 5p(J=0) \rightarrow 4p(J=0), ^1\text{S}]$ Transition

Parameter	GSC 00144-03031	Scale Factor	Predicted Analogue Values	Empirical ^{12}C values	Error
Mass	$\approx 1.75 M_{\odot}$	$1/\Lambda^D$	≈ 12 amu	≈ 12 amu	-
Fund. Mode	radial	-	radial	\approx radial	0
n	$4 \leq n \leq 5$	-	$4 \leq n \leq 5$	$4 \leq n \leq 5$	0
Period	5017.42 sec	$1/\Lambda$	9.65×10^{-15} sec	9.88×10^{-15} sec	0.024

To verify the uniqueness of the discrete self-similar relationship between the specific oscillation periods of GSC 00144–03031 and ^{12}C $[5p(J=0) \rightarrow 4p(J=0), ^1\text{S}]$, the oscillation periods of other transitions in the $3 \leq n \leq 5$ range were checked. The closest alternative match occurred for the $[5p(J=2) \rightarrow 4p(J=2), ^1\text{D}]$ transition. However, its oscillation period of 9.18×10^{-15} sec is about 5% low and the transition is not similar to a fundamental radial mode oscillation. Oscillation periods for other ^{12}C transitions $[3 \leq n \leq 5; ^1\text{S}$ and $^3\text{S}]$ differed from the predicted period of 9.65×10^{-15} sec by 10% or more. Given the good quantitative match between the fundamental period of GSC 00144–03031 and the *single uniquely specified* transition period of ^{12}C , it seems likely that discrete cosmological self-similarity has been shown to apply in this case.

To further demonstrate the uniqueness of our result, we can repeat the same analysis for other atoms such as H, He, Li, Be, B and N. The results of these calculations are summarized in Table 3.

Table 3. Oscillation Periods $[5(l \approx 0) \rightarrow 4(l \approx 0)]$ for Atoms Other Than ^{12}C

Atom	ΔE (cm^{-1})	P (sec)
H	2467.78	1.35×10^{-14}
He	2723.28	1.22×10^{-14}
Li	3287.47	1.02×10^{-14}
Be	4076.88	8.18×10^{-15}
B	5136.47	6.49×10^{-15}
N	4583.11	7.27×10^{-15}

The oscillation periods for the most radial transitions $[5(l=0) \rightarrow 4(l=0)]$ of H, He, Be, B and N are definitely *not* in good agreement with our predicted test period. The closest alternative match is the $5s \rightarrow 4s$ transition for Li with an oscillation period of 1.02×10^{-14} sec, which is higher than our predicted period by about 5.4%, and would require an unreasonable 42% error in the mass estimate for GSC 00144–03031.

5. The Secondary Period of GSC 00144–03031

As an additional check on the uniqueness of the above results, we now explore the secondary oscillation period of GSC 00144–03031, which is 3872.70 sec. At this point in the development of the SSCP, we are still trying to fully understand single-mode pulsators, although research on double-mode pulsation is on-going and seems promising. That preliminary work suggests that *both* oscillation periods of a double-mode pulsator come from the discrete spectrum of allowed transition periods for that system. Therefore, we can predict that for a ^{12}C atom undergoing transitions with $3 \leq n \leq 5$, there will be a transition period very close to $(3873 \text{ sec})(1/\Lambda) \approx 7.45 \times 10^{-15} \text{ sec}$. Actually, when this prediction is tested we find three candidates. The $[5p(J=1) \rightarrow 4p(J=1), ^1P]$ transition comes within 4.4% of the predicted period, but the $[(J=1), ^1P]$ nature of this transition does not have much radial character and we expect the match between predicted period and comparison period to be at the 3% level, or better. The triplet configuration of ^{12}C has a $[1s^2 2s^2 2p(^2P^o) 5s(J=2) \rightarrow 1s^2 2s^2 2p 3d(J=1), ^3P^o]$ transition with a period of $7.48 \times 10^{-15} \text{ sec}$ (only 0.4% high), and the $(J=0)$ version of that transition has a period of $7.55 \times 10^{-15} \text{ sec}$ (1.3% high). These two closely related transitions offer very good quantitative matches, but the acceptability of having simultaneous transitions involving *both* singlet and triplet configurations remains to be more fully explored. The third potential match, and possibly the most interesting, occurs with the $[1s^2 2s^2 2p 4p(J=0), ^1S \rightarrow 1s^2 2s^2 2p 3d(J=2), ^1D^o]$ transition. This transition has an oscillation period of $7.32 \times 10^{-15} \text{ sec}$, which is quite close to the predicted period of $7.45 \times 10^{-15} \text{ sec}$ (1.8% low). It also has the unique feature that it is directly linked to the fundamental pulsation period since both transitions share the $[1s^2 2s^2 2p 4p(J=0), ^1S]$ energy level. Whereas the previous candidates would seem to require some sort of “superposition” of possibly competing transitions, the third candidate suggests an alternative qualitative explanation for double-mode pulsation, wherein the system is undergoing a sequence of two separate, but related, transitions and the second oscillation begins to activate before the first oscillation has completely finished.

Although at present we do not have enough information to definitively choose between the three candidate matches for the secondary oscillation of GSC 00144–03031, we can safely say that this additional check on the uniqueness of the primary results for the dominant oscillation period has yielded encouraging results. Had we not found any period matches at the $< 5\%$ level, then that might have indicated a serious problem with the analysis, or possibly with the whole concept of discrete cosmological self-similarity.

Conclusions

The δ Scuti class of variable stars is a much more heterogeneous class than the RR Lyrae class, corresponding to a collection of Atomic Scale systems with masses in the 10 to 17 amu range. The *high amplitude* δ Scuti stars appear to be limited to low Δn transitions ($\Delta n \approx 1$) primarily within the range $3 \leq n \leq 6$. The substantial heterogeneity

of this class interferes with a simple comparison of sizeable samples of empirical δ Scuti oscillation periods with predicted periods derived from Atomic Scale data, although this may be possible *in principle*. However, we have achieved the specificity required for a meaningful test of discrete cosmological self-similarity by focusing on an individual, well-characterized δ Scuti star. Based purely on physical data for GSC 00144–03031, we have identified:

- (1) a specific Atomic Scale analogue (^{12}C),
- (2) a most likely energy level transition ($1s^2 2s^2 2p^5 p \rightarrow 4p$, $J=0$, ^1S), and
- (3) a uniquely matching self-similar oscillation period (agreement at the 97.6% level).

These new results, combined with the previous successful demonstration (Oldershaw, 2008a,b) of discrete self-similarity between RR Lyrae stars and He atoms undergoing $\Delta n = 1$ Rydberg state transitions, lend further support to our contention that discrete cosmological self-similarity is a fundamental property of nature. It can be predicted that the same methods that have been applied here, and in the case of the RR Lyrae stars, can be successfully applied to other δ Scuti stars if the following criteria are met. The stellar mass must be known to an accuracy of $\leq 0.05 M_{\odot}$, so that the correct Atomic Scale analogue can be identified. Ideally the star should pulsate in a single dominant oscillation mode, although double-mode pulsators can also be analyzed by our methods. Multi-mode pulsators with three or more low-amplitude periods appear to be analogous to excited, highly perturbed, atomic systems that are oscillating at several *potential* transition periods, but are not yet undergoing single specific transitions between energy levels, as will be discussed in a forthcoming paper (Oldershaw, 2008c) on the class of ZZ Ceti variable stars. At any rate, the higher the amplitude of the dominant oscillation period of the δ Scuti star, the more likely it is that we are observing an event that is self-similar to a full-fledged transition between discrete energy levels. Although we may be getting a bit ahead of ourselves here, it is conceivable that a typical single-mode HADS star evolves from a multi-mode LADS star when the latter absorbs a sufficient amount of energy at an appropriate frequency in order to trigger a genuine energy level transition.

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Neutrino Mixings and Magnetic Moments Due to Planck Scale Effects

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Abstract: In this paper, we consider the effect of Planck scale operators on neutrino magnetic moments. We assume that the main part of neutrino masses and mixings arise through GUT scale operators. We further assume that additional discrete symmetries make the neutrino mixing bi-maximal. Quantum gravitational (Planck scale) effects lead to an effective $SU(2)_L \times U(1)$ invariant dimension-5 Lagrangian involving neutrino and Higgs fields, which gives rise to additional terms in neutrino mass matrix. These additional terms can be considered to be perturbation of the GUT scale bi-maximal neutrino mass matrix. We assume that the gravitational interaction is flavor blind and we study the neutrino mixings and magnetic moments due to the physics above the GUT scale.

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1. Introduction

Neutrino magnetic moments is proportional to the neutrino mass as required by the symmetry principles. At present the solar, atmospheric, reactor and accelerator experiments indicates the existence of non zero neutrino masses. Its indicates that neutrino has a magnetic moment. A minimal extension of solar model yields a neutrino magnetic moment [1]

$$\mu_\nu = \frac{3eG_F m_\nu}{8\pi^2\sqrt{2}} = \frac{3G_F m_e m_\nu}{4\pi^2\sqrt{2}} \mu_B, \quad (1)$$

where $\mu_B = e/2m$ is the Bohr magnetron, m_e is the electron mass and m_ν is the neutrino mass.

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The fundamental magnetic moment are associated with the mass eigenstates in the mass eigenstates basis. Dirac neutrino can have diagonal or off diagonal moment, while Majorana neutrino can have transition magnetic moments [2, 3, 4]. The experimental value of neutrino magnetic moment can be determined by only in the recoil electron spectrum from neutrino electron spectrum [5, 6]. In this paper, we study, how Planck scale effects the neutrino magnetic moments. Magnetic moment of neutrinos, in principle, depend on the distance from its source [4]

$$\mu_e^2 = \sum_i \left| \sum_j U_{ej} \mu_{ij} \exp(-iE_j L) \right|^2, \quad (2)$$

where μ_{ij} is the fundamental constant in term of unit μ_B that characterize the coupling of the neutrino mass eigenstate to the electromagnetic field. The expression for μ_e^2 in the case of Dirac neutrino, with only diagonal magnetic moment ($\mu_{ij} = \mu_i \delta_{ij}$); this is used by the Particle Data Group [4]

$$\mu_e^2 = \sum_j |U_{ej}|^2 |\mu_j|^2, \quad (3)$$

In this expression, there is no dependence of L and neutrino energy E . In this one can say the neutrino magnetic moments depend on neutrino mixings. In the case of Majorana neutrino, and we assume three mass eigenstates. Then

$$\mu_e^2 = (|\mu_{12}|^2 + |\mu_{13}|^2)(|U_{e2}|^2 + |U_{e3}|^2). \quad (4)$$

For the Dirac case this implies that at least nondiagonal magnetic moment is as large as the diagonal ones. In the case of Majorana, it implies that two different nondiagonal magnetic moment are of a similar magnitude [16]. Correction to neutrino mixing and neutrino magnetic moments are given in Section 2. In section 3 give the results on neutrino mixing and magnetic moments.

2. Corrections to Mixing Angles and Neutrino Magnetic Moments

The neutrino mass matrix is assumed to be generated by the see saw mechanism [7, 8, 9]. Here we will assume that the dominant part of neutrino mass matrix arises due to GUT scale operators and they lead to bi-maximal mixing. The effective gravitational interaction of neutrinos with Higgs field can be expressed as $SU(2)_L \times U(1)$ invariant dimension-5 operator [10],

$$L_{grav} = \frac{\lambda_{\alpha\beta}}{M_{pl}} (\psi_{A\alpha} \epsilon_{AC} \psi_C) C_{ab}^{-1} (\psi_{Bb\beta} \epsilon_{BD} \psi_D) + h.c. \quad (5)$$

Here and every where below we use Greek indices α, β . for the flavor states and Latin indices i, j, k for the mass states. In the above equation $\psi_\alpha = (\nu_\alpha, l_\alpha)$ is the lepton doublet, $\phi = (\phi^+, \phi^0)$ is the Higgs doublet and $M_{pl} = 1.2 \times 10^{19} GeV$ is the Planck mass.

λ is a 3×3 matrix in flavour space with each element $O(1)$. In eq(4), all indices are explicitly shown. The Lorentz indices $a, b = 1, 2, 3, 4$ are contracted with the charge conjugation matrix C and the $SU(2)_L$ isospin indices $A, B, C, D = 1, 2$ are contracted with ϵ , the Levi-Civita symbol in two dimensions. After spontaneous electroweak symmetry breaking the Lagrangian in eq(4) generates additional terms of neutrino mass matrix

$$L_{mass} = \frac{v^2}{M_{pl}} \lambda_{\alpha\beta} \nu_\alpha C^{-1} \nu_\beta, \quad (6)$$

where $v=174$ GeV is the VEV of electroweak symmetry breaking.

We assume that the gravitational interaction is “flavour blind” , that is $\lambda_{\alpha\beta}$ is independent of α, β indices. Thus the Planck scale contribution to the neutrino mass matrix is

$$\mu \lambda = \mu \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (7)$$

where the scale μ is

$$\mu = \frac{v^2}{M_{pl}} = 2.5 \times 10^{-6} eV. \quad (8)$$

In our calculation, we take eq(6) as a perturbation to the main part of the neutrino mass matrix, that is generated by GUT dynamics. We compute the changes in neutrino mass eigenvalues and mixing angles induced by this perturbation. We assume that GUT scale operators give rise to the light neutrino mass matrix, which in mass eigenbasis, takes the form $M = diag(M_1, M_2, M_3)$, where M_i are real and non negative. We take these to be the unperturbed (0^{th} – order) masses. Let U be the neutrino mixing matrix at 0^{th} – order. Then the corresponding 0^{th} – order mass matrix \mathbf{M} in flavour space is given by

$$\mathbf{M} = U^* M U^\dagger. \quad (9)$$

The 0^{th} – order MNS matrix U is given in this form

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}, \quad (10)$$

where the nine elements are functions of three mixing angles, one Dirac phase and two Majorana phases. In terms of the above elements, the mixing angles are defined by

$$\left| \frac{U_{e2}}{U_{e1}} \right| = \tan\theta_{12}, \quad \left| \frac{U_{\mu3}}{U_{\tau3}} \right| = \tan\theta_{23}, \quad |U_{e3}| = \sin\theta_{13}. \quad (11)$$

In terms of the above mixing angles, the mixing matrix is written as

$$U = \text{diag}(e^{if_1}, e^{if_2}, e^{if_3})R(\theta_{23})\Delta R(\theta_{13})\Delta^* R(\theta_{12})\text{diag}(e^{ia_1}, e^{ia_2}, 1). \quad (12)$$

The matrix $\Delta = \text{diag}(e^{\frac{i\delta}{2}}, 1, e^{-\frac{i\delta}{2}})$ contains the Dirac phase δ . This leads to CP violation in neutrino oscillations. a_1 and a_2 are the so called Majorana phases, which affect the neutrinoless double beta decay. f_1 , f_2 and f_3 are usually absorbed as a part of the definition of the charge lepton field. It is possible to rotate these phases away, if the mass matrix eq(5) is the complete mass matrix. However, since we are going to add another contribution to this mass matrix, these phases of the zeroth order mass matrix can have an impact on the complete mass matrix and thus must be retained. By the same token, the Majorana phases which are usually redundant for oscillations have a dynamical role to play now. Planck scale effects will add other contributions to the mass matrix. Including the Planck scale mass terms, the mass matrix in flavour space is modified as

$$\mathbf{M} \rightarrow \mathbf{M}' = \mathbf{M} + \mu\lambda, \quad (13)$$

with λ being a matrix whose elements are all 1 as discussed in eq(3). Since μ is small, we treat the second term (the Planck scale mass terms) in the above equation as a perturbation to the first term (the GUT scale mass terms). The impact of the perturbation on the neutrino masses and mixing angles can be seen by forming the hermitian matrix

$$\mathbf{M}'^\dagger \mathbf{M}' = (\mathbf{M} + \mu\lambda)^\dagger (\mathbf{M} + \mu\lambda), \quad (14)$$

which is the matrix relevant for oscillation physics. To the first order in the small parameter μ , the above matrix is

$$\mathbf{M}'^\dagger \mathbf{M}' + \mu\lambda^\dagger \mathbf{M} + \mathbf{M}'^\dagger \mu\lambda. \quad (15)$$

