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Editors

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Macroscopically-Discrete Quantum Cosmology

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Abstract: Milne’s Lorentz-group-based cosmological spacetime and Gelfand-Naimark unitary Lorentz-group representation through transformation of Hilbert-space vectors combine to define a Fock space of ‘cosmological preons’—quantum-theoretic universe constituents. Lorentz invariance of ‘age’–*global time*– accompanies Milne’s ‘cosmological principle’ that attributes to each spatial location a Lorentz frame. We divide Milne spacetime—the interior of a forward lightcone— into ‘slices’ of fixed *macroscopic* width in age, with ‘cosmological rays’ defined on (hyperbolic) *slice boundaries*. The Fock space of our macroscopically-discrete quantum cosmology (DQC) is defined *only* at these *exceptional* universe ages. Self-adjoint-operator expectations over the ray at any spacetime-slice boundary prescribe throughout the following slice a non-fluctuating continuous ‘classical reality’ represented by D’Alembertians, of classical electromagnetic (vector) and gravitational (tensor) potentials, that are current densities of locally-conserved electric charge and energy-momentum. The ray at the upper boundary of a slice is determined from the lower-boundary ray by *branched* slice-traversing *stepped* Feynman paths that carry potential-depending action. Path step is at Planck-scale; branching points represent preon creation-annihilation. Each single-preon wave function depends on the coordinates of a 6-dimensional manifold, one of whose ‘extra’ dimensions associates in Dirac sense to a self-adjoint operator that represents the preon’s reversible *local* time. Within a path, local-time *intervals* equal corresponding intervals of monotonically-increasing global time even though, within a (*fixed-age*) ray, the local time of a preon is variable. The operator canonically conjugate to a preon’s local time represents its (total) energy in its (Milne) ‘local frame’. A macroscopically-stable positive-energy single-preon wave function identifies either with a Standard-Model elementary particle or with a graviton. Within intermediate-density sub-Hubble-scale universe regions such as the solar system, where ‘reproducible measurement’ is meaningful, *physical* special relativity—‘Poincaré invariance’—approximates DQC for spacetime scales far above that of Planck.

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1. Introduction

Nonexistence of unitary *finite-dimensional* Lorentz-group representations has heretofore obstructed, for dynamically-*changing* numbers of particles, a Dirac-type quantum theory that, at fixed (global) time, represents entity location, momentum, spin and energy by self-adjoint operators on a (rigged) Hilbert space [1]. The Standard Model, founded not on particles but on quantum fields associated to nonunitary finite-dimensional Lorentz-group representations, and representing action by ill-defined local-field-*product* operators with perturbative renormalization procedures to manage consequent divergences, has failed to accommodate gravity. Problematic, furthermore, is the Standard-Model description of bound states (‘condensed matter’), perturbation theory being unsuited to macroscopically-stationary composite wave functions.

The discrete quantum cosmology (DQC) proposed in Reference [2] and elaborated in certain respects by the present paper postulates at exceptional (global) ‘ages’ a ‘cosmological-preon’ Fock space via *unitary* (infinite-dimensional) single-preon representations of the semi-simple 12-parameter right-left Lorentz-group. Our representations are adaptations of those found by Gelfand and Naimark (G-N) [3].

Cosmological preons are quantum-theoretic universe constituents. We divide global spacetime into ‘slices’ of fixed macroscopic width in age, with ‘cosmological rays’ defined on *slice boundaries*—our Fock space attaching *only* to these exceptional universe ages. The ray at the upper boundary of a slice is determined from that at the lower boundary by action-carrying stepped and branching Feynman paths that traverse the slice.

Self-adjoint-operator expectations at a spacetime-slice boundary prescribe throughout the following slice a non-fluctuating continuous ‘classical reality’—current densities of locally-conserved electric charge and energy-momentum. Real expectations further prescribe action-determining electromagnetic and gravitational potentials. (Fluctuating—path-dependent—potentials also contribute to path action.)

Any cosmological preon (henceforth throughout this paper simply called ‘preon’) is ‘lightlike’—having velocity-magnitude c and a polarization transverse to velocity direction—and carries momentum, angular momentum and energy. However, only very-special preon wave functions exhibit, through expectations (at the exceptional ages) of self-adjoint operators, the relation between energy, momentum, spin and spacetime location that characterizes ‘ordinary matter’. A DQC Fock-space ray, representing the entire universe at some exceptional age, comprises sums of products of preon wave functions whose discrete quantum numbers allow *certain* macroscopically-stable positive-energy single-preon states to be interpreted as lepton, quark, weak boson, photon or graviton. (A massive elementary particle at rest is represented by a unitary preonic analog of Dirac’s nonunitary special-relativistic electron wave function—that superposes opposite lightlike velocities.)

The spacing between exceptional ages provides (precise) cosmological meaning for the

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adjective ‘macroscopic’. Reference (2) addresses ‘creation-annihilation’ Feynman paths that traverse each spacetime slice and that carry gravitational, as well as electromagnetic and weak-strong action.

Although the DQC Fock space comprises Pauli-symmetrized superpositions of products of invariantly-normed single-preon functions, the present paper at the risk of obscuring total relativity chooses to emphasize the unfamiliar labels on which depends a wave function belonging to an *individual* preon. Special functions that represent elementary particles will here be exposed. Another paper will define self-adjoint operators that represent, through superposition of annihilation-creation operators, the (‘familiar’ although cosmological) electromagnetic and gravitational radiation fields.

Any single-preon basis carries six labels—‘extra’ preon dimensions prescribing a velocity of magnitude c whose direction is generally independent of momentum direction. To illustrate let us immediately here compare a DQC basis with 4 continuous and 2 discrete labels to the familiar ‘asymptotic Hilbert-space’ labels of S-matrix theory. This set of preon labels comprises energy, ‘momentum magnitude’, momentum direction (2 angles) and a *pair* of invariant (discrete) helicities. One of the latter is the (familiar) angular momentum in the direction of momentum while the other—here to be called ‘velocity helicity’—is angular momentum in the direction of velocity. The energy of a preon is (c times) the component of its momentum in the direction of its velocity.

Identifiable as ‘extra’ in this label set are the preon’s energy and its velocity-helicity. Although the continuous label called ‘preon-momentum magnitude’ sounds familiar, we warn readers that this label enjoys an association to Lorentz-group Casimirs which parallels the rotation-group-Casimir association of a massive particle’s discrete (integer or half-integer) *spin magnitude*. Only for large values does ‘preon-momentum magnitude’ enjoy the classical meaning suggested by the name we give it.

Commutability of a complete set of 6 self-adjoint operators that represent the foregoing 6 preon attributes derives from commutability of right Lorentz transformations with left transformations. Our usage of the adjectives ‘right’ and ‘left’ will be explained. DQC Hilbert space unitarily represents a *12-parameter* group—the product of right and left Lorentz groups. DQC action is *right*-Lorentz invariant, its right transformations being those employed by Milne to *define* a cosmological spacetime [4] and which we call ‘Milne transformations’ to avoid confusion with the Einstein-Poincaré meaning for a Lorentz transformation. The 6 self-adjoint operators that generate (non-commuting) Milne transformations constitute a second-rank antisymmetric tensor (a 6-vector) whose elements represent preon momentum and angular momentum. DQC dynamics conserves momentum and angular momentum but not energy—which associates to one of the *left* Lorentz generators. Velocity helicity associates to another left generator, one which commutes with preon energy as well as with momentum and angular momentum.

Readers are alerted that two *different* self-adjoint operators represent, for different purposes, *preon energy*; the spectrum of one operator is discrete and positive while that of the other spans the real line. Expectations of the positive-energy operator determine classical reality and gravitational action. The indefinite-energy operator is the generator

of local-time displacement—the ‘Dirac conjugate’ of the individual-preon (local-) time operator. DQC distinguishes local time from global time. Despite local time *in a path* increasing monotonically at the same rate as global time, *in a ray* local time is reversible.

Although DQC *left* transformations lack any 6-parameter-symmetry-group precedent in natural philosophy, a 1-parameter left subgroup associates to *local-time* (not global ‘age’) translation. The generator of this left U(1) subgroup represents indefinite preon energy. (Spacetime slicing precludes meaning for continuous ‘age translation’.) Another U(1) left subgroup, generated by preon velocity helicity, comprises rotations about the preon’s velocity direction. In the DQC algebra of self-adjoint preon operators (which we avoid calling ‘observables’ because cosmology admits no *a priori* meaning for ‘measurement’), the two left-Lorentz-group Casimirs are *equal* to the two right Casimirs (commuting with all 12 group generators).

Reference (2) addresses the DQC action of branched Feynman paths. The present paper is complementary—ignoring path action while addressing various 6-labeled bases for single-preon Hilbert space. Each basis corresponds to a complete set of 6 commuting self-adjoint operators (a 6-csco). Unitary Hilbert-space *regular* representation of the product of right and left Lorentz groups provides a DQC path-contactable basis that parallels the Feynman-path-contacting *coordinate* basis for Dirac’s nonrelativistic quantum theory [1]. Analogs of Dirac’s *momentum* basis associate to unitary *irreducible* (‘unirrep’) SL(2,c) representations.

The algebra of preon self adjoint operators collectively represents in Dirac sense preon spatial location, velocity, polarization, energy, momentum and angular momentum, as well as velocity-helicity and momentum-helicity. Of course not all these operators commute with each other. Each of the DQC bases discussed here associates to a different 6-csco. To deal with a variety of issues the present paper will invoke seven alternative csco’s.

In the ‘path basis’ each preon’s ‘canonical coordinate’ is a product of the 6 continuous coordinates of a manifold traversed by Feynman paths that comprise straight positive-lightlike ‘arcs’ which may be created or annihilated at cubic vertices as a path progresses. The local time along any arc has *unit* derivative with respect to global age. Any path contributes, by Feynman’s prescription, to the determination from some ‘ray’ of the succeeding ray. Each preon of a ray contacts exactly one starting arc of a Feynman path, and each finishing path-arc contacts exactly one preon of the subsequent ray. Any path arc failing to reach its slice’s upper bound is annihilated at a path branching where one or two new arcs are created.

We employ the term ‘ray’ because the norm and overall phase of a DQC wave function lack probabilistic or other significance. Heisenberg uncertainty—leading to ‘many worlds’ in previous attempts to formulate a quantum cosmology—is absent from DQC.

Our quantum-theoretic version of Milne’s ‘cosmological principle’ [4] recognizes time arrow and absoluteness of motion, while respecting Mach’s principle in the sense discussed by Wilczek [5]. Einstein-Poincaré special relativity ignores time arrow and motion absoluteness, although partially accommodating Mach by invariance of maximum velocity.

(General relativity disregards Planck and Dirac as well as Milne.) Application of DQC to *physics* requires scale-based *approximation*; DQC addresses an evolving universe that lacks *a priori* meaning for *reproducible* measurement but which allows scale-dependent approximate meaning. The general meaning of ‘physics’ and of special relativity in particular is confined to spacetime scales tiny compared to that of Hubble while huge compared to that of Planck. Reference (6) presents a Euclidean-group-based physics-scale gravity-less approximation to DQC that may be described as a (Higgsless) ‘sliced-spacetime Standard Model’ (ssSM).

The DQC (global) *age*, invariant under both right and left transformations, plays a discrete role paralleling that of continuous time in nonrelativistic quantum theory. DQC Feynman paths connect successive macroscopically-spaced *exceptional* ages at each of which is defined a cosmological Fock-space ray—in a sense recalling S-matrix theory. DQC spacetime divides into macroscopic-width ‘slices’ whose boundaries locate at the exceptional ages. Path branching—path-arc creation or annihilation—is forbidden at slice boundaries where a ray is defined—occurring at ages *interior* to a slice where rays are not defined. DQC dynamics prescribes quantum propagation in the discrete S-matrix sense of an “*in* state” leading to a subsequent “*out* state” without any wave function being defined between *in* and *out*.

Although not discussed in the present paper, the aggregation of DQC Hilbert space, path rules and initial condition “spontaneously” breaks *C*, *P* and *CP* symmetries. Because the DQC Hilbert space represents a group isomorphic to the *complex* Lorentz group (as does analytic S-matrix theory), some cosmological counterpart to physical CPT symmetry promises to be revealed by future theoretical investigation.

The here-examined infinite-dimensional single-preon Hilbert space comprises normed functions of the continuous coordinates (path-basis labels) that ‘classically locate’ an individual preon within a 6-dimensional manifold which is at once a right *and* a left group manifold (*common* left-right Haar measure). The single-preon infinite-dimensional Hilbert space has as a factor a *finite*-dimensional (6^4 dimensions⁽⁷⁾) Hilbert subspace of *discrete* labels, invariant under both right and left continuous transformations, labels that are largely ignored by the present paper. Discrete labels carried by both path and ray distinguish different preon ‘sectors’ (e.g., electron, up-quark, photon, graviton) by specifying electric charge, color, generation, etc [7].

We *shall* here attend to an invariant 2-valued parity-related “handedness” carried both by path arcs (between path-branching points) and by preons—in contrast to preon helicities that are undefined for a path arc. DQC Hilbert space correlates handedness to the *sign* of momentum and velocity helicities. Both these helicities are right-Lorentz (Milne) invariant.

G-N discussed two different bases for a Hilbert space that represents unitarily the group $SL(2,c)$ [4], without attempting for either basis a natural-philosophical interpretation. The vectors of one basis—analogue to the coordinate basis of nonrelativistic Dirac theory—are normed functions over the 6-dimensional (continuous, left-right) manifold. We call this the ‘path basis’ because DQC Feynman paths traverse this 6-space. G-N’s second

basis—that we here call ‘G-N unirrep’—parallels the Dirac-Fourier-Wigner momentum-spin basis of nonrelativistic quantum theory that unitarily and irreducibly represents the Euclidean group [8]. (The 6-parameter compound Euclidean group is a contraction of the 6-parameter semisimple Lorentz group.) Although G-N’s transformation connecting their two bases differs from Fourier-Wigner, wave-function norm is preserved; the transformation is unitary.

We modify the G-N unirrep basis by (unitary) Fourier transformations of wave-function dependence on a pair of complex *directional* labels, so as to diagonalize simultaneously preon (indefinite) energy—the component of momentum in velocity direction—and components of momentum and angular momentum in some arbitrarily-specified direction. A continuous Casimir label, maintained undisturbed from the G-N unirrep basis, we call ‘magnitude of momentum’. Two discrete labels are helicity interpretable—components of angular momentum in velocity and momentum directions. The modified basis, which facilitates meaning for ‘preon parity reflection’ while remaining a unitary irreducible $SL(2,c)$ representation, we call the ‘energy-momentum unirrep basis’.

The universe spacetime proposed by Milne in the nineteen thirties [4]—an open forward-lightcone interior whose boundary has ‘big-bang’ interpretation—endows ‘Lorentz transformation’ with a cosmological time-arrowed meaning different from the Einstein-Poincaré special-relativistic physics meaning (ignoring time arrow) that augments Lorentz invariance by spacetime-displacement invariance. (Arbitrary spacetime displacements are *not* allowed in Milne spacetime. A sufficiently large spacelike or negative-timelike displacement may move a point of Milne’s spacetime *outside* his universe.) In contrast to an Einstein boost between different ‘rest frames’ that each assigns a different set of velocities to massive entities within some macroscopic region, Milne boosts relate to each other different ‘local frames’ that each associates to a different spatial location. Milne boosts—*right* DQC transformations—are spatial displacements at fixed universe age in a curved (hyperbolic) 3-space.