This hermitian matrix is diagonalized by a new unitary matrix U' . The corresponding diagonal matrix M'^2 , correct to first order in μ , is related to the above matrix by $U' M'^2 U'^\dagger$. Rewriting \mathbf{M} in the above expression in terms of the diagonal matrix M we get

$$U' M'^2 U'^\dagger = U(M^2 + m^\dagger M + M m)U^\dagger \quad (16)$$

where

$$m = \mu U^t \lambda U. \quad (17)$$

Here M and M' are the diagonal matrices with neutrino masses correct to 0th and 1th order in μ . It is clear from eq(15) that the new mixing matrix can be written as:

$$U' = U(1 + i\delta\Theta),$$

$$= \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} + \begin{pmatrix} U_{e2}\delta\Theta_{12}^* + U_{e3}\delta\Theta_{23}^*, & U_{e1}\delta\Theta_{12} + U_{e3}\delta\Theta_{23}^*, & U_{e1}\delta\Theta_{13} + U_{e3}\delta\Theta_{23}^* \\ U_{\mu2}\delta\Theta_{12}^* + U_{\mu3}\delta\Theta_{13}^*, & U_{\mu1}\delta\Theta_{12} + U_{\mu3}\delta\Theta_{23}^*, & U_{\mu1}\delta\Theta_{13} + U_{\mu3}\delta\Theta_{23}^* \\ U_{\tau2}\delta\Theta_{12}^* + U_{\tau3}\delta\Theta_{13}^*, & U_{\tau1}\delta\Theta_{12} + U_{\tau3}\delta\Theta_{23}^*, & U_{\tau1}\delta\Theta_{13} + U_{\tau3}\delta\Theta_{23}^* \end{pmatrix}, \quad (18)$$

where $\delta\theta$ is a hermitian matrix that is first order in μ . From eq(15) we obtain

$$M^2 + m^\dagger M + Mm = M'^2 + [i\delta\Theta, M'^2]. \quad (19)$$

Therefore to first order in μ , the mass squared difference $\Delta M_{ij}^2 = M_i^2 - M_j^2$ get modified [11, 13] as:

$$\Delta M_{ij}^{\prime 2} = \Delta M_{ij}^2 + 2(M_i \text{Re}[m_{ii}] - M_j \text{Re}[m_{jj}]). \quad (20)$$

The change in the elements of the mixing matrix, which we parameterized by $\delta\Theta$, is given by

$$\delta\Theta_{ij} = \frac{i\text{Re}(m_{ij})(M_i + M_j)}{\Delta M_{ij}^{\prime 2}} - \frac{\text{Im}(m_{ij})(M_i - M_j)}{\Delta M_{ij}^{\prime 2}}. \quad (21)$$

The above equation determines only the off diagonal elements of matrix $\delta\Theta_{ij}$. The diagonal elements of $\delta\Theta$ can be set to zero by phase invariance.

The new Majorana neutrino magnetic moments due to Planck scale is given by

$$\mu_x^{\prime 2} = \sum_j \sum_k |U'_{xj}|^2 |\mu_{jk}|^2, \quad (22)$$

where ($x = e, \mu, \tau$) is the flavour indices. In the case of three flavour, the magnetic moment of Majorana electron neutrinos is given by

$$\mu_e^{\prime 2} = (|\mu_{12}|^2 + |\mu_{13}|^2)(|U'_{e2}|^2 + |U'_{e3}|^2). \quad (23)$$

and there is no dependence on the distance L or neutrino energy.

3. Results and Discussions

We assume the largest allowed value of 2 eV for degenerate neutrino mass which comes from tritium beta decay [12]. We also assume normal neutrino mass hierarchy. Thus we have $M_1 = 2$ eV, $M_2 = \sqrt{M_1^2 + \Delta_{21}}$ and $M_3 = \sqrt{M_1^2 + \Delta_{31}}$. As in the case of 0th order mixing angles, we can compute 1st order mixing angles in terms of 1st order mixing

matrix elements [14]. We expect the mixing angles coming from GUT scale operators to be determined by some symmetries. For simplicity, here we assume a bi-maximal mixing pattern, $\theta_{12} = \theta_{23} = \pi/4$ and $\theta_{13} = 0$. We compute the modified mixing angles for the degenerate neutrino mass of 2 eV. We have taken $\Delta_{31} = 0.0025eV^2$ and $\Delta_{21} = 0.00008eV^2$. For simplicity we have set the charge lepton phases $f1 = f2 = f3 = 0$. We have checked that non-zero values for these phases do not change our results. Since we have set $\theta_{13} = 0$, the Dirac phase δ drops out of the 0^{th} order mixing matrix. We consider the Planck scale effects on neutrino mixing and we get the given range of mixing parameter of MNS matrix

$$U' = R(\theta_{23} + \epsilon_3)U_{phase}(\delta)R(\theta_{13} + \epsilon_2)R(\theta_{12} + \epsilon_1), \quad (24)$$

In Planck scale, only $\theta_{12}(\epsilon_1 = \pm 3^\circ)$ have reasonable deviation and θ_{13} , θ_{23} deviation is very small less than 0.3° [14]. In the new mixing at Planck scale we get the given moments of Majorana neutrinos

$$\mu_e'^2 = (|\mu_{12}|^2 + |\mu_{13}|^2)(|U'_{e2}|^2 + |U'_{e3}|^2). \quad (25)$$

$$\mu_\mu'^2 = (|\mu_{21}|^2 + |\mu_{23}|^2)(|U'_{\mu1}|^2 + |U'_{\mu3}|^2). \quad (26)$$

$$\mu_\tau'^2 = (|\mu_{31}|^2 + |\mu_{32}|^2)(|U'_{\tau2}|^2 + |U'_{\tau3}|^2). \quad (27)$$

The best direct limit on the neutrino magnetic moment, $\mu_e \leq 1.8 \times 10^{-10} \mu_B$ at 90% CL [15], coming from neutrino electron scattering with anti-neutrino. However, the limit obtained using the SK data [4], $\mu_e \leq 1.5 \times 10^{-10} \mu_B$. Due to Planck scale effects, mixing angle θ_{12} and θ_{13} will contribute the magnetic moments of neutrinos.

conclusions

We assumed that the main part of neutrino masses and mixings arise from GUT scale operators. We considered these to be 0^{th} order quantities. We further assumed that GUT scale symmetries constrain the neutrino mixing angles to be either bi-maximal or tri-bi-maximal. The gravitational interaction of lepton fields with SM Higgs field gives rise to an $SU(2)_L \times U(1)$ invariant dimension-5 effective lagrangian, given originally by Weinberg [10]. On electroweak symmetry breaking this operator leads to additional mass terms. We consider these to be a perturbation of GUT scale mass terms. We compute the first order corrections to neutrino mass eigenvalues and mixing angles. In [11], it was shown that the change in θ_{13} , due to this perturbation, is small. Here we show that the change in θ_{23} also is small (less than 0.3°) but the change in θ_{12} can be substantial (about $\pm 3^\circ$). The changes in all three mixing angles are proportional to the neutrino mass eigenvalues. To maximize the change we assumed degenerate neutrino masses $\simeq 2.0$ eV. For degenerate neutrino masses, the changes in θ_{13} and θ_{23} are inversely proportional to Δ_{31} and Δ_{32} respectively, whereas the change in θ_{12} is inversely proportional to Δ_{21} . Since $\Delta_{31} \cong \Delta_{32} \gg \Delta_{21}$, the change in θ_{12} is much larger than the changes in θ_{13} and

θ_{23} . In this paper, we write the neutrino magnetic moment expression for three flavour neutrino mixing. For majorana neutrino two non diagonal moment, these expression are Eq(4.0) for vacuum mixing. For majorana neutrino with three flavour, the expression is $\mu_e^2 = (|\mu_{12}|^2 + |\mu_{13}|^2)(|U_{e2}|^2 + |U_{e3}|^2)$. In this paper, finally we wish make a important comment. Due to Planck scale effects mixing angle θ_{12} and θ_{13} contribute the magnetic moments of neutrinos.

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Casimir Force in Confined Crosslinked Polymer Blends

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Abstract: The physical system we consider is a crosslinked polymer blend (or an interpenetrating polymer network), made of two chemically incompatible polymers, which are confined to two parallel plates that are a finite distance L apart, that is $L < \xi^*$. Here, $\xi^* \sim aD^{-1/2}$ (a being the monomer size and D the reticulation dose) denotes the size of the microdomains (mesh size). We assume that these strongly adsorb one or the two polymers, near the spinodal temperature (critical adsorption). The strong fluctuations of composition give rise to an induced force between the walls we are interested in. To compute this force, as a function of the separation L , we elaborate a field model, of which the free energy is a functional of the composition fluctuation (order parameter). Within the framework of this extended de Gennes theory, we exactly compute this induced force, for two special boundary conditions (symmetric and asymmetric plates). Symmetric plates mean that these have the same preference to adsorb one polymer, while asymmetric ones correspond to the situation where one polymer adsorbs onto the first plate and the other onto the second one. Using the *phase portrait* method, we first show that the induced force is *attractive*, for symmetric plates, and *repulsive*, for asymmetric ones. Second, we demonstrate that the force satisfies the scaling laws : $\Pi_a = \Pi_a^0 \cdot \Omega_a(L/\xi^*)$ (symmetric plates) and $\Pi_r = \Pi_r^0 \cdot \Omega_r(L/\xi^*)$ (asymmetric plates). Here, $\Omega_a(x)$ and $\Omega_r(x)$ are *known* universal scaling functions, where $\Pi_a^0 = -E_a L^{-4}$ and $\Pi_r^0 = E_r L^{-4}$ are the induced forces relative to an uncrosslinked polymer blend confined to the same geometry (E_a and E_r are known amplitudes). For very small distances compared to the mesh size ξ^* , we show that, in any case, the force decays exponentially, that is : $\Pi_a \simeq -E_a L^{-4} \exp\{-L^2/\xi^{*2}\}$ and $\Pi_r \simeq E_r L^{-4} \exp\{-L^2/\xi^{*2}\}$. Finally, this work must be regarded as a natural extension of that relative to the uncrosslinked polymer blends.

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1. Introduction

The crosslinked polymer blends (CPBs) and interpenetrating polymer networks (IPNs) constitute new materials that have many potential applications. Among these, we can quote the crosslinked epoxy adhesives, the crosslinked mixtures of bacterial and seaweed polysaccharides gellan and agarose [1], the polysiloxane interpenetrating networks [2], and the IPNs made of polypropylene/poly(*n*-butyl acrylate) [3]. For example, the formers possess a great resistance to acids, bases and many solvents, and exhibit high glass transition temperatures and thermal resistance, while the second ones are used as electronic device encapsulants.

The CPBs and IPNs are made of two chemically incompatible polymers *A* and *B*. Their common feature is that, when they cooled down, below some critical temperature, one assists to the appearance of microdomains alternatively rich in *A* and *B*-polymers. This is the so-called *microphase separation* (MPS), which results from a competition between the tendency that the polymer mixture phase separates completely and the elasticity of the gel that resists to such a separation.

The pioneered theory of MPS was introduced by Gennes [4], followed by several extended works [5 – 21]. The de Gennes' trick was an analogy between the CPB and a dielectric medium. The theoretical predictions were confirmed experimentally by Briber and Bauer [22] by small-angle neutron-scattering experiment on the PS-PVME mixture, which was crosslinked by γ -ray irradiation techniques [23].

Usually, the CPBs and IPNs are trapped not in infinite space but rather in a finite volume. In other words, they are in the presence of geometrical boundaries. To simplify their investigation, we assume that these are confined to two parallel walls. In addition, we suppose that these plates strongly adsorb one or the two crosslinked polymers. This preferential adsorption, termed *critical adsorption* in literature [24 – 40], takes place in the vicinity of the spinodal temperature. Around this temperature, the thermal fluctuations of composition yields an induced force (*critical Casimir force*) we are interested in.

The Casimir effect is one of the fundamental discoveries of the last century. This effect that has been predicted, for the first time, by Hendrick Casimir in 1948 [41], stipulates that the vacuum quantum fluctuations of a confined electromagnetic field induce an attractive force between two parallel uncharged conducting plates. The Casimir effect has been confirmed in more recent experiments by Lamoreaux [42] and by Mohideen and Roy [43].

Thereafter, Fisher and de Gennes [24] pointed out in a short note that the Casimir effect also appears in the context of the critical systems (fluids, simple liquid mixtures, polymer blends, liquid ^4He , liquid-crystals), confined to restricted geometries or in the presence of colloidal particles. For these systems, the critical fluctuations of the order parameter play the role of the vacuum quantum fluctuations, and then, they lead to long-ranged forces between the confining walls or between immersed colloids [44].

The Casimir effect related to the critical systems that are confined between two

parallel walls or in the presence of immersed colloidal particles, has been the subject of numerous theoretical and experimental investigations [24, 25, 44 – 70]. Very recently, these studies were generalized to the critical polymer blends or ternary polymer solutions [71 – 84].

In this paper, we consider a polymer mixture of two polymers A and B of different chemical nature, confined to two parallel plates 1 and 2 that are a finite distance L apart. This mixture is crosslinked in the one-phase region, that is at high-temperature. The system is prepared in such a way that, near the spinodal temperature, the plates strongly adsorb one or the two polymers (critical adsorption) [85]. Near criticality, the crosslinked mixture presents as microdomains, of size $\xi^* \sim aD^{-1/2}$ (a being the monomer size and D the reticulation dose), which are alternatively rich in A and B -polymers. Therefore, the length ξ^* must be compared to the film thickness L . Since the critical fluctuations occurs in domains of size ξ^* , the Casimir effect is relevant only when the latter is much greater than L , that is $L < \xi^*$. The critical adsorption and thermal fluctuations of composition have as consequence to generate an effective force between plates.

In this work, we are interested in the computation of such a force. For the sake of simplicity, we will be concerned only by two boundary conditions, depending on whether the plates have the same preference to attract one polymer (symmetric plates) or they have an opposite preference (asymmetric plates). The latter boundary condition means that the plate 1 adsorbs the polymer A and plate 2 the polymer B .

To determine the induced force, we elaborate an extended model taking into account the plate-polymer interaction. The associated free energy is a functional of the composition fluctuation $\varphi = \Phi_A - \Phi_B$, where Φ_A and Φ_B denote the compositions of A and B -polymers, respectively. With the help of the *phase portrait* method [71 – 74], we compute the resulting force (per unit area), Π , as a function of separation L . This force is attractive for symmetric plates and repulsive for asymmetric ones. For the two boundary conditions, at the spinodal temperature T_s , we find an *exact* form for the expected force : $\Pi_a = \Pi_a^0 \cdot \Omega_a(L/\xi^*)$ (symmetric plates) and $\Pi_r = \Pi_r^0 \cdot \Omega_r(L/\xi^*)$ (asymmetric plates), with known universal scaling functions $\Omega_a(x)$ and $\Omega_r(x)$. Here, $\Pi_a^0 = (N/k_B T_s) \Delta_{++} (L/a)^{-4}$ ($\Delta_{++} \simeq -7$) and $\Pi_r^0 = (N/k_B T_s) \Delta_{+-} (L/a)^{-4}$ ($\Delta_{+-} \simeq 28$) are the induced forces for an uncrosslinked polymer blend confined to the same geometry, and N the common polymerization degree of the crosslinked polymer chains A and B . These scaling forms become simpler when one is concerned with very small distances compared to the mesh size ξ^* , that is : $\Pi_a \simeq \Pi_a^0 \exp\{-L^2/\xi^{*2}\}$ and $\Pi_r \simeq \Pi_r^0 \exp\{-L^2/\xi^{*2}\}$. These closer forms indicate that the induced force corresponding to CPBs is *less* important than that relative to uncrosslinked ones. This result is natural, since the presence of crosslinks considerably reduces the critical fluctuations of composition.

The remaining presentation proceeds as follows. In Sec. II, we briefly recall the essential of the de Gennes theory of MPS of CPBs of infinite extent. The presentation of the used model and computation of the Casimir force are the aim of Sec. III. We draw some concluding remarks in the last section.