General relativity’s association of gravity to spacetime curvature may cause readers to suppose inability of flat Milne spacetime to represent gravity. But DQC gravitational action at a distance, plus creation and annihilation of *soft-gravitonic* arcs at path branching points, enables *discretized* spacetime curvature via ‘gentle’ branchings of stationary-action Feynman paths [2]. Any DQC path, as prescribed in Reference (2), is an ‘event graph’—a set of spacetime-located cubic vertices connected by spacetime-straight arcs of positive-lightlike 4-velocity that carry positive energy as well as discrete attributes. By its disregard of Planck’s constant, general relativity ignores gravitons and might be described as approximating, by a spacetime-curving Feynman-path trajectory (e.g., an electron trajectory), an arc-sector-maintaining stationary-action *sequence* of DQC straight ‘hard’ arcs that are separated at gentle events by ‘soft’ gravitonic-arc absorption or emission.

The adjectives ‘hard’ and ‘soft’ here refer to the energy scale set by Planck’s constant h times the *inverse* of the age width of a spacetime slice—the time interval that *defines* cosmologically the adjective ‘macroscopic’. Both the Standard Model and the Reference (6) ssSM revision thereof attend to the Planck constant, while setting G equal to 0 and

achieving flat 3-space through disregard of the Hubble constant H —suppressing redshift by treating universe age as infinite. The ssSM physics approximation to DQC differs from the Standard Model through recognition of macroscopic spacetime slicing. Even though ignoring gravity and redshift, ssSM by accommodating soft photons *dynamically* represents the macroscopic *electromagnetic* observations that are taken for granted by the S-matrix standardly employed to interpret experiments. *Furthermore*, spacetime slicing frees ssSM from need for elementary Higgs scalar bosons.

In (un-approximated) DQC, the (inverse) *geometric mean* of slice macroscopic width and Planck-scale age step (along any path-arc) [2] establishes a *particle-mass* scale—a scale that in the Standard Model must be supplied by Higgs scalars. DQC particle scale survives into the ssSM approximation because the latter, although ignoring both Hubble and Planck scales, keeps fixed the *ratio* H^2/G as G and H individually approach zero. This ratio sets the universe mean energy density that is required by Mach’s principle in conjunction with universality of maximum velocity [2].

Treated as *a priori* by ssSM are 2 scale parameters: the foregoing particle scale, regardable as counterpart to DQC’s Planck-scale path step [2], and the macroscopic scale that the present paper introduces as an *a priori* DQC feature. The following section notes that number theory may eventually render *non-arbitrary* the integral ratio of spacetime-slice width to path step. Particle scale would thereby, like path step, become set by the trio of universal parameters G , h and c , whose values Planck realized do not require ‘explanation’.

2. Milne Spacetime

The open interior of a forward lightcone—what we call ‘Milne spacetime’—is the product of a lower-bounded one-dimensional ‘age space’ with an unbounded 3-dimensional ‘boost space’. The spacetime displacement from the forward-lightcone vertex (whose spacetime location is meaningless) to any spacetime point is a positive-timelike 4-vector (t, \mathbf{x}) . Defining the “age” τ of a spacetime point to be its Minkowski distance from lightcone vertex—i.e., the Lorentz-invariant modulus $(t^2 - \mathbf{x}^2 c^{-2})^{1/2}$ of its spacetime-location 4-vector—the set of points sharing some common age occupies a 3-dimensional (global) hyperboloid. Any point within such a hyperboloid may be reached from any other by a 3-vector *boost*. Once an origin within boost space is designated, an arbitrary spacetime point is specified by $(\tau, \boldsymbol{\beta})$, where $\boldsymbol{\beta}$ is the 3-vector boost-space displacement from the selected origin to the point. Writing $\boldsymbol{\beta} = \beta \mathbf{n}$, where \mathbf{n} is a unit 3-vector and β is positive,

$$t = \tau \cosh \beta, \quad \mathbf{x} = c\tau \mathbf{n} \sinh \beta. \quad (1)$$

The spatial-location label β will in the following section and in Appendix A be identified within the path basis for DQC Hilbert space. Compatibility of age discretization with Milne’s meaning for Lorentz invariance allows DQC’s spacetime to be temporally discrete for its Feynman-path quantum dynamics even though spatially continuous. Discretization occurs at two very different fundamental scales: (1) Any DQC Feynman path traverses

a ‘slice’ of Milne spacetime—a slice whose width in age is *macroscopic*. (2) Within each slice any (straight and lightlike) arc proceeds in *Planck-scale* age steps whose precise extension is established by Reference (2) from action quantization.

Although consistency requires slice width to be an integral multiple of arc step, the huge-integer ratio will not be addressed by the present paper—which ignores arc steps and number theory. Before attending to path arcs this paper chooses to address the single-preon Hilbert-space path basis. Nevertheless the termination of path arcs at exceptional (ray-age) hyperboloids might be taken as *defining* the path basis of preon Fock space.

To each point of boost space associates a “local” Lorentz frame in which $\beta=0$ i.e., a frame defined up to a rotation by the point’s location 4-vector there being purely timelike. Age change and time change are equal in local frame. In local frame an infinitesimal spatial displacement $d\beta$ by $d\mathbf{x}=c\tau d\beta$. (A phenomenological meaning for ‘local frame’ resides in the approximate isotropy of cosmic background radiation observed in that frame. This meaning parallels that of standard cosmology’s ‘co-moving coordinates’.) The rotational ambiguity of local-frame meaning is related below to arbitrariness of the origin of a 6-dimensional space. Origin location specifies not only a boost-space location but also an attached orthogonal and handed (1, 2, 3) set of 3 reference axes which may be parallel transported along a boost-space geodesic from the origin to any other boost-space location.

The (3-parameter) global orientation ambiguity is accommodated by total relativity—a DQC Fock-space restriction that requires rays to be *globally* rotationally invariant—unchanged when a common Milne rotation is applied to *all* preons. Ray *expectations* of Milne-boost generators are also globally invariant. Total relativity might be said to mean that both the total momentum and the total angular momentum of the universe are zero (6 conditions). The DQC universe is not only rotationally invariant but, associating right-boost generators with infinitesimal spatial displacements at fixed age, the universe is also ‘classically-invariant’ under (non-abelian) *collective-preon* boosts.

3. Path Basis; Preon-Coordinate Matrix

The Dirac ‘classical coordinate’ of a preon is a product of 6 real continuous labels prescribing a *unimodular* 2×2 matrix that we shall call the “preon-coordinate matrix”. Although (in contrast to 4-vectors representable by *hermitian* 2×2 matrices) preon-coordinate matrices are not linearly superposable, they may be *multiplied* either on the left or on the right by unimodular 2×2 matrices. *Right* multiplication corresponds to Milne transformation of preon coordinate—a Lorentz transformation that if applied to *all* preons is merely a change of basis origin and without significance. A right boost applied to an individual preon is a displacement in (hyperbolic) Milne 3-space. A right rotation of a preon has the familiar physics meaning designated by 3 Euler angles that refer to some set of 3 orthogonal and handed reference directions fixed *externally* to the preon.

Left multiplication of a DQC preon’s coordinate matrix represents an ‘internal’ preon modification—by this transformation’s reference to a ‘body-fixed’ preon axis. The reader

is invited to recall the body-fixed symmetry axis of a nonrelativistic symmetrical top; the velocity direction of a DQC preon (*distinct* from its momentum direction) may be regarded as the direction of its symmetry axis. Change of wave function by left transformation of path-basis coordinates, although incapable of altering either momentum or angular momentum, may shift a preon's energy, velocity and velocity helicity. (The mass of an elementary particle—a special preon state—may be altered by left transformation.) The meaning of 'preon' depends on the *inequivalence* of two different $SL(2,c)$ representations—through left and right action on the preon-wave-function argument in the path basis. Preon description relies on the *commutability* of any (self-adjoint) Hilbert-space left-transformation generator with any right-transformation generator.

[The internal versus external meanings of left versus right transformations might be interchanged. The assignment chosen in this paper conforms to an arbitrary choice made by Gelfand and Naimark and our desire to maintain the notation of Reference (3). It is pedagogically unfortunate that Wigner, working with the group $SU(2)$ —an $SL(2,c)$ subgroup—made the opposite choice [8]. He represented rotation of a symmetric top with respect to some external coordinate system, or vice-versa, by *left* multiplication of the top-wave-function's matrix argument. Right multiplication of the Wigner-defined argument has capacity to vary the top's angular momentum *component* in the direction of its symmetry axis while leaving total angular momentum unaltered either in direction or magnitude. Wigner right multiplication may change top (rotational kinetic) energy while left multiplication lacks such capability, even though his left multiplication may alter angular-momentum direction with respect to some fixed set of external axes.]

Because the doubly-covered Lorentz group is isomorphic to the group of unimodular 2×2 matrices—one must be careful to distinguish a preon-coordinate matrix from a matrix that represents a coordinate *transformation* (either left or right). Although coordinate matrices and transformation matrices both depend on 6 real parameters, parameter meanings are of course different. Analog is found in the 3 Euler angles that coordinate a symmetrical top, as opposed to the 3 angles that label a rotation.

The *coordinate matrix* of Preon i or, alternatively, that of Arc i within a Feynman path, will here be denoted by the (boldface) G-N symbol \mathbf{a}_i . Apart from the double covering of rotations, the content of \mathbf{a}_i may be expressed through three Milne 4-vectors represented by hermitian 2×2 matrices defined by the \mathbf{a}_i quadratic forms,

$$\mathbf{x}_i \equiv \tau \mathbf{a}_i^\dagger \mathbf{a}_i = \tau \exp(-\boldsymbol{\sigma} \bullet \boldsymbol{\beta}_i), \quad (2)$$

$$\mathbf{v}_i \equiv \mathbf{a}_i^\dagger (\sigma_0 - \sigma_3) \mathbf{a}_i, \quad (3)$$

$$\mathbf{e}_i \equiv \mathbf{a}_i^\dagger (-\sigma_1) \mathbf{a}_i, \quad (4)$$

where $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ is the standard (handed) set of Pauli hermitian traceless self-inverse 2×2 matrices (determinant -1). The matrix σ_3 is real diagonal while σ_1 and σ_2 are antidiagonal, σ_1 real and σ_2 imaginary. The symbol \bullet in (2) denotes the inner

product of two 3-vectors. The symbol σ_0 denotes the unit 2×2 matrix. Although G-N did not employ Pauli matrices, we have found them convenient. Otherwise we largely maintain the notations of Reference (4). [In checking that (2), (3) and (4) transform as Milne 4-vectors, the reader needs remember that although Milne transformation of a coordinate matrix is multiplication from the *right*, Milne transformation of the hermitian conjugate of a coordinate matrix is a *left* multiplication.]

The positive-definite hermitian matrix \mathbf{x}_i , with determinant τ^2 and positive trace, represents [Formula (1) and Appendix A] a positive-timelike 4-vector whose components are $\tau \cosh \beta_i, \tau \mathbf{n}_i \sinh \beta_i$, where $\boldsymbol{\beta}_i = \beta_i \mathbf{n}_i$ with \mathbf{n}_i a unit 3-vector and β_i a positive real number. We interpret the 4-vector represented by (2) as the spacetime displacement of the preon's location, or a location along some arc of a Feynman path, from the big-bang forward-lightcone vertex—the 'origin' of Milne spacetime. The 3-vector $\boldsymbol{\beta}_i$ is the above-discussed location in Milne boost space.

Components of the 4-vectors \mathbf{v}_i and \mathbf{e}_i may be pulled from the 2×2 matrices (3) and (4) in the usual way. The positive-lightlike 4-vector represented by the zero-determinant, positive trace hermitian matrix \mathbf{v}_i is characterizable, in the path basis of a cosmological ray, as the 4-velocity of Preon i , with unit timelike component in this particle's local frame—where $\boldsymbol{\beta} = 0$. The symbol \mathbf{v}_i also represents the 4-velocity of an i -labeled Feynman-path arc that passes through the 6-space point coordinated by \mathbf{a}_i .

The hermitian matrix \mathbf{e}_i (determinant -1) represents the Preon- i (unit-normalized) *spacelike* transverse-polarization 4-vector that is (right) Lorentz orthogonal to \mathbf{x}_i and \mathbf{v}_i . The same symbol may be used to denote the polarization of Arc i . Formulas (2, 3, 4) specify 4-vectors in that Lorentz frame—belonging to some arbitrarily designated '6-space origin'—in which the coordinate matrix is \mathbf{a}_i .

The *unit* 2×2 matrix σ_0 represents an *origin* of 6-space shared by *all* the preons represented in a cosmological ray. When a preon-coordinate \mathbf{a}_i is a unit matrix, the preon (classically) locates at the origin of boost space with its velocity in the (arbitrarily-selected) 3 direction and its polarization in the 1 direction. The angle that specifies polarization in the general case, in a plane perpendicular to velocity direction, spans a 4π interval with the *sign* of \mathbf{a}_i *reversing* under any 2π continuous displacement of polarization direction.

A path-basis single-preon wave function $\Psi(\mathbf{a}_i)$ has a finite norm

$$\int d\mathbf{a}_i |\Psi(\mathbf{a}_i)|^2, \quad (5)$$

where the 6-dimensional invariant volume element $d\mathbf{a}_i$ (the Haar measure), is specified immediately below for the G-N parameterization of \mathbf{a}_i . The norm (5) is invariant under *both* of the above-discussed (right and left) coordinate transformations. Wave-function norm is preserved by the transformation to G-N unirrep basis as well as to various other bases we shall discuss.

G-N expressed the most general unimodular 2×2 coordinate-matrix \mathbf{a} with $\mathbf{a}_{22} \neq 0$ through 3 *complex* labels s, y, z according to the product of 3 unimodular 2×2 matrices

each representing an abelian 2-parameter subgroup of $SL(2, c)$,

$$\mathbf{a}(s, y, z) = \exp(-\boldsymbol{\sigma}_3 s) \times \exp(\boldsymbol{\sigma}_+ y) \times \exp(\boldsymbol{\sigma}_- z), \quad (6)$$

where $\sigma_{\pm} \equiv 1/2(\sigma_1 \pm i\sigma_2)$. [G-N employed, rather than y , a complex label equal to ye^{-s} .] Because the spatial direction of the 4-velocity \mathbf{v} is completely determined by z (see Formula (3') below) we shall refer to z as the “velocity coordinate”. Considerations in the following two sections reveal it appropriate to call $2 Re s$ either the preon’s ‘longitudinal coordinate’ or its ‘local time’ while y is its ‘transverse coordinate’. Calculation, substituting (6) into (4), shows the angle specifying polarization direction (in a plane transverse to velocity) to be $2 Im s$. The polarization 4-vector \mathbf{e} is independent of $Re s$.

The 6-dimensional volume element,

$$d\mathbf{a} = ds dy dz, \quad (7)$$

is invariant under $\mathbf{a} \rightarrow \mathbf{\Gamma}^{-1}\mathbf{a}$, with $\mathbf{\Gamma}$ a 2×2 unimodular matrix representing an external (right) Milne transformation. It is also invariant under the internal (left) transformation $\mathbf{a} \rightarrow \mathbf{\Gamma}\mathbf{a}$, where $\mathbf{\Gamma}$ is a unimodular 2×2 matrix.

By the symbol $d\xi$, with ξ complex, is meant $d Re \xi \times d Im \xi$. Formula (6) implies periodicity of $\Psi(s, y, z)$ in $Im s$ with period 2π , and the norm (5), with a corresponding interval for $Im s$, is preserved by Fourier-series representation of Ψ dependence on $Im s$. The norm is also preserved by Fourier-integral representation of Ψ dependence on $Re s$ over the real line. Such a 2-dimensional unitary Fourier transformation (*not* to be confused with 6-dimensional transformations to G-N unirrep and energy-unirrep bases) will immediately be discussed. The resulting basis we call “intermediate”.

4. Intermediate Basis—Diagonalizing Preon Energy, Velocity and Velocity Helicity

The intermediate basis is labeled by the two (path-basis) complex-number symbols y , z and two real (positive or negative) Milne-invariant (right Lorentz-invariant) labels—an integer n and a continuous ω . In Dirac sense the label n is ‘conjugate’ to $Im s$ while the label ω is conjugate to $Re s$. Fourier-series representation of wave-function dependence on $Im s$ defines the discrete index n that, before its correlation with ‘extra-Lorentz’ discrete Hilbert-space labels, may take any integral value (positive, negative or zero). Fourier-integral representation of wave-function dependence on $Re s$ analogously defines the continuous label ω that spans the real line. We make the unitary transformation,

$$\Psi(s, y, z) = \sum_n \int d\omega \exp(-in Im s - i\omega Re s) \psi_{n,\omega}(y, z), \quad (8)$$

the integer label n prescribing in \hbar units, as shown below, twice the preon velocity helicity. The label ω , after consideration of DQC Feynman-path arcs, will be seen to prescribe twice preon (total) energy in \hbar/τ units. The norm (5) equals

$$(2\pi)^{-2} \int d\omega dy dz |\psi_{n,\omega}(y, z)|^2. \quad (9)$$

Multiplying the coordinate matrix \mathbf{a} from the left by the unimodular matrix $\lambda_0 \equiv \exp(-\sigma_3 s_0)$, so that s changes to $s + s_0$ with y and z remaining fixed, effectuates a rotation of polarization by an angle $2 \operatorname{Im} s_0$ around velocity direction along with a shift of preon boost-space location along velocity direction by $2 \operatorname{Re} s_0$. The intermediate-basis wave function $\psi_{n,\omega}(w, z)$ is seen from (8) to be changed, under λ_0 left multiplication, merely by the phase factor $\exp(-in \operatorname{Im} s_0 - i\omega \operatorname{Re} s_0)$. For pure rotation (i.e., $\operatorname{Re} s_0 = 0$) by an angle $\phi = 2 \operatorname{Im} s_0$ about velocity direction, this phase factor is $\exp(-in \phi/2)$ —hence our characterization of n 's meaning.

Two self adjoint operators on the G-N Hilbert space generate the foregoing transformations. In the path basis these operators are

$$J_{3L}^\wedge = \frac{1}{2} i \partial / \partial \operatorname{Im} s, \quad (10)$$

$$K_{3L}^\wedge = \frac{1}{2} i \partial / \partial \operatorname{Re} s. \quad (11)$$

J_{3L}^\wedge generates (at fixed $\operatorname{Re} s, y$ and z , which the following section shows to imply fixed preon spatial location) rotation of preon polarization about its velocity direction while K_{3L}^\wedge generates (at fixed velocity and polarization as well as fixed transverse location) spatial displacement in this direction. According to (8) the eigenvalues of J_{3L}^\wedge are $1/2n$ while those of K_{3L}^\wedge are $1/2\omega$. It will be seen later that J_{3L}^\wedge and K_{3L}^\wedge are among the set of 6 commuting self-adjoint operators whose real eigenvalues label the energy-momentum unirrep basis. That is, n and ω are labels *both* of the energy-momentum unirrep basis *and* the intermediate basis.

Odd-even n distinguishes fermion from boson inasmuch as increasing the value of $\operatorname{Im} s$ by π with $\operatorname{Re} s, y, z$ unchanged reverses the sign of the preon-coordinate matrix \mathbf{a} . Writing $\Psi(\mathbf{a}) = \sum_n \Psi_n(\mathbf{a})$, Formula (8) implies

$$\Psi_n(-\mathbf{a}) = (-1)^n \Psi_n(\mathbf{a}). \quad (12)$$

DQC Hilbert space correlates a 2-valued handedness label with the *sign* of n , in restricting *any* preon sector to a *single* value of $n = n$. A fermionic preon has

$n = \pm 1$, a vector-bosonic preon has $n = \pm 2$ and a gravitonic preon $n = \pm 4$. Path-carried sector-specifying discrete labels and Milne-invariant path action perpetuate this reduction of DQC Hilbert space.

The continuous label ω , related to inertial and gravitational action through the positive-energy operator defined in Appendix C, ⁽²⁾ may appropriately be called ‘preon frequency’ or, in units specified below, ‘preon energy’. Both preon velocity-helicity and preon energy, as well as preon momentum-helicity and preon ‘momentum magnitude’, will be shown to be Milne (right-Lorentz) invariant. Energy and momentum-magnitude labels enjoy the meaning familiar for these terms in the preon’s *local* frame.

The following section shows $Re s$ to increase, along any arc of a Feynman path, with *unit* (local-frame) *age* derivative of the *local* time, while the remaining 5 path-basis labels remain fixed. Appendix C associates the label ω to the energy of any ‘beginning arc’—one which contacts that ray at the lower boundary of the spacetime slice occupied by this arc. The ray’s ‘partial-expectation’ of the positive-energy operator defined in Appendix C establishes the energy of any path arc originating at this ray. In the paper’s penultimate section the full expectation of a certain self-adjoint *operator function* of $K_{3L}^?$ prescribes the non-fluctuating near-future energy-momentum density that, by determining gravitational action (for ray propagation to the succeeding ray), constitutes an aspect of ‘mundane reality’. The (indefinite) energy operator $\hbar/\tau K_{3L}^\wedge$ is diagonal (with eigenvalues $\hbar\omega/2\tau$) both in the intermediate basis and in the energy-momentum unirrep basis, although not in the path or G-N unirrep bases.

5. Path Arcs

Each (positive-lightlike) arc of a DQC Feynman path moves (in Planck-scale age steps ⁽²⁾) through the 7-dimensional continuous space coordinated by τ and the 3 complex variables s, y, z . Because the Pauli-matrix identity

$$\sigma_0 = \frac{1}{2}(\sigma_0 + \sigma_3) + \frac{1}{2}(\sigma_0 - \sigma_3), \quad (13)$$

expresses the unit matrix as the sum of two zero-determinant unit-trace hermitian matrices, Formulas (2) and (6), by straightforward calculation, imply any spacetime location (with respect to Milne-lightcone vertex) to be the sum of two “oppositely-directed” positive-lightlike 4-vectors:

$$\mathbf{x} = \tau[|y|^2 e^{-2Re s} \mathbf{h}(z_+ y^{-1}) + e^{2Re s} \mathbf{h}(z)]. \quad (14)$$

The symbol $\mathbf{h}(\varsigma)$ denotes a zero-determinant hermitian positive-trace 2×2 matrix,

$$\mathbf{h}(\varsigma) \equiv (\sigma_0 + \varsigma^* \sigma_+) \sigma_- \sigma_+ (\sigma_0 + \varsigma \sigma_-), \quad (15)$$

a bilinear function of a direction-specifying complex variable. The second term within the square bracket of (14) relates to the following formula, deducible from (3), for the positive-lightlike 4-velocity (unit time-component in local frame) of a preon or arc:

$$\mathbf{v} = 2e^{2Re s} \mathbf{h}(z). \quad (3')$$

Notice absence from (14) of any dependence on $Im s$.

Formula (14) reveals two distinct categories of positive-lightlike fixed- y , fixed- z trajectory through 7-dimensional (τ, \mathbf{a}) space. In one category the first term of (12) is age independent while the second’s magnitude increases with age. In the other category the second term is fixed while the first’s magnitude increases. We call the former category **F** and the latter **B**. Along an **F** trajectory $e^{2Re s}$ is proportional to τ , while along a **B** trajectory $e^{2Re s}$ is *inversely* proportional to τ . DQC achieves unit age derivative of local time along

any Feynman path by requiring every arc to lie along an \mathbf{F} trajectory—identifying the derivative with respect to age of local time as $\tau d(2Re s)/d\tau$.

An \mathbf{F} trajectory is identified by the two complex dimensionless labels z , y and one real label $\tau_F \equiv \tau e^{-2Re s}$ of time dimension. Formula (14_F) represents the most general path spacetime trajectory by a hermitian-matrix that is a second-degree polynomial in age:

$$\mathbf{x}_{z,w}^F(\tau) \equiv \tau_F |y|^2 \mathbf{h}(z + y^{-1}) + \tau^2 \tau_F^{-1} \mathbf{h}(z). \quad (14_F)$$

The age-independent first term of (14_F)—an “initial displacement” from the lightcone vertex that depends on both z and y —locates the \mathbf{F} trajectory’s zero-age *origin* on the (3-dimensional) Milne forward-lightcone boundary. *Any* arc, beginning and ending along *some* such trajectory, lies *inside* the lightcone. The second term of (14_F), positive lightlike like the first term but in the opposite spatial direction and independent of y , leads from the trajectory’s big-bang origin to the age- τ spacetime point. The second term’s direction is determined entirely by z .

6. G-N Unirrep Basis

The unitary *irreducible* $SL(2,c)$ representation found by Gelfand and Naimark provides a Hilbert-space basis, alternative to the path basis, that we shall call the ‘G-N unirrep basis’—comprising functions of 2 real labels and 2 complex labels. A norm-preserving transform connects path and G-N unirrep bases.

The two real labels prescribe eigenvalues of two self-adjoint operators that represent Casimir-quadratics in the generators of *either* right *or* left Lorentz transformations—polynomials invariant under both right and left 6-parameter groups (commuting with all 12 generators). If we denote the trios of right-rotation and left-rotation generators by the 3-vector boldface operator symbols \mathbf{J}_R^\wedge , \mathbf{J}_L^\wedge and the trios of boost generators by the symbols \mathbf{K}_R^\wedge , \mathbf{K}_L^\wedge the Casimir, $\mathbf{K}_R^\wedge \bullet \mathbf{J}_R^\wedge = \mathbf{K}_L^\wedge \bullet \mathbf{J}_L^\wedge$, has eigenvalues $(\rho/2)$ $(m/2)$ while the Casimir, $\mathbf{K}_R^\wedge \bullet \mathbf{K}_R^\wedge - \mathbf{J}_R^\wedge \bullet \mathbf{J}_R^\wedge = \mathbf{K}_L^\wedge \bullet \mathbf{K}_L^\wedge - \mathbf{J}_L^\wedge \bullet \mathbf{J}_L^\wedge$, has eigenvalues $(\rho/2)^2 - (m/2)^2 + 1$. The label m takes positive, negative or zero integer values that are shown below to correspond to *twice* preon helicity (angular momentum in momentum direction). Bosonic preons have even values of m while fermionic preons have odd. The second Casimir label, denoted by the symbol ρ , takes continuous real non-negative values. In $\hbar/c\tau$ units, $\rho/2$ will be called the preon’s (local-frame) ‘magnitude of momentum’.

The remaining labels on which a preon’s G-N unirrep-basis wave function depends are 2 complex variables, z' and z_1 , the former right-invariant and the latter left invariant. In Reference (4) the right-invariant variable z' is denoted by the symbol z ; we have here added a prime superscript to avoid confusion with the path-basis label z . The left-invariant complex label z_1 prescribes in a cosmological ray the *direction* of preon momentum in a sense similar to that above in which the path-basis complex variable z prescribes velocity direction. Loosely speaking, the complex variable z' may be understood as representing the *difference* between momentum and velocity directions. A

precise meaning for z' resides in the difference between ρ and ω and the difference between m and n . In a sense similar to that of Dirac, half these differences will be seen ‘conjugate’, respectively, to $\ln |z'|$ and $\arg z'$.

The set (z', m, ρ, z_1) of 2 real-Casimir and 2 complex directional labels we shall denote by the shorthand index \mathbf{b} , the G-N unirrep-basis wave function $\Phi(\mathbf{b})$ being a transform of the path-basis wave function $\Psi(\mathbf{a})$. The (right-left-invariant) G-N unirrep-basis norm is

$$\int d\mathbf{b} |\Phi(\mathbf{b})|^2, \quad (16)$$

where the symbol $\int d\mathbf{b}$ means (with m assigned some ‘preon-type’ value)

$$\int d\mathbf{b} \equiv (1/2\pi)^4 \int d\rho(m^2 + \rho^2) \int dz' \int dz_1, \quad (17)$$

the integration over $d\rho$ running from zero to + infinity. For Preon Sector of DQC Hilbert space the value of m is taken equal to n -one of the 6 options $\pm 1, \pm 2, \pm 4$.

The norm-preserving transform is achieved by an integration over the 4-dimensional manifold regularly representing those 2×2 unimodular matrices—a 4-parameter $SL(2, c)$ subgroup—that have the form

$$\mathbf{k}(\lambda, y) \equiv \exp(-\sigma_3 \ln \lambda) \times \exp(\sigma_+ y), \quad (18)$$

with λ and y almost-arbitrary complex numbers ($\lambda \neq 0$). The 4-dimensional volume element

$$d_\ell \mathbf{k} = |\lambda|^{-2} d\lambda dy \quad (19)$$

is ‘left-invariant’ in the sense that an integral $\int d_\ell \mathbf{k} f(\mathbf{k})$ over some function $f(\mathbf{k})$ of the matrix \mathbf{k} equals $\int d_\ell \mathbf{k} f(\mathbf{k}_0 \mathbf{k})$, with \mathbf{k}_0 any fixed matrix within the subgroup. The transform from path to G-N unirrep basis is

$$\Phi(\mathbf{b}) = \int d_\ell \mathbf{k} \alpha_{m\rho}(\mathbf{k}) \Psi(\mathbf{z}'^{-1} \mathbf{k} \mathbf{z}_1), \quad (20)$$

where a bold-faced \mathbf{z} symbol in (20) is to be understood as the 2×2 unimodular matrix $\exp(\sigma_- z)$. The function $\alpha_{m\rho}(\mathbf{k})$ —analog of the Fourier transform’s exponential function and a cornerstone of G-N’s analysis—is given by

$$\alpha_{m\rho}(\mathbf{k}) \equiv |k_{22}|^{-m+i\rho-2} k_{22}^m = |k_{22}|^{-2} \exp(im \arg k_{22} + i\rho \ln |k_{22}|). \quad (21)$$

Note that $\alpha_{m\rho}(\mathbf{k})$ is independent of the parameter y in (18), depending only on $k_{22} = \lambda$. Although (20) is written in a form facilitating left transformation, an alternative form facilitates right transformation. Our choice above is arbitrary—motivated by desire to maintain the notation of Reference (4). Later the alternative form will also be invoked.

The G-N reverse transform, expressed in a notation where the path-basis wave-function is understood not as a function of a unimodular matrix \mathbf{a} but as a function of the three complex labels s, y, z defined by (6), is

$$\Psi_m(s, y, z) = (1/2\pi)^4 \int d\rho(m^2 + \rho^2) \int dz_1 |\lambda'|^{-m-i\rho+2} \lambda'^{*m} \Phi(m, \rho, \check{z}, z_1), \quad (22)$$

the symbols λ' and \check{z} denoting quotients that depend on z_1 as well as on s, y, z :

$$\check{z} \equiv e^s \lambda'(z_1 - z), \quad \lambda' \equiv e^s / [1 - (z_1 - z)y]. \quad (23)$$

The right-invariant G-N unirrep-basis complex label z' is constrained in (22) to the value \check{z} , vanishing when momentum and velocity directions are parallel. The point relation (23)—two real point relations—between path-basis and G-N unirrep-basis labels, ‘entangled’ by y involvement, enjoys a ‘disentangled’ counterpart in our use below of the intermediate-basis labels n, ω as also labels on the *energy-momentum* unirrep basis. Disentanglement makes the latter basis appropriate for specification of preon sectors.