2. MPS Theory

Consider two polymers A and B of different chemical nature, which are crosslinked in their one-phase region (at high temperature). When the temperature of the system is lowered, below some critical temperature, one assists to the appearance of microdomains alternatively rich in A and B -polymers. In fact, this MPS is the consequence of a competition between the usual macrophase separation and the elasticity of the polymer network.

To investigate the MPS, the starting point is the de Gennes (bulk) free energy [4]

$$\frac{\mathcal{F}_b[\varphi]}{k_B T} = a^{-3} \int d^3 \mathbf{r} \left(\frac{t}{2} \varphi^2 + \frac{u}{4} \varphi^4 + \kappa (\nabla \varphi)^2 - \frac{C}{2} \varphi \Delta^{-1} \varphi \right), \quad (1)$$

with T the absolute temperature and k_B the Boltzmann constant. The temperature parameter t is

$$t = \frac{1}{2}(\chi_c - \chi). \quad (2)$$

The latter is then proportional to the distance from the consolute point, where χ denotes the standard Flory interaction parameter, and $\chi_c = 2/N$ represents its critical value (N being the common polymerization degree of A and B -chains), when the mixture is uncrosslinked. Notice that the parameter χ has the form : $\chi = \alpha + \beta/T$, where the constants α and β depend on the chemical nature of the components A and B . The gradient term takes into account the interfacial energy between A and B -rich phases, with the coefficient $\kappa = a^2/9$ (a is the monomer size). There, $u = 1/3N$ accounts for the coupling constant, and the field or *order parameter* $\varphi(\mathbf{r})$ for the local fluctuation of composition

$$\varphi(\mathbf{r}) = \Phi(\mathbf{r}) - \Phi_c, \quad (3)$$

where Φ is the composition of one component of the mixture, say A , and $\Phi_c = 1/2$ its critical value. We recall that the first two terms in relation (1) constitute the expansion of the usual Flory-Huggins free energy [86, 87] around the critical composition, and the third one describes the gel elasticity. According to de Gennes [4], the internal rigidity constant C is given by

$$C \sim n^{-2} a^{-2}, \quad (4)$$

where n is the average number of monomers per strand (section of chains between consecutive crosslinks). In Eq. (1), the term $-\varphi \Delta^{-1} \varphi$ plays the role of the squared polarization of the dielectric medium in de Gennes' analogy [4], where Δ^{-1} represents the inverse of the Laplacian operator.

The structure factor, $S(q)$, can be obtained keeping only the quadratic terms in field φ . We simply recall its expression [4]

$$S(q) = \frac{1}{2\kappa q^2 + t + \frac{C}{q^2}}, \quad (5)$$

with $q = (4\pi/\lambda) \sin(\theta/2)$ the module of the wave-vector, where λ is the wavelength of the incident radiation and θ the scattering-angle. This structure factor exhibits a maximum

for

$$q^* = \left(\frac{C}{2\kappa} \right)^{1/4} \sim a^{-1} n^{-1/2}, \quad (6)$$

of which the inverse $q^{*-1} \equiv \xi^* = (C/2\kappa)^{-1/4} \sim a\sqrt{n}$ represents the size of microdomains or mesh size. The maximal value, $S(q^*)$, of the structure factor diverges at the spinodal point for [4]

$$t_s = -2\sqrt{2\kappa C} \sim n^{-1}. \quad (7)$$

In the following section, we will be interested in the computation of the Casimir force between two parallel plates delimitating a CPB.

3. Casimir Forces

3.1 Field Model

Consider a CPB that is confined between two parallel adsorptive plates, separated by a finite distance L . We suppose that, near the spinodal temperature, the plates strongly adsorb one or the two crosslinked polymers. Then, one is in the critical adsorption regime. As noted before, the critical adsorption and thermal fluctuations of composition generate a Casimir force between the parallel plates. The purpose of this work is precisely the computation of such a force, as a function of thickness L . Since the critical fluctuations occur inside microdomains, the expected force is significant only for distances smaller than the mesh size ξ^* ($L < \xi^*$).

To compute the induced force, we start from the free energy (per unit area) that describes the confined CPB

$$\frac{\mathcal{F}[\varphi]}{\mathcal{A}k_B T} = \frac{g}{2} \sum_{i=1}^2 \varphi_i^2 - h \sum_{i=1}^2 \varphi_i + \int_0^L \frac{dz}{a^3} \left[\frac{t}{2} \varphi^2 + \frac{u}{4} \varphi^4 + \kappa \left(\frac{d\varphi}{dz} \right)^2 - \frac{C}{2} \varphi \left(\frac{d^2}{dz^2} \right)^{-1} \varphi \right], \quad (8)$$

which is a generalization of that relative to an uncrosslinked one [72]. In the above definition, \mathcal{A} is the common area of plates, t , κ and C are those bulk parameters defined above, g represents the surface coupling constant measuring the interaction strength between the crosslinked mixture and the two plates, and the field h is the surface chemical potentials difference of polymers A and B (reduced by $k_B T$). There, $u = 1/3N$ is the coupling constant, and φ and φ_i 's are the bulk and surfaces fields, respectively. The critical adsorption emerges in the limit $h \rightarrow \infty$, at fixed coupling constant [36]. This is equivalent to impose the condition that the order parameter at surface goes to infinity: $\varphi_0 \rightarrow \infty$, which is the fixed point of the *normal transition* [36, 37, 88], in the Renormalization-Group (RG) sense. We have $\varphi_1 = \varphi_2 \equiv \varphi_0$, for the symmetric plates, and $\varphi_1 = \varphi_2 = -\varphi_0$, for the asymmetric ones. Notice that, because of the homogeneity property of the two plates, the order parameter φ depends only on the perpendicular distance $z \in [0, L]$ from one plate taken as origin.

The next step consists in the computation of the induced force experienced by the parallel plates.

3.2 Induced Forces

To determine such a force, we will need the knowledge of the properties of the equilibrium profile, $\varphi(z)$. The latter can be obtained through a standard minimization of the above total free energy, that is $\delta\mathcal{F}/\delta\varphi = 0$. Then, the profile is solution to the following non-linear differential equation

$$t\varphi + u\varphi^3 - 2\kappa\frac{d^2\varphi}{dz^2} - C\left(\frac{d^2}{dz^2}\right)^{-1}\varphi = 0, \quad (9)$$

together with the boundary conditions

$$2a^{-3}\kappa\left[\frac{d\varphi}{dz}\right]_{z=0} = g\varphi_0 - h, \quad 2a^{-3}\kappa\left[\frac{d\varphi}{dz}\right]_{z=L} = -g\varphi_0 + h, \quad (10a)$$

for the symmetric plates, and

$$2a^{-3}\kappa\left[\frac{d\varphi}{dz}\right]_{z=0} = g\varphi_0 - h, \quad 2a^{-3}\kappa\left[\frac{d\varphi}{dz}\right]_{z=L} = g\varphi_0 - h, \quad (10b)$$

for the asymmetric ones.

As pointed out in a very recent work [85], it is difficult to extract the necessary information directly from the differential equation (9) due to the non-local character of the inverse Laplacian operator $(d^2/dz^2)^{-1}$. But, when one is interested in distances smaller than the mesh size ξ^* ($z < \xi^*$), it has been shown [85] that the reticulation term in Eq. (9) can be replaced by

$$C\left(\frac{d^2}{dz^2}\right)^{-1}\varphi \sim C\xi^{*2}\varphi. \quad (11)$$

With these considerations, at the spinodal temperature ($t \sim -\xi^{*2}$), the differential equation (9) becomes

$$t^*\varphi + u\varphi^3 - 2\kappa\frac{d^2\varphi}{dz^2} = 0, \quad (12)$$

with the critical temperature parameter $t^* = -\alpha_0(\xi^*/a)^{-2} \sim n^{-1}$, where α_0 is a positive numerical coefficient of the order of unity. We have used the fact that $C \sim a^{-2}(\xi^*/a)^{-4}$.

Now, to compute the expected force, the starting point is the first integral of the above differential equation

$$\kappa\left(\frac{d\varphi}{dz}\right)^2 = G(\varphi) + K, \quad (13)$$

with the notation

$$G(\varphi) = \frac{t^*}{2}\varphi^2 + \frac{u}{4}\varphi^4. \quad (14)$$

In formula (13), K is an integration constant that depends on the bulk parameters (t^* , u), the surface ones (c , h) and the separation L .

3.2.1 Symmetric Plates

In this case, it is easy to see that the expected profile $\varphi(z)$ exhibits a minimum point at the middle of the film [73, 75], that is at $z = L/2$. In this case, the integration constant K is directly related to the minimal value of the bulk free energy density $G(\varphi)$, that is $K = -G(\varphi_m)$. Here, $\varphi_m = \varphi(L/2)$ is the minimal value of the equilibrium profile.

We have now all necessary ingredients for the computation of the Casimir force originating from the thermal fluctuations of composition, which are strong near the spinodal temperature. The key of the problem is the first integral (13), with $K = -G(\varphi_m)$. It is easy to see, from this first integral, that the minimal value φ_m of the equilibrium profile is given by the quadrature formula

$$L = 2\sqrt{\kappa} \int_{\varphi_m}^{\varphi_0} \frac{d\varphi}{\sqrt{G(\varphi) - G(\varphi_m)}} . \quad (15)$$

This relationship then expresses the dependence of the minimal value φ_m on separation L and surface composition φ_0 . On the other hand, the attractive force (per unit area), Π_a , is given by the first derivative of the total free energy, with respect to separation L , that is $\Pi_a = -(1/\mathcal{A}k_B T) \partial\mathcal{F}/\partial L$. As in confined uncrosslinked polymer mixtures case [73, 75], this force is simply given by

$$\Pi_a = -\frac{k_B T_s}{a^3} G(\varphi_m) . \quad (16)$$

(The subscript a on Π_a is for *attractive*). Since the quantity $G(\varphi_m)$ is positive definite [73, 75], the force is *attractive*. Therefore, the induced force is completely determined by the knowledge of the minimal value φ_m of the profile.

In the critical adsorption limit, the upper bound of the integral in Eq. (15) goes to infinity ($\varphi_0 \rightarrow \infty$), and the quadrature formula then reduces to

$$L = 2\sqrt{\kappa} \int_{\varphi_m}^{\infty} \frac{d\varphi}{\sqrt{G(\varphi) - G(\varphi_m)}} . \quad (17)$$

Explicitly, we have

$$L = 4\sqrt{\frac{\kappa}{u}} \int_{\varphi_m}^{\infty} \frac{d\varphi}{\sqrt{(\varphi^2 - \varphi_m^2)(\varphi^2 + \varphi_m^2 - \varphi^{*2})}} . \quad (18)$$

In this equality, the quantity $\varphi^* \equiv \sqrt{-2t^*/u}$ is nothing else but the non-zero value of the order parameter φ that solves $G(\varphi^*) = 0$. Therefore, $\varphi_m > \varphi^* = \sqrt{-2t^*/u}$, to ensure the condition that $G(\varphi_m) > 0$. Making the variable change : $\varphi \rightarrow x = \varphi/\varphi_m$ yields

$$L = \frac{4}{\varphi_m} \sqrt{\frac{\kappa}{u}} \int_1^{\infty} \frac{dx}{\sqrt{(x^2 - 1) \left(x^2 + 1 - \frac{\varphi^{*2}}{\varphi_m^2}\right)}} . \quad (19)$$

Mathematically speaking, the above integral defines an *elliptic function* [89]. To get more information about the variation of the expected induced force, we introduce the

dimensionless parameter

$$\eta = \frac{\varphi^*}{\varphi_m} . \quad (20)$$

Of course, η belongs to the interval $]0, 1[$. In term of this parameter, formulae (16) and (19) rewrite

$$\Pi_a = \Pi_a^0 f_a(\eta) , \quad (21a)$$

$$\frac{L}{\xi^*} = g_a(\eta) , \quad (21b)$$

with the η -dependent functions

$$f_a(\eta) = (1 - \eta^2) \left(\frac{\int_1^\infty \frac{dx}{\sqrt{(x^2-1)(x^2+1-\eta^2)}}}{\int_1^\infty \frac{dx}{\sqrt{x^4-1}}} \right)^4 , \quad (21c)$$

$$g_a(\eta) = \eta \int_1^\infty \frac{dx}{\sqrt{(x^2-1)(x^2+1-\eta^2)}} . \quad (21d)$$

The notation

$$\Pi_a^0 = \frac{Nk_B T_s}{a^3} \frac{\Delta_{++}}{(L/a)^4} < 0 \quad (21e)$$

means the Casimir force in uncrosslinked polymer blends case, at the critical temperature that is identical to the spinodal one T_s . In the above relation, the amplitude Δ_{++} is such that [73, 75] : $\Delta_{++} \simeq -7$. There, $\xi^* \sim an^{1/2}$ accounts for the mesh size.

Come back to the *parametric* equations (21a) and (21b) and notice that they give the *exact* form for the induced force upon separation L , by simple elimination of the parameter η . Also, these relations show that the force obeys the following scaling law

$$\Pi_a = \Pi_a^0 \Omega_a \left(\frac{L}{\xi^*} \right) , \quad L < \xi^* . \quad (22)$$

Here, the universal scaling function $\Omega_a(x)$ identifies to $f_a(\eta)$, where the parameter η can be determined by inversion of relation (21b).

To have an idea about the magnitude of the expected force, we expand the two parametric equations (21a) and (21b), to second order, around $\eta = 0$. After elimination of η between these equations, we find that

$$\Pi_a = \Pi_a^0 \left[1 - \left(1 - 2\frac{J}{I} \right) \frac{L^2}{\xi^{*2}} + \mathcal{O} \left(\frac{L^4}{\xi^{*4}} \right) \right] , \quad (23)$$

with the constants

$$I = \int_1^\infty \frac{dx}{\sqrt{x^4-1}} , \quad J = \int_1^\infty \frac{dx}{(x^2+1)\sqrt{x^4-1}} . \quad (24)$$

The ratio J/I equals 0.27. Of course, there is no difficulty to compute high-order perturbation terms.

By an exponentiation of series (23), we find the *closer* form for the induced force

$$\Pi_a \simeq \Pi_a^0 \exp \left\{ -\frac{L^2}{\xi^{*2}} \right\} . \quad (25)$$

We have rescaled the mesh size ξ^* as follows : $\xi^* \rightarrow \sqrt{1 - 2J/I} \xi^*$. This simple formula calls the following comments.

Firstly, the induced force is *attractive*, because $\Pi_a^0 < 0$.

Secondly, the ratio of the force Π_a to the usual one Π_a^0 is a *universal* scaling function of the renormalized separation L/ξ^* , independently on the chemical details of the crosslinked polymer chains and physical nature of plates.

Thirdly, as it should be, the induced force is directly proportional to the polymerization degree of the chains before they are crosslinked (through Π_a^0 , relation (21e)).

Fourthly, formula (25) suggests that the induced force in confined CPBs is *exponentially small* in comparison to that of their homologous with $C = 0$ (no crosslinks). As a matter of fact, this is not surprising, since the presence of permanent crosslinks considerably reduces the critical fluctuations of composition.

Fifthly, at fixed distance, the force is significant only for weakly CPBs (small reticulation dose).

Sixthly, the force vanishes beyond the scale ξ^* , due to the absence of the critical fluctuations of composition.

Finally, in the limit $C = 0$ ($\xi^* \rightarrow \infty$), we naturally recover the usual force, that is $\Pi_a \rightarrow \Pi_a^0$.

The variation of the attractive force Π_a is depicted in Fig. 1, as a function of thickness L (expressed in a unit). This curve is drawn with the ratio $\xi^*/a = 50$.

In Fig. 2, we make a comparison between the computed attractive force ($C \neq 0$) and that relative to an uncrosslinked polymer mixture ($C = 0$). For the former, the chosen reticulation dose is such that $\xi^*/a = 50$. The fact that the curve with $C \neq 0$ is above that relative to the $C = 0$ case reflects the discussion made before.

Fig. 3 shows a superposition of the variations of the attractive force, for two values of the reticulation dose corresponding to the mesh sizes $\xi_1^* = 50a$ and $\xi_2^* = 80a$. Of course, the curve with $\xi^* = \xi_1^*$ is above that with $\xi^* = \xi_2^*$. Indeed, the first case corresponds to a tightly CPB, while the second to a weakly one.