7. External (Right) and Internal (Left) Transformations

Let us denote by the symbol Γ^{-1} a 2×2 unimodular matrix applied on the *right* of a coordinate matrix so as to effectuate an “external” (Milne) Lorentz transformation $\Psi(\mathbf{a}_1, \mathbf{a}_2 \dots) \rightarrow \Psi^\Gamma(\mathbf{a}_1, \mathbf{a}_2 \dots) \equiv \Psi(\mathbf{a}_1 \Gamma^{-1}, \mathbf{a}_2 \Gamma^{-1}, \dots)$ of a path-basis cosmological wave function—a sum of products of single-preon functions. A corresponding transformation is induced in *any* Hilbert-space basis, although generally a wave-function *factor* accompanies transformation of wave-function argument. (Absence of such factor characterizes the group-manifold path basis). Here ignored are discrete coordinates such as handedness, electric charge and color that are invariant under both right and left coordinate transformation. Although Milne’s cosmological principle associates in DQC to *global* right transformation, the present section addresses the effect on an *individual* preon function of either (external) right or (internal) left transformation.

In terms of the 3 complex path-basis coordinates s, y, z the right transformation is

$$\Psi(s, y, z) \rightarrow \Psi(s^\Gamma, y^\Gamma, z^\Gamma), \quad (24)$$

where by straightforward calculation of $\mathbf{a} \Gamma^{-1}$ with \mathbf{a} given by (6),

$$z^\Gamma = (\Gamma_{22}z - \Gamma_{21})(-\Gamma_{12}z + \Gamma_{11})^{-1}, \quad (24a)$$

$$y^\Gamma = (-\Gamma_{12}z + \Gamma_{11})[(-\Gamma_{12}z + \Gamma_{11})y - \Gamma_{12}], \quad (24b)$$

$$s^\Gamma = s + \ln(-\Gamma_{12}z + \Gamma_{11}). \quad (24c)$$

Notice how right transformation of z is independent of the variables s, y , while all right coordinate transformations involve z and right transformation of $s(y)$ is independent of $y(s)$. [Taken alone, (24a) provides a representation of $SL(2, \mathbb{C})$ by action of this group on the points of a complex plane.]

Notice further how, with respect to the complex coordinate s , the (24c) right transformation is merely a z -dependent displacement, implying right invariance of the intermediate-basis labels n and ω . Right Lorentz transformation of the intermediate basis (with labels

n, ω, y, z) alters the wave-function arguments y and z according to (24b) and (24a) while leaving n and ω undisturbed although multiplying the wave function by a phase, dependent on n, ω and z , that is deducible from (8) and (24c).

Milne (right) transformation induces, by calculation according to (20), in the G-N unirrep basis the transformation

$$\Phi(\mathbf{b}) \rightarrow \Phi^\Gamma(\mathbf{b}) = \alpha_{m\rho}^r [(-\Gamma_{12}z_1 + \Gamma_{11})^{-1}] \Phi(m, \rho, z', z_1^\Gamma), \quad (25)$$

where z_1^Γ is the polynomial quotient (24a) with z_1 in place of z , and

$$\alpha_{m\rho}^r(k_{22}) \equiv |k_{22}|^4 \alpha_{m\rho}(k_{22}). \quad (26)$$

Both the momentum and the angular momentum of a preon may be changed (relative to a fixed origin of Milne 6-space) by right transformation, but helicity and ‘magnitude of momentum’ are invariant as also (we have emphasized above) is energy and velocity helicity. Right-Lorentz transformation leaves undisturbed the complex argument z' of the G-N unirrep-basis wave-function.

The individual-preon transformation from the left in the G-N unirrep basis is as simple as (25). Left transformation alters wave-function dependence on the “internal” parameter z' while dependence on momentum direction z_1 is unchanged. One finds

$$\Phi(\mathbf{b}) \rightarrow^\gamma \Phi(\mathbf{b}) = \alpha_{m\rho}(-\gamma_{12}z' + \gamma_{11}) \Phi(m, \rho, z', z_1), \quad (27)$$

where

$$\gamma z' = (\gamma_{22}z' - \gamma_{21})(-\gamma_{12}z' + \gamma_{11})^{-1}. \quad (28)$$

Neither the momentum nor the angular momentum (nor of course the helicity) changes, as the 6 coordinates (path-basis labels) of some preon are left-shifted (*other* preons remaining unshifted). Left transformation of Preon i may change not only its spatial location and local time but its energy, its velocity and its velocity helicity.

The earlier-considered special left transformation $\gamma = \lambda_0$, where λ_0 is the unimodular matrix $\exp(-\sigma_3 \ln \lambda_0)$, preserves preon velocity while prescribing a rotation around velocity direction and a longitudinal shift of preon location along this direction ($\gamma_{12} = \gamma_{21} = 0$ and $\gamma_{22} = \gamma_{11}^{-1} = \lambda_0$). One finds, from (27) and (28),

$$\gamma \Phi(m, \rho, z', z_1) = \alpha_{m\rho}^{-1}(\lambda_0) \Phi(m, \rho, \lambda_0^2 z', z_1). \quad (29)$$

Although momentum and angular momentum remain undisturbed, there is λ_0 “rescaling” of the variable z' . We have seen earlier that preon energy and velocity helicity are unchanged by the foregoing transformation. A more general $\gamma_{21} = 0$ left transformation, with arbitrary γ_{12} and γ_{22} , continues to preserve velocity while shifting preon spatial location transversely as well as longitudinally. Here, in the path basis,

$$\gamma_z = z, \quad (30a)$$

$$\gamma_y = y + \gamma_{12}\gamma_{22}e^{2s}, \quad (30b)$$

$$\gamma_s = s + \ln\gamma_{22}. \quad (30c)$$

In this case preon energy and velocity helicity are affected if (as finite norm renders unavoidable) the wave function varies with changing y . Extension of the path-basis formulas (30) to the most general left transformation, where the velocity coordinate z also may change, is straightforward. [Formula (30b) is relevant to the elsewhere-discussed transverse Planck-scale 4-stranded (‘cable’) structure of a path arc.]

8. Momentum, Angular-Momentum 6-Vector

$\mathbf{K}_R^{i\wedge}$ and $\mathbf{J}_R^{i\wedge}$, under Milne transformation behaves as a second-rank antisymmetric tensor—a ‘6-vector’—that we denote by the symbol $J_{\mu\nu}^{i\wedge}$. The two preon Casimirs are invariants bilinear in $J_{\mu\nu}^{i\wedge}$. Preon momentum (in certain units) is represented by $\mathbf{K}_R^{i\wedge}$ and preon angular momentum by $\mathbf{J}_R^{i\wedge}$.

Because DQC path action is Milne-transformation invariant, the tensor $J_{\mu\nu}^{i\wedge}$ represents 6 ‘conserved’ preon attributes. The sum $J_{\mu\nu}^\wedge = \sum_i J_{\mu\nu}^{i\wedge}$ of *all* preon 6-vectors is the total momentum and angular momentum of the universe. DQC recognizes Mach’s principle as implying universe rotational invariance—a quantum condition of zero total angular momentum. It is well known that, even though the 3 angular-momentum generators do not commute with each other, a quantum state may be a simultaneous eigenvector of angular-momentum components if the eigenvalue is zero for all three.

Because Hilbert-space vectors have finite norm, the quantum-theoretic notion of zero total momentum for the universe is problematic, but not the classical notion. In DQC certain self-adjoint-operator expectations over a cosmological ray play a central role, defining ‘mundane reality’ and specifying arc energy. We postulate zero expectation for all 3 total-universe-momentum operators—a condition sustained by DQC dynamics.

9. Energy-Momentum Unirrep Basis; Parity Reflection

Returning attention to an *individual* preon, the set of 4 self-adjoint Hilbert-space operators, $J_{3L}^\wedge, K_{3L}^\wedge, J_{3R}^\wedge, K_{3R}^\wedge$ (out of a 12-generator algebra) not only commute with each other but also with the pair of Casimir self adjoint operators—*bilinears* of group-algebra elements that commute with *all* group-algebra elements. The pair of G-N unirrep-basis labels m and ρ together specify Casimir eigenvalues. These two labels attach to *any* unirrep, including the energy-momentum unirrep basis, whose 6 labels we now address. The associated 6-csco supplements the above generator quartet by the Casimir pair.

The symbols m, ρ, n and ω , for four energy-momentum unirrep-basis labels, have already been introduced—specifying, respectively, the eigenvalues of the Casimir pair and of J_{3L}^\wedge and K_{3L}^\wedge . A meaning for n and ω with respect to preon velocity has been achieved in the path basis through left-group action on a velocity eigenvector. Appendix B shows how these two labels are unitarily equivalent to the complex label z' on the G-N

unirrep basis. The two remaining labels on the energy-unirrep basis are the eigenvalues of J_{3R}^\wedge and K_{3R}^\wedge , which Appendix B shows are equivalent to the complex label z_1 .

The self-adjoint operator J_{3R}^\wedge generates rotations about some (arbitrarily-assigned) external 3-direction while K_{3R}^\wedge generates spatial displacements in this direction. The eigenvalues of J_{3R}^\wedge span the same range as those of J_{3L}^\wedge and will be denoted by using the *right* integer symbol m_3 to accompany the *left* integer symbol n . (In both cases a factor $1/2$ connects integer symbol to generator eigenvalue.) Similarly the *right* continuous symbol p_3 (spanning the real line and, with a factor $1/2$, designating the eigenvalue of K_{3R}^\wedge) will accompany the *left* continuous symbol ω . The complete set of energy-unirrep-basis labels is then $n, \omega, m, \rho, m_3, p_3$.

[Our attachment of the subscript 3 to right-generator eigenvalues emphasizes the *arbitrariness* of the designated *external* direction. The *left* labels n and ω refer to the preon's velocity direction. It is because Pauli arbitrarily used the subscript 3 to designate a *diagonal*, rather than antidiagonal, 2×2 hermitian traceless self-inverse matrix that this subscript in Formulas (3) and (6) became associated to the velocity direction. By ignoring Pauli, G-N avoided notational awkwardness.]

One immediate application of the energy-unirrep basis is to define 'preon parity reflection' as sign reversal for n, m and p_3 with ω, ρ and m_3 unchanged. Helicities and momentum direction reverse, while energy, momentum magnitude and angular momentum are unchanged. Implied is velocity-direction reversal. Another immediate application is to the definition of DQC Hilbert-space *sectors*.

10. Path-Arc-Defined Hilbert-Space Sectors

Any preon shares with a path-contacting arc certain *spacetime-independent* ('extra-Lorentz') discrete labels, mostly ignored by the present paper, that in *any* of the 4 DQC bases define preon *sectors* for photons, gravitons, leptons, quarks and weak bosons of 3 generations, 3 colors, 2 weak isospins, 2 chiralities and 2 values of boson handedness. The latter path label divides a photon or graviton sector into a pair of subsectors. (Although gluons are unrepresented in the DQC Hilbert space, certain path arcs that never contact a ray carry gluonic labels.) The DQC Fock space is constrained so that (path-defined) Sector has, in the energy-momentum unirrep basis, a *unique* (twice) helicity $n = m = n$ —i.e., equal components of angular momentum in velocity *and* momentum directions. Sector then needs only 4 energy-momentum unirrep-basis labels ω, ρ, m_3 and p_3 —one discrete (m_3) and three continuous (ω, ρ, p_3) 'spacetime' labels—each of the continuous trio spanning some portion of the real line.

The labels m_3 and p_3 determine, respectively, the components of preon angular momentum and momentum in the 3-direction and not only depend on the choice of this direction but are altered by right (Milne) transformations. We have above discussed how a preon's momentum-angular-momentum 6-vector behaves under right transformation. The two other labels, ω and ρ , are *invariant* under the 6-parameter group of (Milne-interpretable) *right* Lorentz transformations—rotations and spatial displacements—as well

as under a (sector-changing) parity reversal that changes the signs of n and p_3 . We have called (*half* the values of) ω and ρ , respectively, the preon ‘energy’ and ‘momentum magnitude’ in τ -dependent units—both these terms to be understood in the preon’s local frame.

11. Positive-Energy Elementary-Particle Preon Wave Functions

Certain special preon wave functions with sharply-defined *positive* energy are interpretable as Standard-Model elementary particles or gravitons. There is qualitative *difference* between a preon wave function that represents a photon or graviton and a preon wave function that represents any *other* type of elementary particle—which we characterize as ‘massive’.

A photon or graviton is a state reachable by right transformation of a single-preon state that in the energy-momentum unirrep basis has $\omega = \rho = p_3$ and $n = m = m_3 (= n)$. An arbitrary (fixed) positive energy specifies a Hilbert subspace in one of four discrete preon sectors— $n = \pm 2, \pm 4$ —each such sector-subspace representing the right Lorentz group (of Milne transformations). The subspace is isomorphic to normed functions over the complex z_0 plane with the measure dz_0 and an origin corresponding to the foregoing special state. The most general ‘massless particle’ is a superposition of states separately labeled by n , ω , and z_0 , where z_0 designates a direction with respect to an externally-specified direction.

For a massive elementary particle *at rest* (in local frame) the ‘spin-directed’ preon wave function with $m_3 = |n|$ is parity symmetric—with *zero* expectation for momentum, velocity and helicity and with (non-fluctuating) $\omega > 0$. For a Standard-Model massive particle the expectation of (fluctuating) ρ is at Planck-mass scale—hugely larger than the particle-mass scale of (non-fluctuating) ω . Right rotations generate the other m_3 values without changing the foregoing collection of zero expectations. The fluctuating velocity and momentum directions are *not* parallel to each other although rigidly correlated by the value of particle mass—tending to be almost orthogonal for $\omega \ll \rho$. (DQC elementary-particle rest mass does not require Higgs scalars.)

Einstein boosts—transformations in Milne local frame from at-rest to moving massive-particle wave functions—have been specified in the ssSM physics approximation to DQC [6]. The status of Einstein-Poincaré boost as an *approximate* symmetry of the universe at macroscopic and particle scales requires further study. Reference (6) finds essential here the dilation invariance of non-gravitational, non-inertial action.

12. Mundane Reality

The cosmological ray at the lower boundary of some spacetime slice prescribes within that slice, through *expectations* of certain self-adjoint operators, nonfluctuating (‘classical’) current densities of conserved electric charge and energy-momentum together with associated electromagnetic and gravitational (Lorentz-gauge) ‘tethered’ potentials that

specify the *non-radiation* electromagnetic and gravitational action for the Feynman paths traversing this slice. (Path-action-contributing vector and tensor classical *radiation* potentials will be addressed by another paper.) We refer to these current densities, which enjoy one to one correspondence with tethered potentials—hence the adjective ‘tethered’—as ‘mundane reality’. Heisenberg uncertainty is *absent* from DQC—dispensing need for ‘many-worlds’ interpretation.

DQC prescription of the electromagnetic tethered vector potential (in Lorentz gauge) employs a quotient of real polynomials of the 6-csco whose continuous eigenvalues label the path basis. As in our treatment above of these eigenvalues, it proves notationally economical to represent the continuous-product Preon-*i* 6-csco by a unimodular 2×2 complex matrix—a matrix \hat{a}_i of 8 commuting self-adjoint operators, two of which are determined by the remaining six. (The 12, 21 and 22 matrix elements may be regarded as the 6 independent operators, with the 11 matrix element an operator pair determined by the *complete* operator sextet.)

Consider a pair of self-adjoint hermitian 2×2 matrix operators $\hat{\mathbf{x}}_i^\tau$ and $\hat{\mathbf{v}}_i$ which have the real eigenvalues represented by the hermitian 2×2 matrices (2) and (3). That is, the 8 self-adjoint commuting operators represented by the symbols $\hat{\mathbf{x}}_i^\tau$ and $\hat{\mathbf{v}}_i$ are real bilinears of the 8 self-adjoint commuting operators represented by the symbol $\hat{\mathbf{a}}_i$. The operator bilinear coefficients are the same as those of the eigenvalue bilinears. Similarity transformation of $\hat{\mathbf{x}}_i^\tau$ and $\hat{\mathbf{v}}_i$ by our unitary representation of the right Lorentz group allows these operators to be recognized as a pair of commuting Milne 4-vectors whose timelike components are positive. It follows further, if the symbol \bullet denotes the Lorentz inner product of two 4-vectors, that $\hat{\mathbf{x}}_i^\tau \bullet \hat{\mathbf{x}}_i^\tau = \tau^2 I^\wedge$, where I^\wedge is the *unit*-operator (commuting with *all* operators), and $\hat{\mathbf{v}}_i \bullet \hat{\mathbf{v}}_i = 0$. That is, the 4-vector $\hat{\mathbf{x}}_i^\tau$ is positive timelike, while $\hat{\mathbf{v}}_i$ is positive lightlike.

We define a (right-Lorentz) 4-vector Preon-*i* self-adjoint operator $\hat{\mathbf{A}}_i^\tau(\mathbf{x})$ that depends on the preon’s electric charge Q_i and on a spacetime-location 4-vector \mathbf{x} , with $\mathbf{x} \bullet \mathbf{x} > \tau^2$, according to the following operator quotient of a 4-vector and a scalar,

$$\hat{\mathbf{A}}_i^\tau(\mathbf{x}) \equiv Q_i \hat{\mathbf{v}}_i / \hat{\mathbf{v}}_i \bullet (\mathbf{x} I^\wedge - \hat{\mathbf{x}}_i^\tau). \quad (31)$$

For \mathbf{x} within the spacetime slice following the exceptional age τ , we call this operator the ‘electromagnetic vector potential tethered to Preon *i*’. The expectation $\langle \hat{\mathbf{A}}_i^\tau(\mathbf{x}) \rangle$ of (31) over the cosmological ray at age τ is the contribution from Preon *i* to the (full, Lorentz-gauge) tethered vector potential $\tau(\mathbf{x})$ —the sum over *i* of $\langle \hat{\mathbf{A}}_i^\tau(\mathbf{x}) \rangle$. Because the preon-velocity 4-vectors $\hat{\mathbf{v}}_i$ are lightlike, the 4-divergence vanishes for the spacetime-dependent vector-potential-operator (31), as well as for the τ -ray expectation thereof and for the full (sum over *i*) classical tethered vector potential.

The classical tethered electromagnetic potential contributes (often importantly) to the action of Feynman paths traversing the near future of the τ -ray. Further, the D’Alembertian of (non-fluctuating) $\tau(\mathbf{x})$ gives the (divergence-less) 4-current density of conserved electric charge within the slice—one component of DQC ‘mundane reality’.

A Preon-*i*-generated tethered gravitational-tensor potential may analogously be spec-

ified through a discrete-spectrum symmetric-tensor pseudo-self-adjoint operator [9] that parallels (31) but is proportional to the energy rather than to the electric charge of Preon i . A separate paper will define pseudo-self-adjoint single-preon operators corresponding to a positive-lightlike energy-momentum 4-vector and to the thereby-generated symmetric-tensor tethered gravitational potential. Summing the potential operator over all preons and taking the τ -ray expectation gives the classical tethered gravitational potential within the near future of this ray. The gravitational potential's D'Alembertian divided by gravitational constant is the current density of conserved energy-momentum within the slice.

Notice that the photons and gravitons of a cosmological ray contribute to the energy-momentum tensor although not to the electric-charge current density. The former (tensor) current density and the latter (vector) current density, both conserved, collectively comprise what we call 'mundane reality'.

Conclusion

Explored has been a Hilbert space for preons—elementary quantum-theoretic entities in Dirac sense that populate Milne spacetime. Our proposal, based on the right-left Lorentz group rather than the Poincaré group, is suitable for quantum cosmology although not immediately for physics. Our Hilbert space is defined only at macroscopically-spaced exceptional universe ages. The results here support the gravitational-action proposals of Reference (2) and suggest a sliced-spacetime Higgs-less revision of the particle-physics Standard Model—the revised model being related to the present paper's cosmological theory by group contraction, as reported in Reference (6).

Appendix A: A Pair of Real 3-Vector Path-Basis Labels

The most general (6 real-parameter) coordinate matrix of Preon i or Arc i may be expressed, rather than by Formula (6) of the main text, through a 3-vector boost β_i together with a 3-vector rotation χ_i ($0 \leq |\chi_i| \leq 4\pi$) such that the coordinate matrix is

$$\mathbf{a}_i = \exp(i\boldsymbol{\sigma} \cdot \boldsymbol{\chi}_i/2) \times \exp(-\boldsymbol{\sigma} \cdot \boldsymbol{\beta}_i/2). \quad (\text{A.1})$$

The right factor in (A.1) is a *hermitian* unimodular matrix while the left factor is a *unitary* unimodular matrix. (Any square matrix may be written as the product of a hermitian matrix and a unitary matrix.) The choice of matrix order in (A.1) correlates with the definition (2) in the main text of the preon's spacetime-location 4-vector. The 3-vector parameter β_i in (A.1) is the location in Milne boost space of Preon i or Arc i . When the 6-space origin is chosen to coincide in boost space with this location—i.e., when $\beta_i = 0$, the 3-vector χ_i appearing in (A.1), or an equivalent set of 3 angles,

$$0 \leq \theta_i \leq \pi, \quad 0 \leq \phi_i \leq 2\pi, \quad 0 \leq \phi_i' \leq 4\pi, \quad (\text{A.2})$$

specifies Preon-velocity direction by the polar coordinates θ_i , ϕ_i and transverse polarization direction by the angle $-\frac{1}{2}(\phi_i' + \phi_i)$. The relation between χ_i and θ_i , ϕ_i , ϕ_i' is

$$\exp(i\boldsymbol{\sigma} \cdot \boldsymbol{\chi}_i/2) = \exp(i\sigma_3\phi_i'/2) \exp(i\sigma_1\theta_i/2) \exp(i\sigma_3\phi_i/2). \quad (\text{A.3})$$

The invariant 6-dimensional volume element is $d\mathbf{a}_i = d\boldsymbol{\chi}_i d\boldsymbol{\beta}_i$ with

$$d\boldsymbol{\chi}_i = \sin\theta_i d\theta_i d\phi_i d\phi_i', \quad d\boldsymbol{\beta}_i = \sinh^2\beta_i d\beta_i d\mathbf{n}_i, \quad (\text{A.4})$$

the symbol $d\mathbf{n}$ standing for (2-dimensional) solid-angle element.

The parameterization (A.1) is convenient when relating 3 *different* arcs to each other at a Feynman-path vertex. The different arcs at a vertex share the same $\boldsymbol{\beta}$ but have independent $\boldsymbol{\chi}$. The G-N complex coordinates are convenient for arcs *between* (rather than *at*) vertices as well as for wave functions contacted by arcs.

Appendix B: Eigenvectors of the Energy-Momentum Csco.

Formula (25), specifying the outcome of right Lorentz transformation in the G-N unirrep basis, reveals in this basis the eigenfunctions of the two commuting right Lorentz-group generators $J_{3R} \hat{}$ and $K_{3R} \hat{}$. If $\Gamma_{12} = \Gamma_{21} = 0$ so that $\Gamma_{11} = \Gamma_{22}^{-1}$ —the right (Lorentz) transformation here being a boost in (some arbitrarily chosen) 3 direction combined with a rotation about this direction—we find

$$\Phi^\Gamma(\mathbf{b}) = |\Gamma_{22}|^2 \exp(im \arg \Gamma_{22} + i\rho \ln |\Gamma_{22}|) \Phi(m, \rho, z', \Gamma_{22}^2 z_1). \quad (\text{B.1})$$

Formula (B.1) determines both the eigenvalues and the eigenfunctions of the operators that generate right rotations and boosts with respect to the 3 direction. Writing $\Gamma_{22} = e^\varepsilon$ and $\Phi^\Gamma(\mathbf{b}) = \Phi_{m,\rho,z'}^\varepsilon(z_1)$, Formula (B.1) becomes

$$\Phi_{m,\rho,z'}^\varepsilon(z_1) = e^{2\text{Re}\varepsilon + im \text{Im}\varepsilon + i\rho \text{Re}\varepsilon} \Phi_{m,\rho,z'}(e^{2\varepsilon} z_1). \quad (\text{B.2})$$

A function of z_1 proportional to $|z_1|^{-1} \exp(i\ell_1 \arg z_1 + i\rho_1 \ln |z_1|)$ is transformed by (B.2) into itself multiplied by $\exp[i(m+2\ell_1)\text{Im}\varepsilon + i(\rho+2\rho_1)\text{Re}\varepsilon]$. Such a function (not normalizable but admissible in the sense of ‘rigged Hilbert space’)—an eigenvector of both $J_{3R} \hat{}$ and $K_{3R} \hat{}$ —is a state where components in the 3 direction of both angular momentum and of momentum are sharply defined. The eigenvalues of $J_{3R} \hat{}$ are seen to be $m/2 + \ell_1 \equiv m_3/2$ while those of $K_{3R} \hat{}$ are $\rho/2 + \rho_1 \equiv p_3/2$.

A function of z_1 that vanishes except in the neighborhood of $z_1 = 0$ may, according to (B.2), loosely be described as a momentum-*direction* eigenvector—with preon momentum direction in the externally-prescribed 3-direction. In this case Formula (B.2), with $\text{Re}\varepsilon = 0$, supports our calling $m/2$ ‘helicity’—component of angular momentum in the direction of momentum. But for this wave function *no* individual component of momentum is sharply defined.

The curvature of Milne 3-space renders non-commuting the trio of self-adjoint ‘right’-boost generators that represent the 3 components of a preon’s momentum. It is nevertheless reasonable to call $\rho/2$ the ‘magnitude of preon 3-momentum’ in a sense similar to the language which in nonrelativistic quantum theory calls the label j the ‘magnitude of angular momentum’. The relation between self-adjoint operators,

$$\mathbf{K}_R \hat{\bullet} \mathbf{J}_R \hat{=} (\rho \hat{/} 2)(m \hat{/} 2), \quad (\text{B.3})$$

is here helpful to keep in mind, together with the helicity interpretability of $m/2$.

Although the most general internal single-preon transformation of a ray from the *left*, $\Psi(\mathbf{a}_1, \mathbf{a}_2 \dots \mathbf{a}_i \dots) \rightarrow \Psi(\mathbf{a}_1, \mathbf{a}_2, \dots \gamma_i \mathbf{a}_i \dots)$, is less transparent than (24) when expressed in s_i, y_i, z_i coordinates, the transformation induced in the G-N unirrep basis is as simple as (25). The difference from (25) is that left transformation alters wave-function dependence on the “internal” parameter z'_i while dependence on preon helicity and momentum magnitude and direction (m_i, ρ_i, z_{1i}) is unchanged apart from an overall factor. Dropping the subscript i one finds, matching Formula (25),

$$\Phi(\mathbf{b}) \rightarrow {}^\gamma \Phi(\mathbf{b}) = \alpha_{m\rho} (-\gamma_{12} z' + \gamma_{11}) \Phi(m, \rho, {}^\gamma z', z_1), \quad (\text{B.4})$$

where

$$\gamma z' = (\gamma_{22} z' - \gamma_{21}) (-\gamma_{12} z' + \gamma_{11})^{-1}. \quad (\text{B.5})$$

Neither the magnitude nor the direction of momentum changes nor does helicity, as the coordinates (path-basis labels) of some preon *left* shift from one set of values to another (the coordinates of *other* preons remaining unshifted). Change in Preon i is manifested by change in the z'_i dependence of its G-N unirrep-basis wave function. Accompanying the earlier-considered special internal shift $\gamma = \lambda_0$, where λ_0 is the unimodular matrix $\exp(-\sigma_3 \ln \lambda_0)$ that prescribes an orientation rotation around velocity direction together with a longitudinal shift of preon location along velocity direction ($\gamma_{12} = \gamma_{21} = 0$ and $\gamma_{22} = \gamma_{11}^{-1} = \lambda_0$), one finds from (B.4),

$${}^\gamma \Phi(m, \rho, z', z_1) = \alpha_{m\rho}^{-1}(\lambda_0) \Phi(m, \rho, \lambda_0^2 z', z_1). \quad (\text{B.6})$$

Here, although momentum and helicity (together with wave-function norm) remain undisturbed, there is λ_0 “rescaling” of the variable z' .

Because a function $\Phi(m, \rho, z', z_1)$ with z' dependence of the same form as that discussed for z_1 , immediately following (B.2) above, is an eigenvector of $J_{3L} \hat{=}$ and $K_{3L} \hat{=}$, it is possible simultaneously to diagonalize the six self-adjoint operators whose eigenvalues are determined by the labels $n, \omega, m, \rho, m_3, p_3$ —the main-text labels of the energy-momentum unirrep basis.

Appendix C: Starting-Arc Energy at a Spacetime-Slice Lower Boundary

A ‘partial expectation’ of the discrete-spectrum Preon- i positive-lightlike 4-vector energy-momentum pseudo-self-adjoint operator over the ray at some exceptional age τ establishes the energy of a path arc that contacts this preon. ‘Partial expectation’ means at fixed values of $Im s_i, y_i, z_i$, as well as of all coordinates of all other preons. The arc in question is characterized by fixed values of $Im s_i, y_i, z_i$ but not of $Re s_i$.

Shifting local frame in passage along any arc means that local-frame preon energy decreases in inverse proportion to age as age increases (Milne redshift), but measurable redshift–over age intervals much longer than the macroscopic width of a spacetime slice–depends on Feynman-path dynamics. At each (cubic) path vertex the sum of the (1 or 2) ingoing ω values equals the sum of the (2 or 1) outgoing values, so total path invariant energy decreases in inverse proportion to the age increase between path start and finish. But Feynman’s formula for the ray at the upper slice boundary in terms of the ray at the lower boundary does not imply a transfer of finishing path total energy to the upper-boundary ray. Measurable redshift is not that of Milne.

Because the above-noted positive-energy-momentum-operator partial-expectation over a ray, as well as other path-action-influencing self-adjoint-operator expectations, depend *nonlinearly* on this ray, DQC ray propagation over more than one slice fails to be linear. Linearity lacks general macroscopic DQC significance. At each exceptional age there is exactly one universe ray and within each spacetime slice exactly one mundane reality. The special character of *photon* and *graviton* wave functions is nevertheless expected to allow *approximate* physical meaning for linear *super-macroscopic* propagation of electromagnetic and gravitational *radiation*.

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Nonholonomic Ricci Flows: Exact Solutions and Gravity

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Abstract: In a number of physically important cases, the nonholonomically (nonintegrable) constrained Ricci flows can be modelled by exact solutions of Einstein equations with nonhomogeneous (anisotropic) cosmological constants. We develop two geometric methods for constructing such solutions: The first approach applies the formalism of nonholonomic frame deformations when the gravitational evolution and field equations transform into systems of nonlinear partial differential equations which can be integrated in general form. The second approach develops a general scheme when one (two) parameter families of exact solutions are defined by any source-free solutions of Einstein's equations with one (two) Killing vector field(s). A successive iteration procedure results in a class of solutions characterized by an infinite number of parameters for a non-Abelian group involving arbitrary functions on one variable. We also consider nonlinear superpositions of some mentioned classes of solutions in order to construct more general integral varieties of the Ricci flow and Einstein equations depending on infinite number of parameters and three/four coordinates on four/ five dimensional (semi) Riemannian spaces.