3.2.2 Asymmetric Plates

In this case, it is easy to see that the expected profile $\varphi(z)$ is a monotonous decreasing function from $z = 0$ to $z = L$. The induced force experienced by the asymmetric plates is rather repulsive, and exactly identifies to the integration constant K , that is

$$\Pi_r = \frac{k_B T_s}{a^3} K . \quad (26)$$

(The subscript r on Π_r is for *repulsive*). On the other hand, we show that the thickness L can be given by the following quadrature formula

$$L = 2\sqrt{\kappa} \int_0^\infty \frac{d\varphi}{\sqrt{G(\varphi) + K}}, \quad (27)$$

with $G(\varphi)$ that quantity defined in Eq. (14).

Some algebra yields the two parametric equations defining the induced force

$$\Pi_r = \Pi_r^0 f_r(\eta), \quad (28a)$$

$$\frac{L}{\xi^*} = \eta g_r(\eta), \quad (28b)$$

with the functions

$$f_r(\eta) = \left(\frac{\int_0^\infty \frac{dx}{\sqrt{x^4 + \eta^2 x^2 + 1}}}{\int_0^\infty \frac{dx}{\sqrt{x^4 + 1}}} \right)^4, \quad (28c)$$

$$g_r(\eta) = \eta \int_0^\infty \frac{dx}{\sqrt{x^4 + \eta^2 x^2 + 1}}, \quad (28d)$$

where the new parameter η is defined by : $\eta = (4K/u)^{-1/4} \sqrt{t^*/2} < 1$. There, the quantity

$$\Pi_r^0 = \frac{Nk_B T_s}{a^3} \frac{\Delta_{+-}}{(L/a)^4} > 0 \quad (28e)$$

denotes the Casimir force for an uncrosslinked polymer blend, at the critical temperature that is identical to the spinodal one T_s , with the amplitude [73, 75] : $\Delta_{+-} \simeq 28$.

As for symmetric plates, the induced force rewrites on the scaling form

$$\Pi_r = \Pi_r^0 \Omega_r \left(\frac{L}{\xi^*} \right), \quad L < \xi^*, \quad (29)$$

with the *universal* scaling function $\Omega_r(x)$ that identifies to $f_r(\eta)$, where the parameter η can be determined by inversion of relation (28b). To second order in L/ξ^* , we find

$$\Pi_r = \Pi_r^0 \left[1 - 2 \frac{J'}{I'} \frac{L^2}{\xi^{*2}} + \mathcal{O} \left(\frac{L^4}{\xi^{*4}} \right) \right], \quad (30)$$

with the constants

$$I' = \int_0^\infty \frac{dx}{\sqrt{x^4 + 1}}, \quad J' = \int_0^\infty \frac{x^2}{(x^4 + 1)^{3/2}} dx. \quad (31)$$

Numerically, the ratio J'/I' equals 0.30.

As for symmetric plates, we propose the following *closer* form for the expected induced force

$$\Pi_r \simeq \Pi_r^0 \exp \left\{ - \frac{L^2}{\xi^{*2}} \right\}. \quad (32)$$

We have rescaled the mesh size ξ^* as follows : $\xi^* \rightarrow 2\sqrt{J'/I'} \xi^*$. The above scaling law tells us that the Casimir force in confined CPBs is *exponentially small*, when it is compared to its homologous in the absence of crosslinks.

Emphasize that similar comments could be made for asymmetric plates. In addition, we note that, as for confined uncrosslinked polymer mixtures, the repulsive force remains more important than the attractive one. This fact is explained in Refs. [73] and [75].

In Fig. 4, we report the variation of the repulsive force Π_r upon separation L (expressed in a unit). This curve is drawn with the ratio $\xi^*/a = 50$.

Fig. 5 shows a comparison between the computed repulsive force ($C \neq 0$) and that relative to an uncrosslinked polymer mixture ($C = 0$). For the former, the chosen reticulation dose is such that $\xi^*/a = 50$. Of course, the curve with $C \neq 0$ is below that with $C = 0$, since the thermal fluctuations of composition are reduced due to the presence of permanent reticulations.

In Fig. 6, we superpose the variations of the repulsive force, for two values of the reticulation dose corresponding to the mesh sizes $\xi_1^* = 50a$ and $\xi_2^* = 80a$. The fact that the curve with $\xi^* = \xi_1^*$ is below that with $\xi^* = \xi_2^*$ is a natural tendency, since the critical fluctuations of the order parameter are less important when one is interested in tightly CPBs.

Conclusions

The goal of the present work was the computation of the Casimir force between two parallel plates delimitating a critical CPB. Near the spinodal temperature, the plates strongly adsorb one or the two polymers. The critical fluctuations of composition imply an induced force between the parallel walls. Within the framework of an extended de Gennes theory, we *exactly* computed this force, for two special boundary conditions (symmetric and asymmetric plates).

With the help of the phase portrait method, we have shown, first, that the induced force is attractive for symmetric plates and repulsive for asymmetric ones. Second, we found that the force obeys the scaling laws : $\Pi_a = \Pi_a^0 \Omega_a(L/\xi^*)$ (symmetric plates) and $\Pi_r = \Pi_r^0 \Omega_r(L/\xi^*)$ (asymmetric plates), with known universal scaling functions $\Omega_a(x)$ and $\Omega_r(x)$. Here, $\Pi_a^0 \sim -L^{-4}$ and $\Pi_r^0 \sim L^{-4}$ account for the induced force of an uncrosslinked polymer blend confined to the same geometry. For very small distances compared to the mesh size $\xi^* \sim aD^{-1/2}$ (a being the monomer size and $D = 1/n$ the reticulation dose), we have shown that the force obeys an exponential decay, that is : $\Pi_a \sim -L^{-4} \exp\{-L^2/\xi^{*2}\}$ and $\Pi_r \sim L^{-4} \exp\{-L^2/\xi^{*2}\}$, with known amplitudes.

We note that these results also remain valid for confined IPNs. Therefore, the expression of the Casimir force does not depend on the nature of reticulations.

Emphasize that the used model is a mean-field theory, which is reliable only for high-molecular-weight polymer mixture, as demonstrated, many years ago, by de Gennes [90]. For low-molecular-weight polymer mixtures, however, the mean-field theory is no longer valid. To go beyond this approach, and in order to get correct results, it would be

necessary to use the RG-techniques [91, 92].

Also, we point out that it would be interesting to introduce a good solvent as a third component and study its effects on the induced force. As a matter of fact, the solvent adds new fluctuations of composition, and then, a drastic change of the force expression is expected.

Finally, this work must be considered as a natural extension of that relative to un-crosslinked polymer blends [73, 75].

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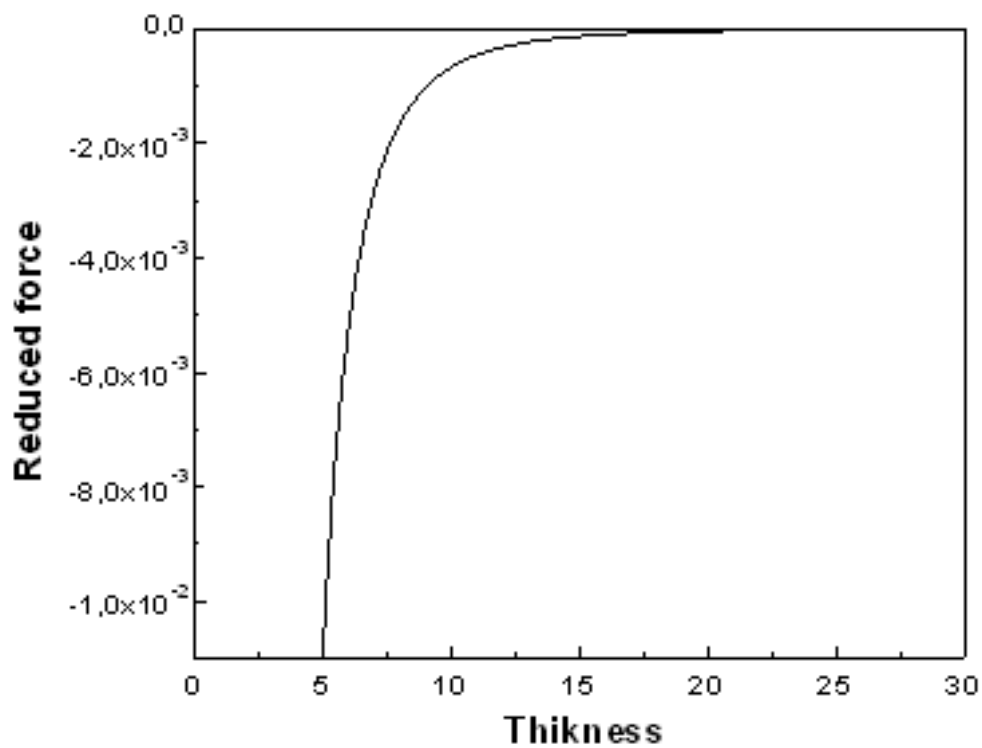


Fig. 1 Reduced attractive force, $a^3\Pi_a/Nk_B T_s$, versus thickness L (expressed in a unit). This curve is drawn with the ratio $\xi^*/a = 50$.

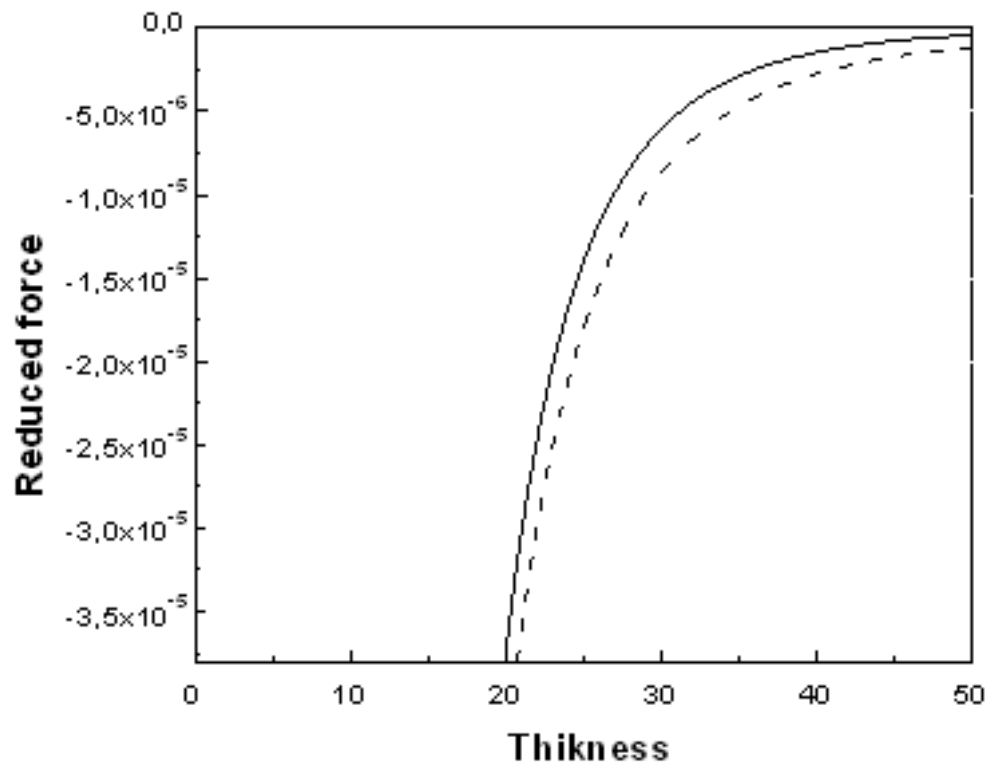


Fig. 2 Superposition of the variations of the computed (reduced) attractive force (solid line) and that relative to an uncrosslinked polymer mixture (dashed line). For the former, the chosen reticulation dose is such that $\xi^*/a = 50$.

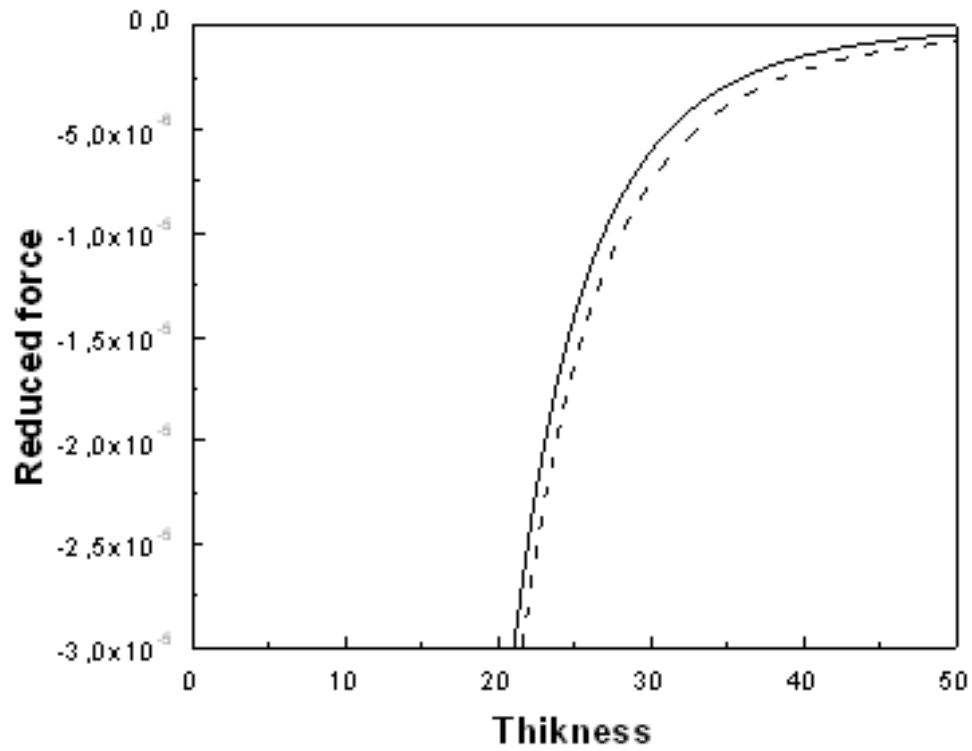


Fig. 3 Superposition of the variations of the computed (reduced) attractive force upon thickness L (expressed in a unit), for two reticulation doses corresponding to the mesh sizes : $\xi_1^* = 50a$ (solid line) and $\xi_2^* = 80a$ (dashed line).

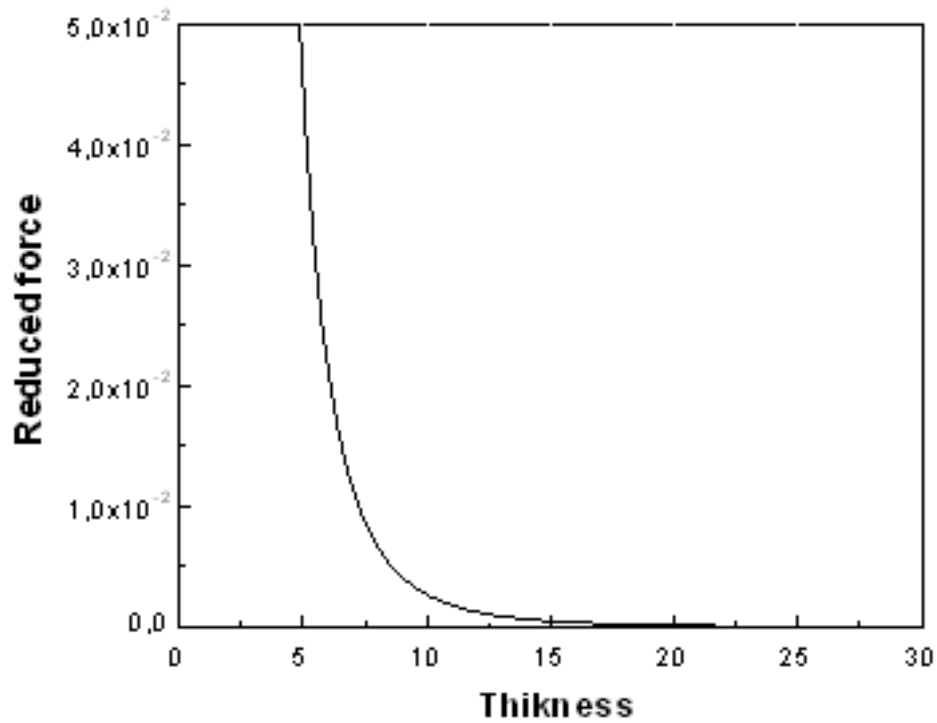


Fig. 4 Reduced repulsive force, $a^3\Pi_r/Nk_B T_s$, versus thickness L (expressed in a unit). This curve is drawn with the ratio $\xi^*/a = 50$.

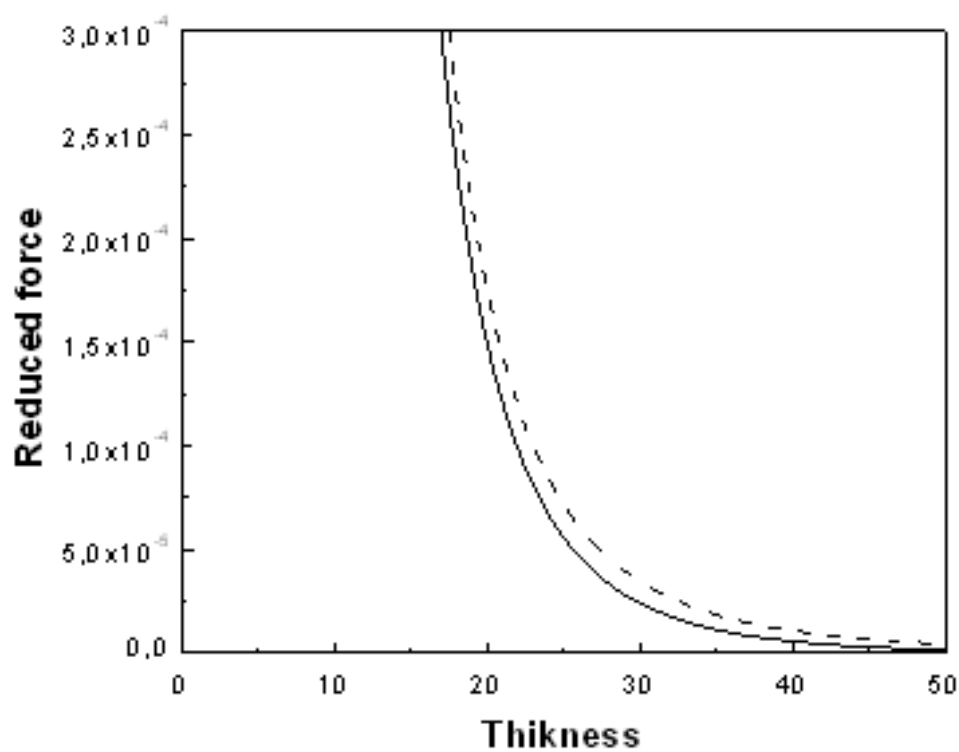


Fig. 5 Superposition of the variations of the computed (reduced) repulsive force (solid line) and that relative to an uncrosslinked polymer mixture (dashed line). For the former, the chosen reticulation dose is such that $\xi^*/a = 50$

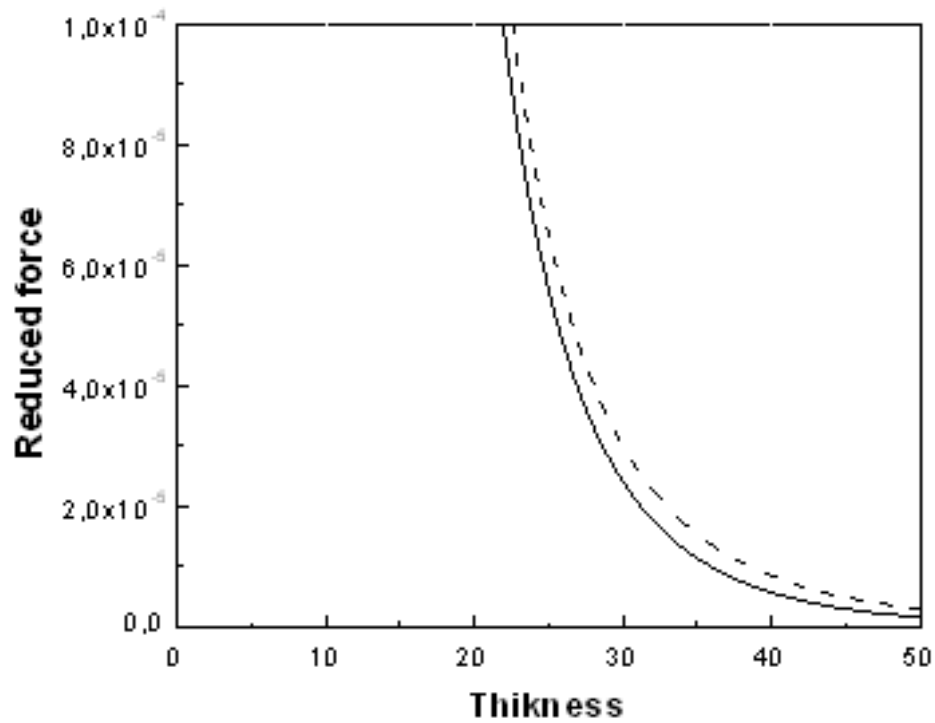


Fig. 6 Superposition of the variations of the computed (reduced) repulsive force upon thickness L (expressed in a unit), for two reticulation doses corresponding to the mesh sizes : $\xi_1^* = 50a$ (solid line) and $\xi_2^* = 80a$ (dashed line).

Transport Properties of Thermal Shot Noise Through Superconductor-Ferromagnetic /2DEG Junction

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Abstract: We study transport properties of thermal shot noise, thermo power and thermal conductance through superconductor-ferromagnetic /2DEG junction under the effect of Fermi energy, number of open channels and excitation energy. Thermal shot noise, $P_{Thermal}$ is directly related to the conductance through the fluctuation- dissipation theorem; the model consists of a 2DEG region inserted between two identical superconductor electrodes. Ferromagnetic strips are placed onto top of each superconductor/2DEG junction and voltage applied across the model. The results show an oscillatory behavior of the dependence of the thermal shot noise on Fermi energy. These results agree with existing experiments. This research is very important for using a model as a high-frequency shot noise detector.

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Keywords: Ferromagnetic Strips; Excitation Energy; Andreev Reflections; Thermal Shot Noise; Thermo Power

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1. Introduction

On chip noise detection schemes, where device and detector are coactively coupled within sub millimeter length scales, can benefit from large frequency bandwidths. This results in a good sensitivity and allows one to study the quantum limit of noise, where an asymmetry can occur in the spectrum between positive and negative frequencies [1, 2]. Shot noise measurements allow us to access the dynamical properties of a resonant tunneling device which are not accessible by measuring solely the average current [2]. Shot noise is more interesting, because it contains information on the temporal correlation of the

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electrons which is not contained in the conductance [3-4].

In this paper we study transport properties of thermal shot noise, thermo power and the thermal conductance through superconductor-ferromagnetic /2DEG junction.

2. Method of Calculation

The model consists of a 2DEG region inserted between two identical superconductor electrodes. Ferromagnetic strips are placed onto top of each superconductor/2DEG junction and voltage applied across the model [4]. In order to study electron transport in the model we make use of the Bogolubov-de Gennes equation [5]. Within the Landauer-Buttiker scattering approach, the conductance through the system biased at the across applied voltage V_a can be written as [6]

$$G(\varepsilon) = \left(\frac{e}{h}\right) \int_{-\infty}^{\infty} d\varepsilon \gamma(\varepsilon) \left(-\frac{\partial f(\varepsilon - eV_a)}{\partial \varepsilon}\right) \quad (1)$$

Where $\gamma(\varepsilon)$ is tunneling probability and given by

$$\gamma(\varepsilon) = [N(\varepsilon) - R_0(\varepsilon) + R_A(\varepsilon)] \quad (2a)$$

and

$$\left(-\frac{\partial f(\varepsilon - eV_a)}{\partial \varepsilon}\right) = (4K_B T)^{-1} \text{Cosh} \left(\frac{\varepsilon - \varepsilon_F - eV_a}{2K_B T}\right) \quad (2b)$$

Where $R_0(\varepsilon)$ and $R_A(\varepsilon)$ are normal, Andreev reflection and $N(\varepsilon)$ is the number of open channels in the system, V_a applied voltage, f is the Fermi distribution function, ε is the excitation energy, Δ_0 (Niobium)=0.0015 eV is the superconducting energy gap, critical temperature T_c (Niobium)= 9.3 K, critical magnetic field B_c (Niobium)= 0.1985 Tesla and coherence length ξ_0 (Niobium)=38 nm.

Thermal shot noise, $P_{Thermal}$ is directly related to the conductance through the fluctuation-dissipation theorem [7]

$$P_{Thermal} = (4K_B T) \left(\frac{e}{h}\right) \int_{-\infty}^{\infty} d\varepsilon \gamma(\varepsilon) \left(-\frac{\partial f(\varepsilon - eV_a)}{\partial \varepsilon}\right) \quad (3)$$

Thermal shot noise is more interesting, because it contains information on the temporal correlation of the electrons which is not contained in the conductance. The thermo power S , of the system is given by [8]

$$S = -L/G \quad (4)$$

Where L , is thermo-electric coefficient and G , is the conductance.

The thermal conductance, K , is given by

$$K(\varepsilon) = -\left(\frac{\pi^2 k_B^2}{3e}\right) \left(\frac{T}{h}\right) \int_{-\infty}^{\infty} d\varepsilon \gamma(\varepsilon) \left(-\frac{\partial f(\varepsilon - eV_a)}{\partial \varepsilon}\right) \quad (5)$$

3. Results and Discussions

The thermal shot noise, $P_{Thermal}$, thermo power, S , and the thermal conductance, K , Eqs. 3, 4 and 5 has been computed respectively over Fermi energy, excitation energy, magnetic field and applied voltage. The calculations were performed for the cases: Fig. (1) Show that thermal shot noise as a function of the Fermi energy at different value of the temperature, this results show an oscillatory behavior of the dependence of the thermal shot noise on Fermi energy, the coincidence of peaks in the thermal shot noise with steps in the Fermi energy is clearly visible; these oscillations are due to Coulomb blockade effect [9-11]. Fig. (2) Show that Periodic suppression of thermal conductance as a function of the number of open channels at different applied voltage, the value of thermal conductance it is very high with the increasing the number of open channels. Fig. (3) Show that thermo power as a function of the magnetic field at different temperatures, the value of thermo power decreasing with increasing the magnetic field, the results show an oscillatory behavior of the dependence of the thermo power on magnetic field [11, 12]. Fig. (4) Show that thermo power as a function of the excitation energy, ε , a crossover from the quantization behavior of the thermo power to a δ -function behavior, might be explained as the probability for normal reflections $R_0(\varepsilon)$ be more dominant over the Andreev reflections $R_A(\varepsilon)$ for large numbers of open channels in the system $N(\varepsilon)$ [12-14]. Our results are in good concordant with those in the literature [12-18].

Conclusion

In this paper we study transport properties of thermal shot noise, thermal conductance and thermo power through superconductor-ferromagnetic /2DEG junction under the effect of Fermi energy, number of open channels and excitation energy. These results agree with existing experiments. This research is very important for using a model as a high-frequency shot noise detector.

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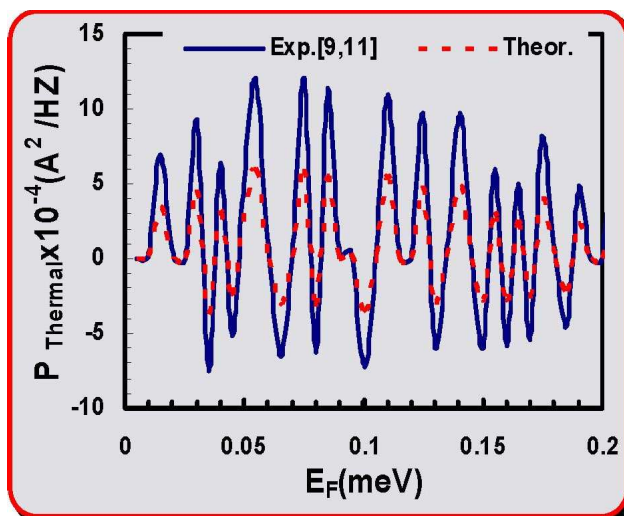


Fig. 1 Thermal shot noise as a function of the Fermi energy.

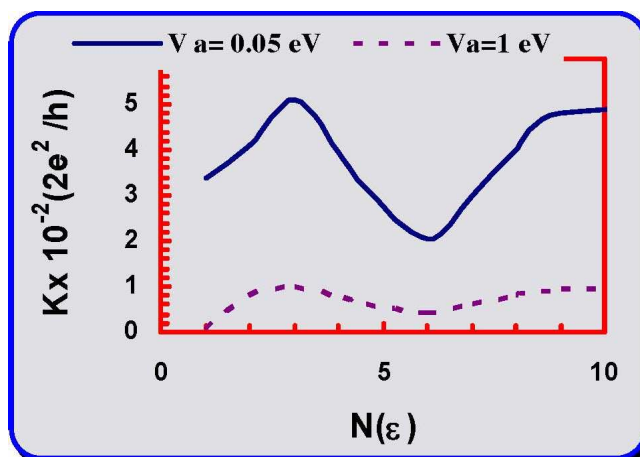


Fig. 2 Thermal conductance as a function of the number of open channels at different applied voltage.

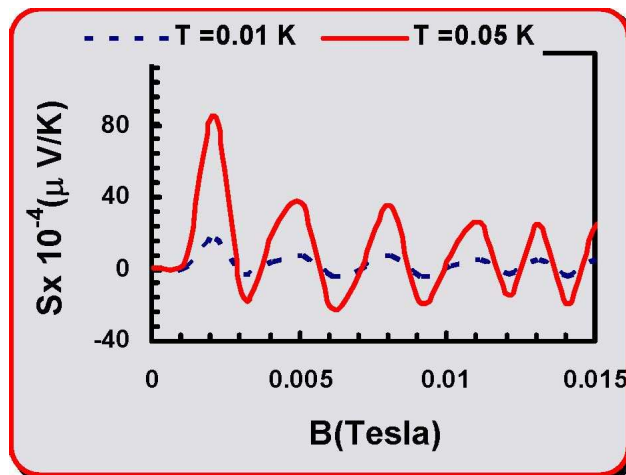


Fig. 3 Thermo power as a function of the magnetic field at different temperatures.

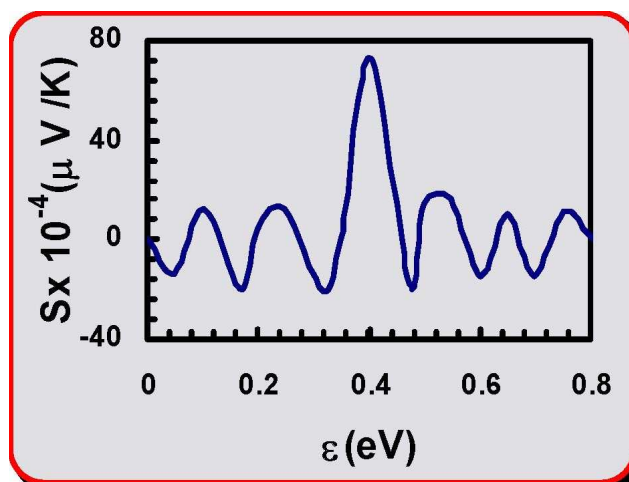


Fig. 4 Thermo power as a function of the excitation energy, ε , a crossover from the quantization behavior of the thermo power to a δ -function behavior.

On the Genuine Bound States of a Non-Relativistic Particle in a Linear Finite Range Potential

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Abstract: We explore the energy spectrum of a non-relativistic particle bound in a linear finite range, attractive potential, envisaged as a quark-confining potential. The intricate transcendental eigenvalue equation is solved numerically to obtain the explicit eigen-energies. The linear potential, which resembles the triangular well, has potential significance in particle physics and exciting applications in electronics

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1. Introduction

A challenging problem in particle physics in recent years is that of quark confinement. It is presently known that mesons are not elementary particles, but are composed of quarks, as are the nucleons.