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1. Introduction

In recent years, much work has been done on Ricci flow theory and fundamental problems in mathematics [1, 2, 3] (see [4, 5, 6] for reviews and references therein). In this context, a number of possible applications in modern gravity and mathematical physics were proposed, for instance, for low dimensional systems and gravity [7, 8, 9, 10] and black holes and cosmology [11, 12]. Such special cases were investigated following certain low dimensional or approximative solutions of the evolution equations.

There were also examined possible connections between Ricci flows, solitonic configurations and Einstein spaces [13, 14, 15]. In our works, we tackled the problem of constructing exact solutions in Ricci flow and gravity theories in a new way. Working with general (pseudo) Riemannian spaces and moving frames, we applied certain methods from the geometry of Finsler–Lagrange spaces and nonholonomic manifolds provided with nonlinear connection structure (N–connection) [16, 17, 18, 19].

Prescribing on a manifold some preferred systems of reference and symmetries, it is equivalent to define some nonintegrable (nonholonomic, equivalently, anholonomic) distributions with associated N–connections. From this point of view, of the geometry of so–called nonholonomic manifolds, it is possible to elaborate a unified formalism for locally fibred manifolds and vector/tangent bundles when the geometric constructions are adapted to the N–connection structure. We can consider different classes of metric and N–connection ansatz and model, for instance, a Finsler, or Lagrange, geometry in a (semi) Riemannian (in particular, Einstein) space. Inversely, we can define some effective Lagrangian, or Finsler like, fundamental functions for lifts of geometric objects for a theory of gravity to tangent bundles in order to elaborate a geometric mechanics model for such gravitational and/or gauge field interactions, see examples and details in Refs. [19, 20, 21, 22, 23]. It was also proved that constraining some classes of Ricci flows of (semi) Riemannian metrics we can model Finsler like geometries and, inversely, we can transform Finsler–Lagrange metrics and connections into Riemannian, or Riemann–Cartan ones [13, 14, 24].

The most important idea in constructing exact solutions by geometric methods is that we can consider such nonholonomic deformations of the frame and connection structures when the Cartan structure equations, Ricci flow and/or Einstein equations transform into systems of partial differential equations which can be integrated in general form, or one can be derived certain bi–Hamilton and solitonic equations with corresponding hierarchies and conservation laws, see [19, 15] and references therein.

The first examples of physically valuable exact solutions of nonholonomic Ricci flow evolution equations and gravitational field equations were constructed following the so–called anholonomic frame method [25, 26, 27]. We analyzed two general classes of solutions of evolution equations on time like and/or extra dimension coordinate (having certain nontrivial limits to exact solutions in gravity theories): The first class was elaborated for solitonic and pp–wave nonholonomic configurations. The second class was connected to a study of nonholonomic Ricci flow evolutions of three and four dimensional

(in brief, 3D and 4D) Taub–NUT metrics. Following those constructions and further geometric developments in Refs. [13, 14], we concluded that a number of important for physical considerations solutions of Ricci flow equations can be defined by nonholonomically generalized Einstein spaces with effective cosmological constant running on evolution parameter, or (for more general and/or normalized evolution flows) by ‘nonhomogeneous’ (locally anisotropic) cosmological constants.

This is the fourth paper in a series of works on nonholonomic Ricci flows modelled by nonintegrable constraints on the frame structure and evolution of metrics [13, 14, 15]. It is devoted to geometric methods of constructing generic off–diagonal exact solutions in gravity and Ricci flow theory.² The goal is to elaborate a general scheme when starting with certain classes of metrics, frames and connections new types of exact solutions are constructed following some methods from nonholonomic spaces geometry [20, 21, 22, 23, 18] and certain group ideas [31, 32]. The approach to generating vacuum Einstein metrics by parametric nonholonomic transforms was recently formulated in Ref. [33] (this article proposes a “Ricci flow development” of sections 2 and 3 in that paper). Such results seem to have applications in modern gravity and nonlinear physics: In the fifth partner paper [28], we show how nonholonomic Ricci flow evolution scenarios of physically valuable metrics can be modelled by parametric deformations of solitonic pp–waves and Schwarzschild solutions.

One should be noted that even there were found a large number of exact solutions in different models of gravity theory [29, 30, 19, 20, 21, 22, 23], and in certain cases in the Ricci flow theory [7, 8, 9, 10, 25, 26, 27, 15], one has been elaborated only a few general methods for generating new physical solutions from a given metric describing a real physical situation. For quantum fields, there were formulated some approximated approaches when (for instance, by using Feynman diagrams, the formalism of Green’s functions, or quantum integrals) the solutions are constructed to represent a linear or nonlinear prescribed physical situation. Perhaps it is unlikely that similar computation techniques can be elaborated in general form in gravity theories and related evolution equations. Nevertheless, certain new possibilities seem to be opened after formulation of the anholonomic frame method with parametric deformations for the Ricci flow theory. Although many of the solutions resulting from such methods have no obvious physical interpretation, one can be formulated some criteria selecting explicit classes of solutions with prescribed symmetries and physical properties.

The paper has the following structure: In section 2, we outline some results on nonholonomic manifolds and Ricci flows. Section 3 is devoted to the anholonomic frame method for constructing exact solutions of Einstein and Ricci flow equations. There are analyzed the conditions when such solutions define four and five dimensional foliations related to Einstein spaces and Ricci flows for the canonical distinguished connection and the Levi Civita connection. In section 4, we consider how various classes of metrics can be subjected to nonholonomic deformations and multi–parametric transforms (with Killing

² We shall follow the conventions from the first two partner works in the series; the reader is recommended to study them in advance.

symmetries) resulting in new classes of solutions of the Einstein/ Ricci flow equations. We consider different ansatz for metrics and two examples with multi-parametric families of Einstein spaces and related Ricci flow evolution models. The results are discussed in section 5. The reader is suggested to see Appendices before starting the main part of the paper: Appendix A outlines the geometry of nonlinear connections and the anholonomic frame method of constructing exact solutions. Appendix B summarizes some results on the parametric (Geroch) transforms of vacuum Einstein equations.

Notation remarks: It is convenient to use in parallel two types of denotations for the geometric objects subjected to Ricci flows by introducing "left-up" labels like ${}^x\gamma = \gamma(\dots, \chi)$. Different left-up labels will be also considered for some classes of metrics defining Einstein spaces, vacuum solutions and so on. We shall also write "boldface" symbols for geometric objects and spaces adapted to a nonholonomic (N-connection) structure, for instance, $\mathbf{V}, \mathbf{E}, \dots$. A nonholonomic distribution with associated N-connection structure splits the manifolds into conventional horizontal (h) and vertical (v) subspaces. The geometric objects, for instance, a vector \mathbf{X} can be written in abstract form $\mathbf{X} = (hX, vX)$, or in coefficient forms, $\mathbf{X}^\alpha = (X^i, X^a) = (X^{\underline{i}}, X^{\underline{a}})$, equivalently decomposed with respect to a general nonholonomic frame $\mathbf{e}_\alpha = (e_i, e_a)$ or coordinate frame $\partial_{\underline{\alpha}} = (\partial_{\underline{i}}, \partial_{\underline{a}})$ for local h- and v-coordinates $u = (x, y)$, or $u^\alpha = (x^i, y^a)$ when $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$ and $\partial_{\underline{i}} = \partial/\partial x^{\underline{i}}$, $\partial_{\underline{a}} = \partial/\partial x^{\underline{a}}$. The h-indices $i, j, k, \dots = 1, 2, \dots, n$ will be used for nonholonomic vector components and the v-indices $a, b, c, \dots = n+1, n+2, \dots, n+m$ will be used for holonomic vector components. Greek indices of type α, β, \dots will be used as cumulative ones. We shall omit labels, indices and parametric/ coordinate dependencies if it does not result in ambiguities.

2. Preliminaries

In this section we present some results on nonholonomic manifolds and Ricci flows [13, 14] selected with the aim to outline a new geometric method of constructing exact solutions. The anholonomic frame method and the geometry of nonlinear connections (N-connections) are considered, in brief, in Appendix A. The ideas on generating new solutions from one/ two Killing vacuum Einstein spacetimes [31, 32] are summarized in Appendix B.

2.1 Nonholonomic (pseudo) Riemannian Spaces

We consider a spacetime as a (necessary smooth class) manifold V of dimension $n+m$, when $n \geq 2$ and $m \geq 1$ (a splitting of dimensions being defined by a N-connection structure, see (A.3)). Such manifolds (equivalently, spaces) are provided with a metric, $g = g_{\alpha\beta} e^\alpha \otimes e^\alpha$, of any (pseudo) Euclidean signature and a linear connection $D = \{\Gamma_{\beta\gamma}^\alpha e^\beta\}$ satisfying the metric compatibility condition $Dg = 0$.³ The components of geometrical

³ in this work, the Einstein's summation rule on repeating "upper-lower" indices will be applied if the contrary will not be stated

objects, for instance, $g_{\alpha\beta}$ and $\Gamma_{\beta\gamma}^\alpha$, are defined with respect to a local base (frame) e_α and its dual base (co-base, or co-frame) e^α for which $e_\alpha \lrcorner e^\beta = \delta_\alpha^\beta$, where " \lrcorner " denotes the interior product induced by g and δ_α^β is the Kronecker symbol. For a local system of coordinates $u^\alpha = (x^i, y^a)$ on V (in brief, $u = (x, y)$), we write respectively

$$e_\alpha = (e_i = \partial_i = \frac{\partial}{\partial x^i}, e_a = \partial_a = \frac{\partial}{\partial y^a}) \text{ and } e^\beta = (e^j = dx^j, e^b = dy^b),$$

for $e_\alpha \lrcorner e^\tau = \delta_\alpha^\tau$; the indices run correspondingly values of type: $i, j, \dots = 1, 2, \dots, n$ and $a, b, \dots = n + 1, n + 2, \dots, n + m$ for any conventional splitting $\alpha = (i, a), \beta = (j, b), \dots$

Any local (vector) basis e_α and dual basis e^β can be decomposed with respect to other local bases $e_{\alpha'}$ and $e^{\beta'}$ by considering frame transforms,

$$e_\alpha = A_{\alpha'}^{\alpha'}(u)e_{\alpha'} \text{ and } e^\beta = A^{\beta'}_{\beta'}(u)e^{\beta'}, \quad (1)$$

where the matrix $A^{\beta'}_{\beta'}$ is the inverse to $A_{\alpha'}^{\alpha'}$. It should be noted that an arbitrary basis e_α is nonholonomic (equivalently, anholonomic) because, in general, it satisfies certain anholonomy conditions

$$e_\alpha e_\beta - e_\beta e_\alpha = W_{\alpha\beta}^\gamma e_\gamma \quad (2)$$

with nontrivial anholonomy coefficients $W_{\alpha\beta}^\gamma(u)$. For $W_{\alpha\beta}^\gamma = 0$, we get holonomic frames: for instance, if we fix a local coordinate basis, $e_\alpha = \partial_\alpha$.

Denoting the covariant derivative along a vector field $X = X^\alpha e_\alpha$ as $D_X = X \lrcorner D$, we can define the torsion

$$\mathcal{T}(X, Y) \doteq D_X Y - D_Y X - [X, Y], \quad (3)$$

and the curvature

$$\mathcal{R}(X, Y)Z \doteq D_X D_Y Z - D_Y D_X Z - D_{[X, Y]}Z, \quad (4)$$

tensors of connection D , where we use "by definition" symbol " \doteq " and $[X, Y] \doteq XY - YX$. The components $\mathcal{T} = \{T_{\beta\gamma}^\alpha\}$ and $\mathcal{R} = \{R_{\beta\gamma\tau}^\alpha\}$ are computed by introducing $X \rightarrow e_\alpha, Y \rightarrow e_\beta, Z \rightarrow e_\gamma$ into respective formulas (3) and (4).

The Ricci tensor is constructed $Ric(D) = \{R_{\beta\gamma} \doteq R_{\beta\gamma\alpha}^\alpha\}$. The scalar curvature R is by definition the contraction with $g^{\alpha\beta}$ (being the inverse to the matrix $g_{\alpha\beta}$), $R \doteq g^{\alpha\beta} R_{\alpha\beta}$, and the Einstein tensor is $\mathcal{E} = \{E_{\alpha\beta} \doteq R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\}$. The vacuum (source-free) Einstein equations are

$$\mathcal{E} = \{E_{\alpha\beta} = R_{\alpha\beta}\} = 0. \quad (5)$$

In general relativity theory, one chooses a connection $D = \nabla$ which is uniquely defined by the coefficients of a metric, $g_{\alpha\beta}$, following the conditions of metric compatibility, $\nabla g = 0$, and of zero torsion, $\mathcal{T} = 0$. This is the so-called Levi Civita connection $\lrcorner D = \nabla$. We shall respectively label its curvature tensor, Ricci tensor, scalar curvature and Einstein tensor in the form $\lrcorner \mathcal{R} = \{\lrcorner R_{\beta\gamma\tau}^\alpha\}$, $\lrcorner Ric(\nabla) = \{\lrcorner R_{\alpha\beta} \doteq \lrcorner R_{\beta\gamma\alpha}^\alpha\}$, $\lrcorner R \doteq g^{\alpha\beta} \lrcorner R_{\alpha\beta}$ and $\lrcorner \mathcal{E} = \{\lrcorner E_{\alpha\beta}\}$.

Modern gravity theories consider extra dimensions and connections with nontrivial torsion. For instance, in string gravity [35, 36], the torsion coefficients are induced by the

so-called anti-symmetric H -fields and contain additional information about additional interactions in low-energy string limit. A more special class of gravity interactions are those with effective torsion when such fields are induced as a nonholonomic frame effect in a unique form (3) by prescribing a nonholonomic distribution (A.3), defining a nonlinear connection structure, N-connection, on a (pseudo) Riemannian manifold \mathbf{V} enabled with a metric structure (A.9). Such spaces with local fibred structure are called nonholonomic (in more special cases, when the nonholonomy is defined by a N-connection structures, the manifolds are called N-anholonomic) [17, 19]. The N-anholonomic spaces can be described in equivalent form by two linear connections ∇ (A.16) and $\widehat{\mathbf{D}}$ (A.17), both metric compatible and completely stated by a metric (A.10), equivalently (A.8).⁴ As a matter of principle, the general relativity theory can be formulated in terms of both connections, ∇ and $\widehat{\mathbf{D}}$; the last variant being with nonholonomic constraints on geometrical objects. One must be emphasized that the standard approach follows the formulation of gravitational field equations just for the Einstein tensor ${}_{\mathcal{E}}$ for ∇ which, in general, is different from the Einstein tensor $\widehat{\mathbf{E}}$ for $\widehat{\mathbf{D}}$.

A surprising thing found in our works is that for certain classes of generic off-diagonal metric ansatz (A.9) it is possible to construct exact solutions in general form by using the connection $\widehat{\mathbf{D}}$ but not the connection ∇ . Here we note that having defined certain integral varieties for a first class of linear connections we can impose some additional constraints and generate solutions for a class of Levi Civita connections, for instance, in the Einstein and string, or Finsler like, generalizations of gravity. Following a geometric N-adapted formalism (the so-called anholonomic frame method), such solutions were constructed and studied in effective noncommutative gravity [18], various locally anisotropic (Finsler like and more general ones) extensions of the Einstein and Kaluza–Klein theory, in string an brane gravity [20, 21, 23] and for Lagrange–Fedosov manifolds [34], see a summary in [19].

The anholonomic frame method also allows us to construct exact solutions in general relativity: One defines a more general class of solutions for $\widehat{\mathbf{D}}$ and then imposes certain subclasses of nonholonomic constraints when such solutions solve the four dimensional Einstein equations for ∇ . Here we note that by nonholonomic deformations we were able to study nonholonomic Ricci flows of certain classes of physically valuable exact solutions like solitonic pp-waves [25] and Taub NUT spaces [26, 27]. In this work, we develop the approach by applying new group methods.