In literature, several approximation methods are available relating to quark confinement. The *lattice model* [1] suggests that at large distance between quarks, the interaction increases approximately linearly with separation. The *bag model*, where quarks and gluons are confined in a bag, is not suitable for calculating the hadronic properties of heavy quarks or in computing the energy levels of excited states. *String model*, on the other hand, proposes quark-antiquark pair at the ends of an open string and creation of quark-antiquark pair when the string breaks. In recent years, *potential models* [2] are best justified theoretically to describe heavy quarkonia and seem to be most powerful in

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calculating the static properties.

Several authors [3 - 6] have addressed the bound states of various kinds of linear potential. Chiu [7] has examined the quarkonium systems with the regulated linear plus Coulomb potential in momentum space. Deloff [8] has used a semi-spectral Chebyshev method for numerically solving integral equations and has applied the same to the quarkonium bound state problem in momentum space.

Rao and Kagali [9 - 11] have investigated extensively the bound states of both spin and spinless particles in a screened Coulomb potential, having linear behaviour near the origin. In the present work, we propose a finite, short-ranged linearly rising potential, envisaged as a quark-confining potential and explore the non-relativistic bound states.

2. The Schrodinger Equation with the Linear Potential

Several attempts have been made to study the meson spectra using the non-relativistic Schrodinger equation with a linear potential. Intuitively, we construct a simple linear rising, finite range potential of the form [12]

$$V(x) = -\frac{V_0}{a}(a - |x|), \quad (1)$$

in which the well depth V_0 and range $2a$ are positive and adjustable parameters. The linear potential with its boundary regions is illustrated in Fig.1 and owing to its shape, this potential could also be called the triangular potential well.

Obviously in regions I and IV, the particle is free and the allowed solutions of the free particle Schrodinger equation are

$$\psi_1(x) = C_1 e^{\alpha x} \quad -\infty < x \leq -a \quad (2)$$

$$\psi_4(x) = C_6 e^{-\alpha x} \quad a \leq x < \infty, \quad (3)$$

consistent with the requirement $\psi(x)$ vanishes as $|x| \rightarrow \infty$.

Here $\alpha^2 = -\frac{2mE}{\hbar^2}$ is implied. Since $E < 0$ for bound states, α is positive. To discuss the nature of the solution within the potential region, $-a < x < a$, we insert the potential described in Eqn.(1) in the celebrated Schrodinger equation and obtain

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} \left[E + V_0 \left(1 - \frac{|x|}{a} \right) \right] \psi(x) = 0. \quad (4)$$

Setting $\frac{x}{a} = y$ and defining $\bar{E} = \frac{E}{\hbar^2/2ma^2}$ and $\bar{V}_0 = \frac{V_0}{\hbar^2/2ma^2}$ we obtain the dimensionless form of the Schrodinger equation

$$\frac{d^2\psi(y)}{dy^2} + [\bar{E} + \bar{V}_0(1 - y)] \psi(y) = 0, \quad (5)$$

which may further be written as

$$\frac{d^2\psi}{dy^2} - Ay\psi + B\psi = 0. \quad (6)$$

The constants $A = \bar{V}_0$ and $B = \bar{E} + \bar{V}_0$ are also dimensionless. Introducing an auxiliary function

$$w = A^{\frac{1}{3}} \left(y - \frac{B}{A} \right) \quad (7)$$

yields

$$\frac{d^2\psi}{dw^2} - w\psi = 0. \quad (8)$$

The solutions of this differential equation are the well-known Airy functions $Ai(w)$ and $Bi(w)$ [13], having oscillatory and damping nature.

The Eigenvalue Equation

The admissible solutions in the four regions, consistent with physical reality, are

$$\psi_1(x) = C_1 e^{\alpha x} \quad -\infty < x \leq -a \quad (9)$$

$$\psi_2(x) = C_2 Ai(w) + C_3 Bi(w) \quad -a \leq x \leq 0 \quad (10)$$

$$\psi_3(x) = C_4 Ai(-w) + C_5 Bi(-w) \quad 0 \leq x \leq a \quad (11)$$

$$\psi_4(x) = C_6 e^{-\alpha x} \quad a \leq x < \infty \quad (12)$$

where C_1 to C_6 are the normalisation constants. Imposing on the solutions in equations (9) to (12) the requirements that ψ and $\frac{d\psi}{dx}$ be continuous at the origin and also at the potential boundaries ($x = \pm a$) leads to the eigenvalue equation.

At $x = -a$, $\psi_1(x) = \psi_2(x)$ and $\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx}$

This leads to

$$C_1 e^{-\alpha a} = C_2 Ai(w_1) + C_3 Bi(w_1) \quad (13)$$

$$\alpha C_1 e^{-\alpha a} = \frac{A^{\frac{1}{3}}}{a} [C_2 Ai'(w_1) + C_3 Bi'(w_1)] \quad (14)$$

where

$$w_1 = A^{\frac{1}{3}} \left(-1 - \frac{B}{A} \right) \quad (15)$$

On simplification, we obtain

$$\alpha = \frac{A^{\frac{1}{3}}}{a} \left[\frac{P Ai'(w_1) + Bi'(w_1)}{P Ai(w_1) + Bi(w_1)} \right] \quad (16)$$

where $P = \frac{C_2}{C_3}$. Similarly, the continuity condition at $x = 0$ demands

$$\psi_2(x) = \psi_3(x) \quad \text{and} \quad \frac{d\psi_2}{dx} = \frac{d\psi_3}{dx},$$

from which we obtain

$$C_2 Ai(w_0) + C_3 Bi(w_0) = C_4 Ai(-w_0) + C_5 Bi(-w_0) \quad (17)$$

$$C_2 Ai'(w_0) + C_3 Bi'(w_0) = C_4 Ai'(-w_0) + C_5 Bi'(-w_0) \quad (18)$$

with

$$w_0 = A^{\frac{1}{3}} \left(\frac{-B}{A} \right). \quad (19)$$

As before,

$$\frac{P Ai'(w_0) + Bi'(w_0)}{P Ai(w_0) + Bi(w_0)} = \frac{Q Ai'(-w_0) + Bi'(-w_0)}{Q Ai(-w_0) + Bi(-w_0)} \quad (20)$$

where $Q = \frac{C_4}{C_5}$ is another constant. Adapting similar procedure at the boundary $x = +a$, demanding $\psi_3(x) = \psi_4(x)$ and $\frac{d\psi_3}{dx} = \frac{d\psi_4}{dx}$ one would on similar grounds obtain

$$-\alpha = \frac{A^{\frac{1}{3}}}{a} \left[\frac{Q Ai'(-w_2) + Bi'(-w_2)}{Q Ai(-w_2) + Bi(-w_2)} \right] \quad (21)$$

with

$$w_2 = A^{\frac{1}{3}} \left(1 - \frac{B}{A} \right). \quad (22)$$

It is worthwhile mentioning that the arguments of the Airy function w_0 , w_1 , and w_2 are dependent both on the energy as well as the potential and are related by the simple equation

$$w_0 = \frac{w_1 + w_2}{2}. \quad (23)$$

It is straightforward to check that

$$P = \frac{\beta Bi(w_1) - A^{\frac{1}{3}} Bi'(w_1)}{A^{\frac{1}{3}} Ai'(w_1) - \beta Ai(w_1)}, \quad (24)$$

and

$$Q = - \left[\frac{\beta Bi(-w_2) + A^{\frac{1}{3}} Bi'(-w_2)}{\beta Ai(-w_2) + A^{\frac{1}{3}} Ai'(-w_2)} \right], \quad (25)$$

where $\beta = \alpha a$ is implied.

Formally on eliminating P and Q in Eqn.(20) we obtain the eigenvalue equation as

$$\left[\frac{\left\{ \beta Bi(\omega_1) - A^{\frac{1}{3}} Bi'(\omega_1) \right\} Ai'(\omega_0) + \left\{ A^{\frac{1}{3}} Ai'(\omega_1) - \beta Ai(\omega_1) \right\} Bi'(\omega_0)}{\left\{ \beta Bi(\omega_1) - A^{\frac{1}{3}} Bi'(\omega_1) \right\} Ai(\omega_0) + \left\{ A^{\frac{1}{3}} Ai'(\omega_1) - \beta Ai(\omega_1) \right\} Bi(\omega_0)} \right] =$$

$$\left[\frac{\left\{ \beta Bi(-\omega_2) + A^{\frac{1}{3}} Bi'(-\omega_2) \right\} Ai'(-\omega_0) - \left\{ A^{\frac{1}{3}} Ai'(-\omega_2) + \beta Ai(-\omega_2) \right\} Bi'(-\omega_0)}{\left\{ \beta Bi(-\omega_2) + A^{\frac{1}{3}} Bi'(-\omega_2) \right\} Ai(-\omega_0) - \left\{ A^{\frac{1}{3}} Ai'(-\omega_2) + \beta Ai(-\omega_2) \right\} Bi(-\omega_0)} \right] \quad (26)$$

This intricate and fairly complicated transcendental eigenvalue equation involving the Airy function and its derivatives is solved both graphically and numerically using Mathematica[14]. The real roots, which correspond to the eigenenergies, are listed in Table 1 for a typical value of the range parameter(a). Energy (E) and well-depth (V_0) are both expressed in units of $\frac{\hbar^2}{2ma^2}$.

3. Results and Discussion

One of the distinctive characteristics of quantum mechanics, in contrast to classical mechanics, is the existence of bound states corresponding to discrete energy levels. It is well-known in quantum mechanics that bound states exist for all attractive potentials, the exact number depending on the specific form of the potential and the dimensionality of the space.

More specifically, as is seen from the spectrum of energies listed in Table 1, for a finite range of the potential, deeper wells admit excited state energies, consistent with the wisdom of quantum mechanics. Such studies, apart from being pedagogical in nature, are potentially exciting and significant as it is concerned with quark confinement.

Quantum chromodynamics, which governs the quark-antiquark interaction is widely accepted as a good theory of strongly interacting particles. One can explore the hadronic properties by investigating the bound states of quarks. Our investigation concerning the linear potential is seeming interesting and can be regarded as a model to describe the quarkonia.

Further, the linear potential well or in other words, the triangular well has potential applications in electronics. Interestingly, in many semiconductor devices, it is believed that electrons are confined in almost triangular quantum wells [15]. Examples of such devices are Si MOSFETs (Metal Oxide Semiconductor Field Effect Transistors) widely used in digital applications and GaAs/AlGaAs MODFETs (Modulation Doped Field Effect Transistors) used for high speed applications. The bound states of the linear potential is a subject of renewed interest and intensive research and we have extended the study of this naive potential to the relativistic domain, which will be reported shortly.

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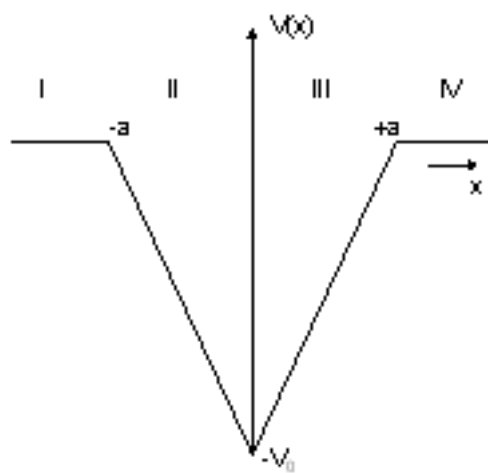


Fig. 1

Table 1**Eigenenergies of a non-relativistic particle in a linear potential**

$$(a = 1\lambda)$$

\bar{V}_0	\bar{E}_0	\bar{E}_1	\bar{E}_2
0.01	- 0.0000976		
0.05	- 0.0022242		
0.10	- 0.0080220		
0.20	- 0.0268461		
0.30	- 0.0519188		
0.40	- 0.0808526		
0.50	- 0.1122377		
0.60	- 0.1451770		
0.70	- 0.1790694		
0.80	- 0.2134967		
0.90	- 0.2480162		
1.0	- 0.2828516		
2.0	- 0.6121732		
3.0	- 0.9072505		
4.0	- 1.2056918		
4.28	- 1.2984370	- 0.0000991	
5.0	- 1.5770587	- 0.1653382	
10.0	- 5.8335771	- 1.3641773	
20.0	-16.4420518	- 2.3126501	
20.62	-17.0607820	- 2.3662130	-0.0000343
25.0	-21.2732190	- 3.6112921	-2.4865189
30.0	-25.7497380	- 9.0650370	-2.9501500
35.0	-29.8836000	-15.0604510	-3.2815680
40.0	-33.7181130	-21.1744464	-3.5904940

Exact Non-traveling Wave and Coefficient Function Solutions for (2+1)-Dimensional Dispersive Long Wave Equations

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Abstract: In this paper, a new generalized F-expansion method is proposed to seek exact solutions of nonlinear evolution equations. With the aid of symbolic computation, we choose the (2+1)-dimensional dispersive long wave equations to illustrate the validity and advantages of the proposed method. As a result, many new and more general exact non-traveling wave and coefficient function solutions are obtained including single and combined non-degenerate Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions, each of which contains two arbitrary functions. The arbitrary functions provide us with enough freedom to discuss the behaviors of solutions. As an illustrative example, new spatial structures of two solutions are shown. Compared with the most existing F-expansion methods, the new generalized F-expansion method gives not only more general exact solutions but also new formal exact solutions. The proposed method can also be applied to other nonlinear evolution equations in mathematical physics.

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Keywords: F-expansion Method, Jacobi Elliptic Function Solutions, Non-traveling Wave and Coefficient Function Solutions, Soliton-Like Solutions, Trigonometric Function Solutions
PACS (2008): 02.30.Jr; 04.20.Jb; 05.45.Yv

1. Introduction

Since the soliton phenomena were first observed by Scott Russell ¹ in 1834 and the KdV equation was solved by the inverse scattering method by Gardner et al. ² in 1967, finding exact solutions of nonlinear evolution equations (NLEEs) has become one of the most exciting and extremely active areas of research investigation. Many effective methods

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for obtaining exact solutions of NLEEs have been presented such as Hirota's bilinear method,³ Bäcklund transformation,⁴ Painlevé expansion,⁵ sine-cosine method,⁶ homogenous balance method,⁷ homotopy perturbation method,^{8–10} variational method,^{11–14} asymptotic methods,^{15,16} non-perturbative methods,¹⁷ Adomian Pade approximation,¹⁸ tanh-function method,^{19–32} algebraic method,^{33–38} auxiliary equation method,^{39–45} Exp-function method,^{46–55} and so on.

Recently, many exact solutions expressed by various Jacobi elliptic functions of many NLEEs in mathematical physics have been obtained by Jacobi elliptic function expansion method^{56–58} and F-expansion method^{59–61} which can be thought of as a generalization of Jacobi elliptic function expansion method. F-expansion method was later extended in different manners^{62–70}. Very recently, we proposed a generalized F-expansion method^{71–73} to obtain more general exact solutions which contain not only the results obtained by using the methods^{59–61,67–70} but also a series of new and more general exact solutions, in which the restriction on $\xi(x, y, z, \dots, t)$ as merely a linear function of x, y, z, \dots, t and the restriction on the coefficients being constants are removed.

The present paper is motivated by the desire to further improve and develop our method^{71–73} by taking the (2+1)-dimensional dispersive long wave (DLW) equations as an example. As a result, many new and more general exact non-traveling wave and coefficient function solutions are obtained including single and combined non-degenerate Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions, each of which contains two arbitrary functions.

The rest of this paper is organized as follows. In Section 2, we give the description of the new generalized F-expansion method. In Section 3, we use this method to obtain more general exact solutions of the (2+1)-dimensional DLW equations. In Section 4, some conclusions are given.

2. Description of the New Generalized F-expansion Method

For a given NLEE with independent variables $X = (x, y, z, \dots, t)$ and dependent variable u :

$$F(u, u_t, u_x, u_y, u_z, \dots, u_{xt}, u_{yt}, u_{zt}, \dots, u_{tt}, u_{xx}, u_{yy}, u_{zz}, \dots) = 0, \quad (1)$$

we seek its solutions in the new and more general form^{71–73}:

$$u = a_0 + \sum_{i=1}^n \{a_i F^i(\xi) + b_i F^{-i}(\xi) + c_i F^{i-1}(\xi) F'(\xi) + d_i F^{-i}(\xi) F'(\xi)\}, \quad (2)$$

where $a_0 = a_0(X)$, $a_i = a_i(X)$, $b_i = b_i(X)$, $c_i = c_i(X)$, $d_i = d_i(X)$ ($i = 1, 2, \dots, n$) and $\xi = \xi(X)$ are undetermined functions, $F(\xi)$ and $F'(\xi)$ in (2) satisfy

$$F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R, \quad (3)$$

and hence holds for $F(\xi)$ and $F'(\xi)$

$$\left\{ \begin{array}{l} F''(\xi) = 2PF^3(\xi) + QF(\xi), \\ F'''(\xi) = (6PF^2(\xi) + Q)F'(\xi), \\ F^{(4)}(\xi) = 24P^2F^5(\xi) + 20PQF^3(\xi) + (Q^2 + 12PR)F(\xi), \\ F^{(5)}(\xi) = (120P^2F^4(\xi) + 60PQF^2(\xi) + Q^2 + 12PR)F'(\xi), \\ \dots \end{array} \right. \quad (4)$$

where P , Q and R are all parameters, the prime denotes $d/d\xi$. Given different values of P , Q and R , the different Jacobi elliptic function solutions $F(\xi)$ can be obtained from Eq. (3) (see Appendix A for its various Jacobi elliptic function solutions including some new ones which have not been listed in the method ^{71–73}). To determine u explicitly, we take the following four steps:

Step 1. Determine the integer n by balancing the highest order nonlinear term(s) and the highest order partial derivative term of u in Eq. (1).

Step 2. Substitute (2) along with (3) and (4) into Eq. (1) and collect all coefficients of $F^i(\xi)F^j(\xi)$ ($i = 0, 1; j = 0, \pm 1, \pm 2, \dots$), then set each coefficient to zero to derive a set of over-determined partial differential equations for a_0, b_i, c_i, d_i ($i = 1, 2, \dots, n$) and ξ .

Step 3. Solve the system of over-determined partial differential equations obtained in Step 2 for a_0, a_i, b_i, c_i, d_i and ξ by use of Mathematica.

Step 4. From Appendix A select P, Q, R and $F(\xi)$, then substitute them along with a_0, a_i, b_i, c_i, d_i and ξ obtained in Step 3 into (2) to obtain Jacobi elliptic function solutions of Eq. (1) (see Appendix B for $F'(\xi)$), from which soliton-like solutions and trigonometric function solutions can be obtained when $m \rightarrow 1$ and $m \rightarrow 0$.

Remark 1. In order to determine the explicit solutions of the partial differential equations derived in Step 2, we may choose special forms of a_0, a_i, b_i, c_i, d_i and ξ as we do in Section 3.

3. Exact Solutions of the (2+1)-dimensional DLW Equations

$$u_{yt} + v_{xx} + \frac{1}{2}(u^2)_{xy} = 0, \quad (5)$$

$$v_t + (uv + u + u_{xy})_x = 0. \quad (6)$$

This system was firstly obtained by Boiti et al. ⁷⁴ as a compatibility condition for a “weak” Lax pair. Paquin and Winternitz ⁷⁵ showed that the symmetry algebra of it is infinite-dimensional and has a Kac–Moody–Virasoro structure. Lou ⁷⁶ gave the more general symmetry algebra, W_∞ symmetry algebra. Lou ⁷⁷ found nine types of two-dimensional similarity reductions and thirteen types of ordinary differential equation reductions. Though the model equation system is Lax or IST integrable, it does not pass

the Painleve test.⁷⁸ Recently, He¹⁴ established a variational model by the semi-inverse method. We⁶⁸ obtained periodic wave solutions by using the extended F-expansion method. Xia and Chen⁷⁹ obtained dromion solutions by means of the homogenous balance method. Zhao and Bai⁸⁰ obtained non-traveling wave solutions via the extended mapping transformation method.

According to Step 1, we get $n_1 = 1$ for u and $n_2 = 2$ for v . In order to search for explicit solutions, we suppose that the solutions of Eqs. (5) and (6) can be expressed as:

$$u = a_0 + a_1F(\xi) + b_1F^{-1}(\xi) + c_1F'(\xi)F(\xi) + d_1F'(\xi)F^{-1}(\xi), \quad (7)$$

$$v = A_0 + A_1F(\xi) + A_2F^2(\xi) + B_1F^{-1}(\xi) + B_2F^{-2}(\xi) + C_1F'(\xi) \\ + C_2F'(\xi)F(\xi) + D_1F'(\xi)F^{-1}(\xi) + D_2F'(\xi)F^{-2}(\xi), \quad (8)$$

where $a_0 = a_0(y, t)$, $a_1 = a_1(y, t)$, $b_1 = b_1(y, t)$, $c_1 = c_1(y, t)$, $d_1 = d_1(y, t)$, $A_0 = A_0(y, t)$, $A_1 = A_1(y, t)$, $A_2 = A_2(y, t)$, $B_1 = B_1(y, t)$, $B_2 = B_2(y, t)$, $C_1 = C_1(y, t)$, $C_2 = C_2(y, t)$, $D_1 = D_1(y, t)$, $D_2 = D_2(y, t)$, $\xi = kx + \eta$, $\eta = \eta(y, t)$, k is a nonzero constant.

With the aid of Mathematica, substituting (7) and (8) along with (3) and (4) into Eqs. (5) and (6), the left-hand sides of Eqs. (5) and (6) are converted into two polynomials of $F^i(\xi)F^j(\xi)$ ($i = 0, 1; j = 0, \pm 1, \pm 2, \dots$), then setting each coefficient to zero, we get a set of over-determined partial differential equations for $a_0, a_1, b_1, c_1, d_1, A_0, A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ and η . Solving these over-determined partial differential equations by use of Mathematica, we get the following results:

Case 1

$$a_1 = \pm k\sqrt{P}, \quad d_1 = \pm k, \quad C_1 = \pm k\sqrt{P}f'(y), \quad D_2 = \pm k\sqrt{R}f'(y),$$

$$a_0 = -\frac{g'(t)}{k}, \quad b_1 = \pm k\sqrt{R}, \quad A_2 = -kPf'(y), \quad B_2 = -kRf'(y),$$

$$A_0 = -1 \pm 2k\sqrt{PR}f'(y), \quad c_1 = A_1 = B_1 = C_2 = D_1 = 0, \quad \eta = f(y) + g(t),$$

where $f(y)$ and $g(t)$ are arbitrary functions of y and t respectively, $f'(y) = df(y)/dy$, $g'(t) = dg(t)/dt$. The sign “ \pm ” means that all possible combinations of “+” and “-” can be taken in a_1, d_1, C_1 and D_2 . If the same sign is used in d_1 and D_2 , then “-” must be used in b_1 , furthermore the different signs must be used in a_1 and A_0 . If different signs are used in d_1 and D_2 , then “+” must be used in b_1 , furthermore the same sign must be used in a_1 and A_0 . Hereafter, the sign “ \pm ” always stands for this meaning in the similar circumstances.

Case 2

$$a_0 = -\frac{g'(t)}{k}, \quad d_1 = \pm 2k, \quad A_2 = -2kPf'(y), \quad B_2 = -2kRf'(y),$$

$$A_0 = -1, \quad a_1 = b_1 = c_1 = A_1 = B_1 = C_1 = C_2 = D_1 = D_2 = 0, \quad \eta = f(y) + g(t),$$

where $f(y)$ and $g(t)$ are arbitrary functions of y and t respectively, $f'(y) = df(y)/dy$, $g'(t) = dg(t)/dt$.

Case 3

$$a_0 = -\frac{g'(t)}{k}, \quad a_1 = \pm 2k\sqrt{P}, \quad b_1 = \pm 2k\sqrt{R}, \quad A_2 = -2kPf'(y),$$

$$A_0 = -1 - kQf'(y) \pm 2k\sqrt{PR}f'(y), \quad \eta = f(y) + g(t),$$

$$B_2 = -2kPf'(y), \quad c_1 = d_1 = A_1 = B_1 = C_1 = C_2 = D_1 = D_2 = 0,$$

where $f(y)$ and $g(t)$ are arbitrary functions of y and t respectively, $f'(y) = df(y)/dy$, $g'(t) = dg(t)/dt$. The sign “ \pm ” means that all possible combinations of “+” and “-” can be taken in a_1 and b_1 . If the same sign is used in a_1 and b_1 , then “+” must be used in A_0 . If different signs are used in a_1 and b_1 , then “-” must be used in A_0 . Hereafter, the sign “ \pm ” always stands for this meaning in the similar circumstances.

Case 4

$$a_0 = -\frac{g'(t)}{k}, \quad a_1 = \pm 2k\sqrt{P}, \quad A_2 = -2kPf'(y), \quad \eta = f(y) + g(t),$$

$$A_0 = -1 - kQf'(y), \quad b_1 = c_1 = d_1 = A_1 = B_1 = B_2 = C_1 = C_2 = D_1 = D_2 = 0,$$

where $f(y)$ and $g(t)$ are arbitrary functions of y and t respectively, $f'(y) = df(y)/dy$, $g'(t) = dg(t)/dt$.

Substituting Cases 1–4 into (7) and (8) respectively, we have four kinds of formal solutions of Eqs. (5) and (6):

$$u = -\frac{g'(t)}{k} \pm k\sqrt{P}F(\xi) \pm k\sqrt{R}F^{-1}(\xi) \pm kF'(\xi)F^{-1}(\xi), \quad (9)$$

$$v = -1 \pm 2k\sqrt{PR}f'(y) - kPf'(y)F^2(\xi) - kRf'(y)F^{-2}(\xi) \pm k\sqrt{P}f'(y)F'(\xi) \\ \pm k\sqrt{R}f'(y)F'(\xi)F^{-2}(\xi), \quad (10)$$

where $\xi = kx + f(y) + g(t)$.

$$u = -\frac{g'(t)}{k} \pm 2kF'(\xi)F^{-1}(\xi), \quad (11)$$

$$v = -1 - 2kPf'(y)F^2(\xi) - 2kRf'(y)F^{-2}(\xi), \quad (12)$$

where $\xi = kx + f(y) + g(t)$.

$$u = -\frac{g'(t)}{k} \pm 2k\sqrt{P}F(\xi) \pm 2k\sqrt{R}F^{-1}(\xi), \quad (13)$$

$$v = -1 - kQf'(y) \pm 2k\sqrt{PR}f'(y) - 2kPf'(y)F^2(\xi) - 2kPf'(y)F^{-2}(\xi), \quad (14)$$

where $\xi = kx + f(y) + g(t)$.

$$u = -\frac{g'(t)}{k} \pm 2k\sqrt{P}F(\xi), \quad (15)$$

$$v = -1 - kQf'(y) - 2kPf'(y)F^2(\xi), \quad (16)$$

where $\xi = kx + f(y) + g(t)$.

From (9)–(16) and Appendices A, B and C, we can obtain Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions of Eqs. (5) and (6). For example, from Appendix A, selecting $F(\xi) = \operatorname{sn}\xi$, $P = m^2$, $Q = -(1 + m^2)$, $R = 1$, inserting them into (19) and (20) and using Appendix B, we can obtain combined non-degenerate Jacobi elliptic function solutions of Eqs. (5) and (6):

$$u = -\frac{g'(t)}{k} \pm kmsn\xi \pm kns\xi \pm kcn\xi ds\xi, \quad (17)$$

$$v = -1 \pm 2kmf'(y) - km^2f'(y)\operatorname{sn}^2\xi - kf'(y)\operatorname{ns}^2\xi \\ \pm kmf'(y)\operatorname{cn}\xi \operatorname{dn}\xi \pm kf'(y)\operatorname{cs}\xi ds\xi, \quad (18)$$

where $\xi = kx + f(y) + g(t)$.

When $m \rightarrow 1$, from (17) and (18) we obtain soliton-like solutions of Eqs. (5) and (6):

$$u = -\frac{g'(t)}{k} \pm ktanh\xi \pm kcoth\xi \pm k\operatorname{sech}\xi \operatorname{csch}\xi, \quad (19)$$

$$v = -1 \pm 2kf'(y) - kf'(y)\operatorname{tanh}^2\xi - kf'(y)\operatorname{coth}^2\xi \\ \pm kf'(y)\operatorname{sech}^2\xi \pm kf'(y)\operatorname{csch}^2\xi, \quad (20)$$

where $\xi = kx + f(y) + g(t)$.

Selecting $F(\xi) = \operatorname{ns}\xi$, $P = 1$, $Q = -(1 + m^2)$, $R = m^2$, we obtain

$$u = -\frac{g'(t)}{k} \pm kns\xi \pm kmsn\xi \mp kcn\xi ds\xi, \quad (21)$$

$$v = -1 \pm 2kmf'(y) - kf'(y)\operatorname{ns}^2\xi - km^2f'(y)\operatorname{sn}^2\xi \\ \mp kf'(y)\operatorname{cs}\xi ds\xi \mp kmf'(y)\operatorname{cn}\xi \operatorname{dn}\xi, \quad (22)$$

where $\xi = kx + f(y) + g(t)$.

When $m \rightarrow 0$, from (21) and (22) we obtain trigonometric function solutions of Eqs. (5) and (6):

$$u = -\frac{g'(t)}{k} \pm kcsc\xi \mp kcot\xi, \quad (23)$$

$$v = -1 - kf'(y)\operatorname{csc}^2\xi \mp kf'(y)\operatorname{cot}\xi \operatorname{csc}\xi, \quad (24)$$

where $\xi = kx + f(y) + g(t)$.

Selecting $F(\xi) = \operatorname{sn}\xi \pm \operatorname{icn}\xi$, $P = \frac{m^2}{4}$, $Q = \frac{m^2-2}{2}$, $R = \frac{m^2}{4}$, we obtain

$$u = -\frac{g'(t)}{k} \pm \frac{km}{2}(\operatorname{sn}\xi \pm \operatorname{icn}\xi) \pm \frac{km}{2(\operatorname{sn}\xi \pm \operatorname{icn}\xi)} \pm k(\mp \operatorname{idn}\xi), \quad (25)$$

$$v = -1 \pm \frac{km^2}{2}f'(y) - \frac{km^2}{4}f'(y)(\operatorname{sn}\xi \pm \operatorname{icn}\xi)^2 - \frac{km^2f'(y)}{4(\operatorname{sn}\xi \pm \operatorname{icn}\xi)^2} \\ \pm \frac{km}{2}f'(y)(\operatorname{cn}\xi \operatorname{dn}\xi \mp \operatorname{isn}\xi \operatorname{dn}\xi) \pm \frac{kmf'(y)(\mp \operatorname{dn}\xi)}{2(\operatorname{sn}\xi \pm \operatorname{icn}\xi)}, \quad (26)$$

where $\xi = kx + f(y) + g(t)$.

When $m \rightarrow 1$, from (25) and (26) we obtain soliton-like solutions of Eqs. (5) and (6):

$$u = -\frac{g'(t)}{k} \pm \frac{k}{2}(\tanh\xi \pm \operatorname{sech}\xi) \pm \frac{k}{2(\tanh\xi \pm \operatorname{sech}\xi)} \pm k(\pm \operatorname{sech}\xi), \quad (27)$$

$$v = -1 \pm \frac{k}{2}f'(y) - \frac{k}{4}f'(y)(\tanh\xi \pm \operatorname{sech}\xi)^2 - \frac{kf'(y)}{4(\tanh\xi \pm \operatorname{sech}\xi)^2} \\ \pm \frac{k}{2}f'(y)(\operatorname{sech}^2\xi \mp i \tanh\xi \operatorname{sech}\xi) \pm \frac{kf'(y)(\pm \operatorname{sech}\xi)}{2(\tanh\xi \pm \operatorname{sech}\xi)}, \quad (28)$$

where $\xi = kx + f(y) + g(t)$.