Certain nontrivial limits to the vacuum Einstein gravity can be selected if we impose on the nonholonomic structure such constraints when

$$\mathbf{E} = {}_{\mathcal{E}} \quad (6)$$

even, in general, $\mathbf{D} \neq \nabla$. We shall consider such conditions when \mathbf{D} and ∇ have the same components with respect to certain preferred bases and the equality (6) can be satisfied

⁴ In this work, we shall use only ∇ and the canonical d-connection $\widehat{\mathbf{D}}$ and, for simplicity, we shall omit "hat" writing \mathbf{D} if that will not result in ambiguities.

for some very general classes of metric ansatz.⁵

We shall use left–up labels ” \circ ” or ” λ ” for a metric,

$$\circ g = \circ g_{\alpha\beta} e^\alpha \otimes e^\beta \text{ or } \lambda g = \lambda g_{\alpha\beta} e^\alpha \otimes e^\beta$$

being (correspondingly) a solution of the vacuum Einstein (or with cosmological constant) equations $\mathcal{E} = 0$ (5) or of the Einstein equations with a cosmological constant λ , $R_{\alpha\beta} = \lambda g_{\alpha\beta}$, for a linear connection D with possible torsion $\mathcal{T} \neq 0$. In order to emphasize that a metric is a solution of the vacuum Einstein equations, in any dimension $n + m \geq 3$, for the Levi Civita connection ∇ , we shall write

$$\circ g = \circ g_{\alpha\beta} e^\alpha \otimes e^\beta \text{ or } \lambda g = \lambda g_{\alpha\beta} e^\alpha \otimes e^\beta,$$

where the left–low label ” \circ ” will distinguish the geometric objects for the Ricci flat space defined by a Levi Civita connection ∇ .

Finally, in this section, we note that we shall use ”boldface” symbols, for instance, if $\lambda \mathbf{g} = \lambda \mathbf{g}_{\alpha\beta} \mathbf{e}^\alpha \otimes \mathbf{e}^\beta$ defines a nonholonomic Einstein space as a solution of

$$\mathbf{R}_{\alpha\beta} = \lambda \mathbf{g}_{\alpha\beta} \tag{7}$$

for the canonical d–connection \mathbf{D} .

2.2 Evolution Equations for Nonholonomic Ricci Flows

The normalized (holonomic) Ricci flows [3, 4, 5, 6] for a family of metrics $\underline{g}_{\alpha\beta}(\chi) = \underline{g}_{\alpha\beta}(u^\mu, \chi)$, parametrized by a real parameter χ , with respect to the coordinate base $\underline{\partial}_\alpha = \partial/\partial u^\alpha$, are described by the equations

$$\frac{\partial}{\partial \chi} \underline{g}_{\alpha\beta} = -2 \circ R_{\alpha\beta} + \frac{2r}{5} \underline{g}_{\alpha\beta}, \tag{8}$$

where the normalizing factor $r = \int \circ R dV/dV$ is introduced in order to preserve the volume V .⁶

⁵ We emphasize that different linear connections may be subjected to different rules of frame and coordinate transforms. It should be noted here that tensors and nonlinear and linear connections transform in different ways under frame and coordinate changing on manifolds with locally fibred structures, see detailed discussions in [16, 19].

⁶ The Ricci flow evolution equations were introduced by R. Hamilton [1], as evolution equations

$$\frac{\partial \underline{g}_{\alpha\beta}(\chi)}{\partial \chi} = -2 \circ \underline{R}_{\alpha\beta}(\chi),$$

for a set of Riemannian metrics $\underline{g}_{\alpha\beta}(\chi)$ and corresponding Ricci tensors $\circ \underline{R}_{\alpha\beta}(\chi)$ parametrized by a real χ (we shall underline symbols or indices in order to emphasize that certain geometric objects/ equations are given with the components defined with respect to a coordinate basis). For our further purposes, on generalized Riemann–Finsler spaces, it is convenient to use a different system of denotations than those considered by R. Hamilton or Grisha Perelman on holonomic Riemannian spaces.

For N–anholonomic Ricci flows, the coefficients $g_{\underline{\alpha}\underline{\beta}}$ are parametrized in the form (A.9), see proofs and discussion in Refs. [13, 25, 26, 27]. With respect to N–adapted frames (A.4) and (A.5), the Ricci flow equations (8), redefined for $\nabla \rightarrow \widehat{\mathbf{D}}$ and, respectively, ${}_{\perp}R_{\alpha\beta} \rightarrow \widehat{\mathbf{R}}_{\alpha\beta}$ are

$$\frac{\partial}{\partial\chi}g_{ij} = 2 \left[N_i^a N_j^b \left(\widehat{\mathbf{R}}_{ab} - \lambda g_{ab} \right) - \widehat{\mathbf{R}}_{ij} + \lambda g_{ij} \right] - g_{cd} \frac{\partial}{\partial\chi} (N_i^c N_j^d), \quad (9)$$

$$\frac{\partial}{\partial\chi}g_{ab} = -2 \left(\widehat{\mathbf{R}}_{ab} - \lambda g_{ab} \right), \quad (10)$$

$$\widehat{\mathbf{R}}_{ia} = 0 \text{ and } \widehat{\mathbf{R}}_{ai} = 0, \quad (11)$$

where $\lambda = r/5$ the Ricci coefficients $\widehat{\mathbf{R}}_{ij}$ and $\widehat{\mathbf{R}}_{ab}$ are computed with respect to coordinate coframes. The equations (11) constrain the nonholonomic Ricci flows to result in symmetric metrics.⁷

Nonholonomic deformations of geometric objects (and related systems of equations) on a N–anholonomic manifold \mathbf{V} are defined for the same metric structure \mathbf{g} by a set of transforms of arbitrary frames into N–adapted ones and of the Levi Civita connection ∇ into the canonical d–connection $\widehat{\mathbf{D}}$, locally parametrized in the form

$$\partial_{\underline{\alpha}} = (\partial_{\underline{i}}, \partial_{\underline{a}}) \rightarrow \mathbf{e}_{\alpha} = (\mathbf{e}_i, \mathbf{e}_a); \quad g_{\underline{\alpha}\underline{\beta}} \rightarrow [g_{ij}, g_{ab}, N_i^a]; \quad {}_{\perp}\Gamma^{\gamma}_{\alpha\beta} \rightarrow \widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}.$$

A rigorous proof for nonholonomic evolution equations is possible following a N–adapted variational calculus for the Perelman’s functionals presented in Refs. [14]. For a five dimensional space with diagonal d–metric ansatz (A.10), when $g_{ij} = \text{diag}[\pm 1, g_2, g_3]$ and $g_{ab} = \text{diag}[g_4, g_5]$, we considered [25] the nonholonomic evolution equations

$$\frac{\partial}{\partial\chi}g_{ii} = -2 \left[\widehat{\mathbf{R}}_{ii} - \lambda g_{ii} \right] - g_{cc} \frac{\partial}{\partial\chi} (N_i^c)^2, \quad (12)$$

$$\frac{\partial}{\partial\chi}g_{aa} = -2 \left(\widehat{\mathbf{R}}_{aa} - \lambda g_{aa} \right), \quad (13)$$

$$\widehat{\mathbf{R}}_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta, \quad (14)$$

with the coefficients defined with respect to N–adapted frames (A.4) and (A.5). This system can be transformed into a similar one, like (9)–(11), by nonholonomic deformations.

3. Off–Diagonal Exact Solutions

We consider a five dimensional (5D) manifold \mathbf{V} of necessary smooth class and conventional splitting of dimensions $\dim \mathbf{V} = n + m$ for $n = 3$ and $m = 2$. The local coordinates are labelled in the form $u^{\alpha} = (x^i, y^a) = (x^1, x^{\widehat{i}}, y^4 = v, y^5)$, for $i = 1, 2, 3$ and $\widehat{i} = 2, 3$ and $a, b, \dots = 4, 5$. Any coordinates from a set u^{α} can be a three dimensional (3D) space, time, or extra dimension (5th) one. Ricci flows of geometric objects will be parametrized by a real χ .

⁷ In Refs. [13, 24], we discuss this problem related to the fact that the tensor $\widehat{\mathbf{R}}_{\alpha\beta}$ is not symmetric which results, in general, in Ricci flows of nonsymmetric metrics.

3.1 Off–diagonal Ansatz for Einstein Spaces and Ricci Flows

The ansatz of type (A.10) is parametrized in the form

$$\begin{aligned} \mathbf{g} &= g_1 dx^1 \otimes dx^1 + g_2(x^2, x^3) dx^2 \otimes dx^2 + g_3(x^2, x^3) dx^3 \otimes dx^3 \\ &\quad + h_4(x^k, v) \delta v \otimes \delta v + h_5(x^k, v) \delta y \otimes \delta y, \\ \delta v &= dv + w_i(x^k, v) dx^i, \quad \delta y = dy + n_i(x^k, v) dx^i \end{aligned} \quad (15)$$

with the coefficients defined by some necessary smooth class functions

$$\begin{aligned} g_1 &= \pm 1, g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^i, v), \\ w_i &= w_i(x^i, v), n_i = n_i(x^i, v). \end{aligned}$$

The off–diagonal terms of this metric, written with respect to the coordinate dual frame $du^\alpha = (dx^i, dy^a)$, can be redefined to state a N–connection structure $\mathbf{N} = [N_i^4 = w_i(x^k, v), N_i^5 = n_i(x^k, v)]$ with a N–elongated co–frame (A.5) parametrized as

$$\begin{aligned} e^1 &= dx^1, \quad e^2 = dx^2, \quad e^3 = dx^3, \\ \mathbf{e}^4 &= \delta v = dv + w_i dx^i, \quad \mathbf{e}^5 = \delta y = dy + n_i dx^i. \end{aligned} \quad (16)$$

This coframe is dual to the local basis

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - w_i(x^k, v) \frac{\partial}{\partial v} - n_i(x^k, v) \frac{\partial}{\partial y^5}, \quad e_4 = \frac{\partial}{\partial v}, \quad e_5 = \frac{\partial}{\partial y^5}. \quad (17)$$

We emphasize that the metric (15) does not depend on variable y^5 , i.e. it posses a Killing vector $e_5 = \partial/\partial y^5$, and distinguishes the dependence on the so–called ”anisotropic” variable $y^4 = v$.

The above considered ansatz and formulas can be generalized in order to model Ricci flows,

$$\begin{aligned} {}^x\mathbf{g} &= g_1 dx^1 \otimes dx^1 + g_2(x^2, x^3, \chi) dx^2 \otimes dx^2 + g_3(x^2, x^3, \chi) dx^3 \otimes dx^3 \\ &\quad + h_4(x^k, v, \chi) \quad {}^x\delta v \otimes {}^x\delta v + h_5(x^k, v, \chi) \quad {}^x\delta y \otimes {}^x\delta y, \\ {}^x\delta v &= dv + w_i(x^k, v, \chi) dx^i, \quad {}^x\delta y = dy + n_i(x^k, v, \chi) dx^i \end{aligned} \quad (18)$$

with corresponding flows for N–adapted bases,

$$\begin{aligned} \mathbf{e}_\alpha &= (\mathbf{e}_i, e_a) \rightarrow {}^x\mathbf{e}_\alpha = ({}^x\mathbf{e}_i, e_a) = \mathbf{e}_\alpha(\chi) = (\mathbf{e}_i(\chi), e_a), \\ \mathbf{e}^\alpha &= (e^i, \mathbf{e}^a) \rightarrow {}^x\mathbf{e}^\alpha = (e^i, {}^x\mathbf{e}^a) = \mathbf{e}^\alpha(\chi) = (e^i, \mathbf{e}^a(\chi)) \end{aligned}$$

defined by $w_i(x^k, v) \rightarrow w_i(x^k, v, \lambda)$, $n_i(x^k, v) \rightarrow n_i(x^k, v, \lambda)$ in (17), (16).

Computing the components of the Ricci and Einstein tensors for the metric (18) (see main formulas in Appendix and details on tensors components’ calculus in Refs. [18, 19]), one proves that the corresponding family of Ricc tensors for the canonical d–connection with respect to N–adapted frames are compatible with the sources (they can be any matter fields, string corrections, Ricci flow parameter derivatives of metric, ...)

$$\begin{aligned} \Upsilon_\beta^\alpha &= [\Upsilon_1^1 = \Upsilon_2 + \Upsilon_4, \Upsilon_2^2 = \Upsilon_2(x^2, x^3, v, \chi), \Upsilon_3^3 = \Upsilon_2(x^2, x^3, v, \chi), \\ \Upsilon_4^4 &= \Upsilon_4(x^2, x^3, \chi), \Upsilon_5^5 = \Upsilon_4(x^2, x^3, \chi)] \end{aligned} \quad (19)$$

transform into this system of partial differential equations:

$$R_2^2 = R_3^3(\chi) \quad (20)$$

$$= \frac{1}{2g_2g_3} \left[\frac{g_2^*g_3^*}{2g_2} + \frac{(g_3^*)^2}{2g_3} - g_3^{**} + \frac{g_2'g_3'}{2g_3} + \frac{(g_2')^2}{2g_2} - g_2'' \right] = -\Upsilon_4(x^2, x^3, \chi),$$

$$S_4^4 = S_5^5(\chi) = \frac{1}{2h_4h_5} \left[h_5^* \left(\ln \sqrt{|h_4h_5|} \right)^* - h_5^{**} \right] = -\Upsilon_2(x^2, x^3, v, \chi), \quad (21)$$

$$R_{4i} = -w_i(\chi) \frac{\beta(\chi)}{2h_5(\chi)} - \frac{\alpha_i(\chi)}{2h_5(\chi)} = 0, \quad (22)$$

$$R_{5i} = -\frac{h_5(\chi)}{2h_4(\chi)} [n_i^{**}(\chi) + \gamma(\chi)n_i^*(\chi)] = 0, \quad (23)$$

where, for $h_{4,5}^* \neq 0$,

$$\alpha_i(\chi) = h_5^*(\chi) \partial_i \phi(\chi), \quad \beta(\chi) = h_5^*(\chi) \phi^*(\chi), \quad (24)$$

$$\gamma(\chi) = \frac{3h_5^*(\chi)}{2h_5(\chi)} - \frac{h_4^*(\chi)}{h_4(\chi)}, \quad \phi(\chi) = \ln \left| \frac{h_5^*(\chi)}{\sqrt{|h_4(\chi)h_5(\chi)|}} \right|, \quad (25)$$

when the necessary partial derivatives are written in the form $a^\bullet = \partial a / \partial x^2$, $a' = \partial a / \partial x^3$, $a^* = \partial a / \partial v$. In the vacuum case, we must consider $\Upsilon_{2,4} = 0$. We note that we use a source of type (19) in order to show that the anholonomic frame method can be applied also for non-vacuum configurations, for instance, when $\Upsilon_2 = \lambda_2 = const$ and $\Upsilon_4 = \lambda_4 = const$, defining local anisotropies generated by an anisotropic cosmological constant, which in its turn, can be induced by certain ansatz for the so-called H -field (absolutely antisymmetric third rank tensor) in string theory [18, 19]. We note that the off-diagonal gravitational interactions and Ricci flows can model locally anisotropic configurations even if $\lambda_2 = \lambda_4$, or both values vanish.