Selecting $F(\xi) = \frac{\operatorname{sn}\xi}{1 \pm \operatorname{cn}\xi}$, $P = \frac{1}{4}$, $Q = \frac{1-2m^2}{2}$, $R = \frac{1}{4}$, we obtain

$$u = -\frac{g'(t)}{k} \pm \frac{k \operatorname{sn}\xi}{2(1 \pm \operatorname{cn}\xi)} \pm \frac{k(1 \pm \operatorname{cn}\xi)}{2 \operatorname{sn}\xi} \pm \frac{k[\operatorname{cn}\xi \operatorname{dn}\xi(1 \pm \operatorname{cn}\xi) \pm \operatorname{sn}^2\xi \operatorname{dn}\xi]}{\operatorname{sn}\xi(1 \pm \operatorname{cn}\xi)}, \quad (29)$$

$$v = -1 \pm \frac{kf'(y)}{2} - \frac{kf'(y)\operatorname{sn}^2\xi}{4(1 \pm \operatorname{cn}\xi)^2} - \frac{kf'(y)(1 \pm \operatorname{cn}\xi)^2}{4\operatorname{sn}^2\xi} \\ \pm \frac{kf'(y)[\operatorname{cn}\xi \operatorname{dn}\xi(1 \pm \operatorname{cn}\xi) \pm \operatorname{sn}^2\xi \operatorname{dn}\xi]}{2(1 \pm \operatorname{cn}\xi)^2} \pm \frac{kf'(y)[\operatorname{cn}\xi \operatorname{dn}\xi(1 \pm \operatorname{cn}\xi) \pm \operatorname{sn}^2\xi \operatorname{dn}\xi]}{2\operatorname{sn}^2\xi}, \quad (30)$$

where $\xi = kx + f(y) + g(t)$.

When $m \rightarrow 1$, from (29) and (30) we obtain soliton-like solutions of Eqs. (5) and (6):

$$u = -\frac{g'(t)}{k} \pm \frac{k \tanh\xi}{2(1 \pm \operatorname{sech}\xi)} \pm \frac{k(1 \pm \operatorname{sech}\xi)}{2 \tanh\xi} \pm \frac{k[\operatorname{sech}^2\xi(1 \pm \operatorname{sech}\xi) \pm \tanh^2\xi \operatorname{sech}\xi]}{\tanh\xi(1 \pm \operatorname{sech}\xi)}, \quad (31)$$

$$v = -1 \pm \frac{kf'(y)}{2} - \frac{kf'(y)\tanh^2\xi}{4(1 \pm \operatorname{sech}\xi)^2} - \frac{kf'(y)(1 \pm \operatorname{sech}\xi)^2}{4 \tanh^2\xi} \\ \pm \frac{kf'(y)[\operatorname{sech}^2\xi(1 \pm \operatorname{sech}\xi) \pm \tanh^2\xi \operatorname{sech}\xi]}{2(1 \pm \operatorname{sech}\xi)^2} \\ \pm \frac{kf'(y)[\operatorname{sech}^2\xi(1 \pm \operatorname{sech}\xi) \pm \tanh^2\xi \operatorname{sech}\xi]}{2 \tanh^2\xi}, \quad (32)$$

where $\xi = kx + f(y) + g(t)$.

When $m \rightarrow 0$, from (29) and (30) we obtain trigonometric function solutions of Eqs. (5) and (6):

$$u = -\frac{g'(t)}{k} \pm \frac{k \sin\xi}{2(1 \pm \cos\xi)} \pm \frac{k(1 \pm \cos\xi)}{2 \sin\xi} \pm \frac{k[\cos\xi(1 \pm \cos\xi) \pm \sin^2\xi]}{\sin\xi(1 \pm \cos\xi)}, \quad (33)$$

$$v = -1 \pm \frac{kf'(y)}{2} - \frac{kf'(y)\sin^2\xi}{4(1 \pm \cos\xi)^2} - \frac{kf'(y)(1 \pm \cos\xi)^2}{4 \sin^2\xi} \\ \pm \frac{kf'(y)[\cos\xi(1 \pm \cos\xi) \pm \sin^2\xi \cos\xi]}{2(1 \pm \cos\xi)^2} \pm \frac{kf'(y)[\cos\xi(1 \pm \cos\xi) \pm \sin^2\xi \cos\xi]}{2 \sin^2\xi}, \quad (34)$$

where $\xi = kx + f(y) + g(t)$.

Selecting $F(\xi) = \frac{cn\xi}{1 \pm sn\xi}$, $P = \frac{1-m^2}{4}$, $Q = \frac{1+m^2}{2}$, $R = \frac{1-m^2}{4}$, we obtain

$$u = -\frac{g'(t)}{k} \pm \frac{k\sqrt{1-m^2}cn\xi}{2(1 \pm sn\xi)} \pm \frac{k\sqrt{1-m^2}(1 \pm sn\xi)}{2cn\xi} \mp \frac{k[sn\xi dn\xi(1 \pm sn\xi) \pm cn^2\xi dn\xi]}{cn\xi(1 \pm sn\xi)}, \quad (35)$$

$$v = -1 \pm \frac{k(1-m^2)f'(y)}{2} - \frac{k(1-m^2)f'(y)cn^2\xi}{4(1 \pm sn\xi)^2} - \frac{k(1-m^2)f'(y)(1 \pm sn\xi)^2}{4cn^2\xi} \mp \frac{k\sqrt{1-m^2}f'(y)[sn\xi dn\xi(1 \pm sn\xi) \pm cn^2\xi dn\xi]}{2(1 \pm sn\xi)^2} \mp \frac{k\sqrt{1-m^2}f'(y)[sn\xi dn\xi(1 \pm sn\xi) \pm cn^2\xi dn\xi]}{2cn^2\xi}, \quad (36)$$

where $\xi = kx + f(y) + g(t)$.

When $m \rightarrow 0$, from (35) and (36) we obtain trigonometric function solutions of Eqs. (5) and (6):

$$u = -\frac{g'(t)}{k} \pm \frac{k\cos\xi}{2(1 \pm \sin\xi)} \pm \frac{k(1 \pm \sin\xi)}{2\cos\xi} \mp \frac{k[\sin\xi(1 \pm \sin\xi) \pm \cos^2\xi]}{\cos\xi(1 \pm \sin\xi)}, \quad (37)$$

$$v = -1 \pm \frac{kf'(y)}{2} - \frac{kf'(y)\cos^2\xi}{4(1 \pm \sin\xi)^2} - \frac{kf'(y)(1 \pm \sin\xi)^2}{4\cos^2\xi} \mp \frac{kf'(y)[\sin\xi(1 \pm \sin\xi) \pm \cos^2\xi]}{2(1 \pm \sin\xi)^2} \mp \frac{kf'(y)[\sin\xi(1 \pm \sin\xi) \pm \cos^2\xi]}{2\cos^2\xi}, \quad (38)$$

where $\xi = kx + f(y) + g(t)$.

From (9)–(16) and Appendices A, B and C, we can also obtain other Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions of Eqs. (5) and (6), we omit them here for simplicity.

The obtained solutions and the omitted ones contain two arbitrary functions which can make us discuss the behaviors of solutions and also provide us with enough freedom to construct solutions that may be related to real physical problems. As an illustrative example, we choose $f(y) = \text{sn}(y|0.3)$, $g(t) = \text{cn}(t|0.3)$, $k = 1$, $t = 0$, $m = 0.5$, and select the sign “ \pm ” in the way of $a_1(+)$, $b_1(+)$, $d_1(+)$, $C_1(-)$, $D_2(-)$, $A_0(+)$, then new spatial structures of solutions (21) and (22) are shown in Figs. 1 and 2 respectively.

Remark 2. All the results reported in this paper have been checked with Mathematica. All solutions presented above can not be obtained by the method.^{59–70} Solutions (17)–(28) can be found by the method.^{71–73} However, solutions (29)–(38) can not be obtained by the method.^{71–73} It shows that the proposed method is more powerful in searching for exact solutions of NLEEs in mathematical physics.

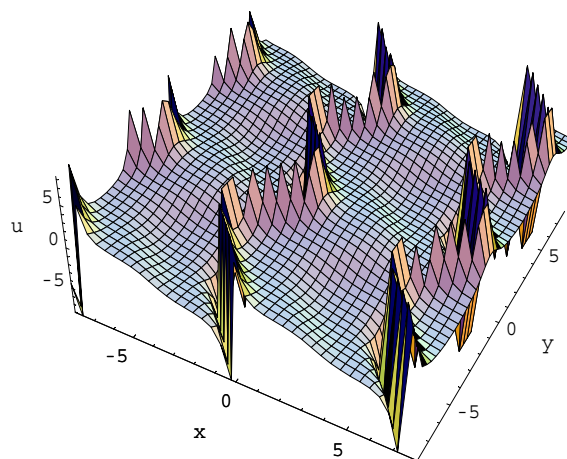


Fig. 1. Spatial structure of solution (21).

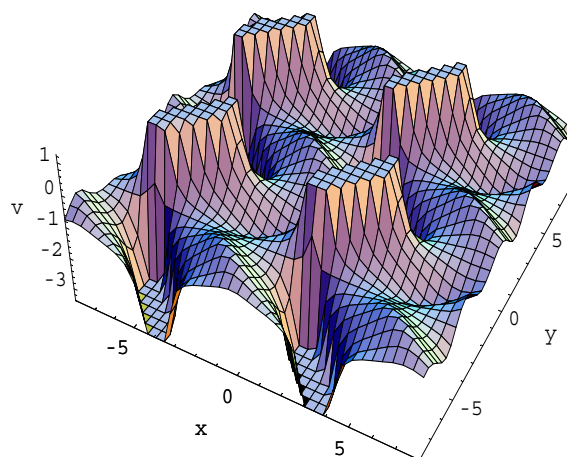


Fig. 2. Spatial structure of solution (22).

Conclusion

In this paper, we have proposed a new generalized F-expansion method to improve and develop the most existing F-expansion method. Applying the proposed method to the (2+1)-dimensional DLW equations, we have successfully obtained new and more general exact non-traveling wave and coefficient function solutions including combined non-degenerate Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions, each of which contains two arbitrary functions. The arbitrary functions in the obtained solutions imply that these solutions have rich local structures. To the best of our knowledge, these solutions have not been reported in literature. It may be important to explain some physical phenomena. Compared with the most existing F-expansion methods [59–73], the new generalized F-expansion method gives both more general exact solutions and new formal solutions. More importantly, the method with the help of symbolic computation provides a powerful mathematical tool for solving a great many NLEEs in mathematical physics, such as the (3+1)-dimensional Kadomtsev–Petviashvili (KP) equation, the (2+1)-dimensional Broer–Kaup–Kupershmidt (BKK) equations, breaking soliton (BS) equations, Nizhnik–Novikov–VesselovIt (NNV) equations and so on. Its

applications are worth further studying.

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Appendix A

Relations between values of (P, Q, R) and corresponding $F(\xi)$ in Eq. (3):

P	Q	R	$F(\xi)$
m^2	$-(1+m^2)$	1	$\text{sn}\xi, \quad \text{cd}\xi = \frac{\text{cn}\xi}{\text{dn}\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$\text{cn}\xi$
-1	$2-m^2$	m^2-1	$\text{dn}\xi$
1	$-(1+m^2)$	m^2	$\text{ns}\xi = (\text{sn}\xi)^{-1}, \quad \text{dc}\xi = \frac{\text{dn}\xi}{\text{cn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$\text{nc}\xi = (\text{cn}\xi)^{-1}$
m^2-1	$2-m^2$	-1	$\text{nd}\xi = (\text{dn}\xi)^{-1}$
$1-m^2$	$2-m^2$	1	$\text{sc}\xi = \frac{\text{sn}\xi}{\text{cn}\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\text{sd}\xi = \frac{\text{sn}\xi}{\text{dn}\xi}$
1	$2-m^2$	$1-m^2$	$\text{cs}\xi = \frac{\text{cn}\xi}{\text{sn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\text{ds}\xi = \frac{\text{dn}\xi}{\text{sn}\xi}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\text{ns}\xi \pm \text{cs}\xi, \quad \frac{\text{cn}\xi}{\sqrt{1-m^2\text{sn}\xi \pm \text{dn}\xi}}$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\text{nc}\xi \pm \text{sc}\xi$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\text{ns}\xi \pm \text{ds}\xi$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\text{sn}\xi \pm \text{icn}\xi, \quad \frac{\text{dn}\xi}{\sqrt{1-m^2\text{sn}\xi \pm \text{cn}\xi}}$
$-\frac{1}{4}$	$\frac{1+m^2}{2}$	$-\frac{(1-m^2)^2}{4}$	$m\text{cn}\xi \pm \text{dn}\xi$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$m\text{sn}\xi \pm \text{idn}\xi, \quad \frac{\text{sn}\xi}{1 \pm \text{cn}\xi}$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{\text{sn}\xi}{1 \pm \text{dn}\xi}$
$\frac{m^2-1}{4}$	$\frac{1+m^2}{2}$	$\frac{m^2-1}{4}$	$\frac{\text{dn}\xi}{1 \pm m\text{sn}\xi}$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\frac{\text{cn}\xi}{1 \pm \text{sn}\xi}$
$\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1}{4}$	$\frac{\text{sn}\xi}{\text{cn}\xi \pm \text{dn}\xi}$
$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{\text{cn}\xi}{\sqrt{1-m^2 \pm \text{dn}\xi}}$

Appendix B

Derivatives of Jacobi elliptic functions

$$\text{sn}'\xi = \text{cn}\xi \text{dn}\xi, \quad \text{cd}'\xi = -(1-m^2)\text{sd}\xi \text{nd}\xi, \quad \text{cn}'\xi = -\text{sn}\xi \text{dn}\xi, \quad \text{dn}'\xi = -m^2 \text{sn}\xi \text{cn}\xi,$$

$$\text{ns}'\xi = -\text{cs}\xi \text{ds}\xi, \quad \text{dc}'\xi = (1-m^2)\text{nc}\xi \text{sc}\xi, \quad \text{nc}'\xi = \text{sc}\xi \text{dc}\xi, \quad \text{nd}'\xi = m^2 \text{cd}\xi \text{sd}\xi,$$

$$\text{sc}'\xi = \text{dc}\xi \text{nc}\xi, \quad \text{cs}'\xi = -\text{ns}\xi \text{ds}\xi, \quad \text{ds}'\xi = -\text{cs}\xi \text{ns}\xi, \quad \text{sd}'\xi = \text{nd}\xi \text{cd}\xi.$$

Appendix C

Jacobi elliptic functions degenerate into hyperbolic functions when $m \rightarrow 1$:

$$\begin{aligned} \operatorname{sn}\xi &\rightarrow \tanh\xi, & \operatorname{cn}\xi &\rightarrow \operatorname{sech}\xi, & \operatorname{dn}\xi &\rightarrow \operatorname{sech}\xi, & \operatorname{sc}\xi &\rightarrow \sinh\xi, \\ \operatorname{sd}\xi &\rightarrow \sinh\xi, & \operatorname{cd}\xi &\rightarrow 1, & \operatorname{ns}\xi &\rightarrow \coth\xi, & \operatorname{nc}\xi &\rightarrow \cosh\xi, \\ \operatorname{nd}\xi &\rightarrow \cosh\xi, & \operatorname{cs}\xi &\rightarrow \operatorname{csch}\xi, & \operatorname{ds}\xi &\rightarrow \operatorname{csch}\xi, & \operatorname{dc}\xi &\rightarrow 1. \end{aligned}$$

Jacobi elliptic functions degenerate into trigonometric functions when $m \rightarrow 0$:

$$\begin{aligned} \operatorname{sn}\xi &\rightarrow \sin\xi, & \operatorname{cn}\xi &\rightarrow \cos\xi, & \operatorname{dn}\xi &\rightarrow 1, & \operatorname{sc}\xi &\rightarrow \tan\xi, & \operatorname{sd}\xi &\rightarrow \sin\xi, & \operatorname{cd}\xi &\rightarrow \cos\xi, \\ \operatorname{ns}\xi &\rightarrow \csc\xi, & \operatorname{nc}\xi &\rightarrow \sec\xi, & \operatorname{nd}\xi &\rightarrow 1, & \operatorname{cs}\xi &\rightarrow \cot\xi, & \operatorname{ds}\xi &\rightarrow \csc\xi, & \operatorname{dc}\xi &\rightarrow \sec\xi. \end{aligned}$$

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