Summarizing the results for an ansatz (15) with arbitrary signatures $\epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)$ (where $\epsilon_\alpha = \pm 1$) and $h_4^* \neq 0$ and $h_5^* \neq 0$, for a fixed value of χ , one proves, see details in [18, 19], that any off-diagonal metric

$$\begin{aligned} \circ \mathbf{g} = & \epsilon_1 dx^1 \otimes dx^1 + \epsilon_2 g_2(\hat{x}^i) dx^2 \otimes dx^2 + \epsilon_3 g_3(\hat{x}^i) dx^3 \otimes dx^3 \\ & + \epsilon_4 h_0^2(x^i) [f^*(x^i, v)]^2 |_{\varsigma}(x^i, v) | \delta v \otimes \delta v \\ & + \epsilon_5 [f(x^i, v) - f_0(x^i)]^2 \delta y^5 \otimes \delta y^5, \\ \delta v = & dv + w_k(x^i, v) dx^k, \quad \delta y^5 = dy^5 + n_k(x^i, v) dx^k, \end{aligned} \quad (26)$$

with the coefficients being of necessary smooth class and the indices with "hat" running the values $\hat{i}, \hat{j}, \dots = 2, 3$, where $g_{\hat{k}}(\hat{x}^{\hat{i}})$ is a solution of the 2D equation (20) for a given source $\Upsilon_4(\hat{x}^{\hat{i}})$,

$$\varsigma(x^i, v) = \varsigma_{[0]}(x^i) - \frac{\epsilon_4}{8} h_0^2(x^i) \int \Upsilon_2(x^{\hat{k}}, v) f^*(x^i, v) [f(x^i, v) - f_0(x^i)] dv,$$

and the N-connection coefficients $N_i^4 = w_i(x^k, v)$, $N_i^5 = n_i(x^k, v)$ are computed following

the formulas

$$w_i = -\frac{\partial_i \varsigma(x^k, v)}{\varsigma^*(x^k, v)} \quad (27)$$

$$n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{[f^*(x^i, v)]^2}{[f(x^i, v) - f_0(x^i)]^3} \varsigma(x^i, v) dv, \quad (28)$$

define an exact solution of the system of Einstein equations (7). It should be emphasized that such solutions depend on arbitrary functions $f(x^i, v)$, with $f^* \neq 0$, $f_0(x^i)$, $h_0^2(x^i)$, $\varsigma_{[0]}(x^i)$, $n_{k[1]}(x^i)$, $n_{k[2]}(x^i)$ and $\Upsilon_2(x^{\hat{k}}, v)$, $\Upsilon_4(x^{\hat{i}})$. Such values for the corresponding signatures $\epsilon_\alpha = \pm 1$ have to be stated by certain boundary conditions following some physical considerations.⁸

The ansatz of type (15) with $h_4^* = 0$ but $h_5^* \neq 0$ (or, inversely, $h_4^* \neq 0$ but $h_5^* = 0$) consist more special cases and request a bit different methods for constructing exact solutions. Nevertheless, such solutions are also generic off-diagonal and they may be of substantial interest (the length of paper does not allow us to include an analysis of such particular cases).

3.2 Generalization of Solutions for Ricci Flows

For families of solutions parametrized by χ , we consider flows of the generating functions, $g_2(x^i, \chi)$, or $g_3(x^i, \chi)$, and $f(x^i, v, \chi)$, and various types of integration functions and sources, for instance, $n_{k[1]}(x^i, \chi)$ and $n_{k[2]}(x^i, \chi)$ and $\Upsilon_2(x^{\hat{k}}, v, \chi)$, respectively, in formulas (27) and (28). Let us analyze an example of exact solutions of equations (12)–(14):

We search a class of solutions of with

$$\begin{aligned} g_2 &= \epsilon_2 \varpi(x^2, x^3, \chi), g_3 = \epsilon_3 \varpi(x^2, x^3, \chi), \\ h_4 &= h_4(x^2, x^3, v), h_5 = h_5(x^2, x^3, v), \end{aligned}$$

for a family of ansatz (18) with any prescribed signatures $\epsilon_\alpha = \pm 1$ and non-negative functions ϖ and h . Following a tensor calculus, adapted to the N-connection, for the canonical d-connection,⁹ we express the integral variety for a class of nonholonomic Ricci flows as

$$\begin{aligned} \epsilon_2 (\ln |\varpi|)^{\bullet\bullet} + \epsilon_3 (\ln |\varpi|)'' &= 2\lambda - h_5 \partial_\chi (n_2)^2, \\ h_4 &= h \varsigma_4 \end{aligned} \quad (29)$$

⁸ Our classes of solutions depending on integration functions are more general than those for diagonal ansatz depending, for instance, on one radial like variable like in the case of the Schwarzschild solution (when the Einstein equations are reduced to an effective nonlinear ordinary differential equation, ODE). In the case of ODE, the integral varieties depend on integration constants to be defined from certain boundary/ asymptotic and symmetry conditions, for instance, from the constraint that far away from the horizon the Schwarzschild metric contains corrections from the Newton potential. Because our ansatz (15) transforms (7) in a system of nonlinear partial differential equations transforms, the solutions depend not on integration constants but on integration functions.

⁹ similar computations are given in [18] and Chapter 10 of [19]

for

$$\begin{aligned} \varsigma_4(x^2, x^3, v) &= \varsigma_{4[0]}(x^2, x^3) - \frac{\lambda}{4} \int \frac{hh_5}{h_5^*} dv \\ \sqrt{|h|} &= h_{[0]}(x^i) \left(\sqrt{|h_5(x^2, x^3, v)|} \right)^* \end{aligned} \quad (30)$$

and, for $\varphi = -\ln \left| \sqrt{|h_4 h_5|} / |h_5^*| \right|$,

$$\begin{aligned} w_1 &= 0, w_2 = (\varphi^*)^{-1} \varphi^\bullet, w_3 = (\varphi^*)^{-1} \varphi', \\ n_1 &= 0, n_2 = n_3 = n_{[1]}(x^2, x^3, \chi) + n_{[2]}(x^2, x^3, \chi) \int dv h_4 / \left(\sqrt{|h_5|} \right)^3, \end{aligned} \quad (31)$$

where the partial derivatives are denoted in the form $\varphi^\bullet = \partial\varphi/\partial x^2$, $\varphi' = \partial\varphi/\partial x^3$, $\varphi^* = \partial\varphi/\partial v$, $\partial_\chi = \partial/\partial x^2$, and arbitrary h_5 when $h_5^* \neq 0$. For $\lambda = 0$, we shall consider $\varsigma_{4[0]} = 1$ and $h_{[0]}(x^i) = \text{const}$ in order to solve the vacuum Einstein equations. There is a class of solutions when

$$h_5 \int dv h_4 / \left(\sqrt{|h_5|} \right)^3 = C(x^2, x^3),$$

for a function $C(x^2, x^3)$. This is compatible with the condition (30), and we can chose such configurations, for instance, with $n_{[1]} = 0$ and any $n_{[2]}(x^2, x^3, \chi)$ and $\varpi(x^2, x^3, \chi)$ solving the equation (29).

Putting together (29)–(31), we get a class of solutions of the system (12)–(14) (the equations being expressed equivalently in the form (20)–(23)) for nonholonomic Ricci flows of metrics of type (18),

$$\begin{aligned} {}^x\mathbf{g} &= \epsilon_1 dx^1 \otimes dx^1 + \varpi(x^2, x^3, \chi) [\epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3] \\ &\quad + h_4(x^2, x^3, v) \delta v \otimes \delta v + h_5(x^2, x^3, v) {}^x\delta y \otimes {}^x\delta y, \\ \delta v &= dv + w_2(x^2, x^3, v) dx^2 + w_3(x^2, x^3, v) dx^3, \\ {}^x\delta y &= dy + n_2(x^2, x^3, v, \chi) [dx^2 + dx^3]. \end{aligned} \quad (32)$$

Such solutions describe in general form the Ricci flows of nonholonomic Einstein spaces constrained to relate in a mutually compatible form the evolution of horizontal part of metric, $\varpi(x^2, x^3, \chi)$, with the evolution of N–connection coefficients $n_2 = n_3 = n_2(x^2, x^3, v, \chi)$. We have to impose certain boundary/ initial conditions for $\chi = 0$, beginning with an explicit solution of the Einstein equations, in order to define the integration functions and state an evolution scenario for such classes of metrics and connections.

3.3 4D and 5D Einstein Foliations and Ricci Flows

The method of constructing 5D solutions can be restricted to generate 4D nonholonomic configurations and generic off–diagonal solutions in general relativity. In order to consider reductions $5D \rightarrow 4D$ for the ansatz (15), we can eliminate from the formulas the variable x^1 and consider a 4D space \mathbf{V}^4 (parametrized by local coordinates (x^2, x^3, v, y^5)) trivially embedded into a 5D spacetime \mathbf{V} (parametrized by local coordinates (x^1, x^2, x^3, v, y^5))

with $g_{11} = \pm 1, g_{1\hat{\alpha}} = 0, \hat{\alpha} = 2, 3, 4, 5$). In this case, there are possible 4D conformal and anholonomic transforms depending only on variables (x^2, x^3, v) of a 4D metric $g_{\hat{\alpha}\hat{\beta}}(x^2, x^3, v)$ of arbitrary signature. To emphasize that some coordinates are stated just for a 4D space we might use "hats" on the Greek indices, $\hat{\alpha}, \hat{\beta}, \dots$ and on the Latin indices from the middle of the alphabet, $\hat{i}, \hat{j}, \dots = 2, 3$; local coordinates on \mathbf{V}^4 are parametrized $u^{\hat{\alpha}} = (x^{\hat{i}}, y^a) = (x^2, x^3, y^4 = v, y^5)$, for $a, b, \dots = 4, 5$. The ansatz

$$\mathbf{g} = g_2 dx^2 \otimes dx^2 + g_3 dx^3 \otimes dx^3 + h_4 \delta v \otimes \delta v + h_5 \delta y^5 \otimes \delta y^5, \quad (33)$$

is written with respect to the anholonomic co-frame $(dx^{\hat{i}}, \delta v, \delta y^5)$, where

$$\delta v = dv + w_{\hat{i}} dx^{\hat{i}} \text{ and } \delta y^5 = dy^5 + n_{\hat{i}} dx^{\hat{i}} \quad (34)$$

is the dual of $(\delta_{\hat{i}}, \partial_4, \partial_5)$, for

$$\delta_{\hat{i}} = \partial_{\hat{i}} + w_{\hat{i}} \partial_4 + n_{\hat{i}} \partial_5, \quad (35)$$

and the coefficients are necessary smoothly class functions of type:

$$\begin{aligned} g_{\hat{j}} &= g_{\hat{j}}(x^{\hat{k}}), h_{4,5} = h_{4,5}(x^{\hat{k}}, v), \\ w_{\hat{i}} &= w_{\hat{i}}(x^{\hat{k}}, v), n_{\hat{i}} = n_{\hat{i}}(x^{\hat{k}}, v); \hat{i}, \hat{k} = 2, 3. \end{aligned}$$

In the 4D case, a source of type (19) should be considered without the component Υ_1^1 in the form

$$\Upsilon_{\hat{\beta}}^{\hat{\alpha}} = \text{diag}[\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_2(x^{\hat{k}}, v), \Upsilon_4^4 = \Upsilon_5^5 = \Upsilon_4(x^{\hat{k}})]. \quad (36)$$

The Einstein equations with sources of type (36) for the canonical d-connection (A.16) defined by the ansatz (33) transform into a system of nonlinear partial differential equations very similar to (20)–(23). The difference for the 4D equations is that the coordinate x^1 is not contained into the equations and that the indices of type $i, j, \dots = 1, 2, 3$ must be changed into the corresponding indices $\hat{i}, \hat{j}, \dots = 2, 3$. The generated classes of 4D solutions are defined almost by the same formulas (26), (27) and (28).

Now we describe how the coefficients of an ansatz (33) defining an exact vacuum solution for a canonical d-connection can be constrained to generate a vacuum solution in Einstein gravity: We start with the conditions (A.20) written (for our ansatz) in the form

$$\frac{\partial h_4}{\partial x^{\hat{k}}} - w_{\hat{k}} h_4^* - 2w_{\hat{k}}^* h_4 = 0, \quad (37)$$

$$\frac{\partial h_5}{\partial x^{\hat{k}}} - w_{\hat{k}} h_5^* = 0, \quad (38)$$

$$n_{\hat{k}}^* h_5 = 0. \quad (39)$$

These equations for nontrivial values of $w_{\hat{k}}$ and $n_{\hat{k}}$ constructed for some defined values of h_4 and h_5 must be compatible with the equations (21)–(23) for $\Upsilon_2 = 0$. One can be taken nonzero values for $w_{\hat{k}}$ in (22) if and only if $\alpha_{\hat{i}} = 0$ because the the equation (21)

imposes the condition $\beta = 0$. This is possible, for the sourceless case and $h_5^* \neq 0$, if and only if

$$\phi = \ln \left| h_5^* / \sqrt{|h_4 h_5|} \right| = \text{const}, \quad (40)$$

see formula (25). A very general class of solutions of equations (37), (38) and (40) can be represented in the form

$$\begin{aligned} h_4 &= \epsilon_4 h_0^2 (b^*)^2, h_5 = \epsilon_5 (b + b_0)^2, \\ w_{\hat{k}} &= (b^*)^{-1} \frac{\partial(b + b_0)}{\partial x^{\hat{k}}}, \end{aligned} \quad (41)$$

where $h_0 = \text{const}$ and $b = b(x^{\hat{k}}, v)$ is any function for which $b^* \neq 0$ and $b_0 = b_0(x^{\hat{k}})$ is an arbitrary integration function.

The next step is to satisfy the integrability conditions (A.18) defining a foliated space-times provided with metric and N-connection and d-connection structures [18, 19, 34] (we note that (pseudo) Riemannian foliations are considered in a different manner in Ref. [17]) for the so-called Schouten – Van Kampen and Vranceanu connections not subjected to the condition to generate Einstein spaces). It is very easy to show that there are nontrivial solutions of the constraints (A.18) which for the ansatz (33) are written in the form

$$\begin{aligned} w'_2 - w_3^\bullet + w_3 w_2^* - w_2 w_3^* &= 0, \\ n'_2 - n_3^\bullet + w_3 n_2^* - w_2 n_3^* &= 0. \end{aligned} \quad (42)$$

We solve these equations for $n_2^* = n_3^* = 0$ if we take any two functions $n_{2,3}(x^{\hat{k}})$ satisfying

$$n'_2 - n_3^\bullet = 0 \quad (43)$$

(this is possible by a particular class of integration functions in (28) when $n_{\hat{k}[2]}(x^{\hat{i}}) = 0$ and $n_{\hat{k}[1]}(x^{\hat{i}})$ are constraint to satisfy just the conditions (43)). Then we can consider any function $b(x^{\hat{k}}, v)$ for which $w_{\hat{k}} = (b^*)^{-1} \partial_{\hat{k}}(b + b_0)$ solve the equation (42). In a more particular case, one can be constructed solutions for any $b(x^3, v)$, $b^* \neq 0$, and $n_2 = 0$ and $n_3 = n_3(x^3, v)$ (or, inversely, for any $n_2 = n_2(x^2, v)$ and $n_3 = 0$). We also note that the conditions (A.19) are solved in a straightforward form by the ansatz (33).

We conclude that for any sets of coefficients

$$h_4(x^{\hat{k}}, v), h_5(x^{\hat{k}}, v), w_{\hat{k}}(x^{\hat{k}}, v), n_{2,3}(x^{\hat{k}})$$

respectively generated by functions $b(x^{\hat{k}}, v)$ and $n_{\hat{k}[1]}(x^{\hat{i}})$, see (41), and satisfying (43), the generic off-diagonal metric (33) possess the same coefficients both for the Levi Civita and canonical d-connection being satisfied the conditions (6) of equality of the Einstein tensors. Here we note that any 2D metric can be written in a conformally flat form, i. e. we can chose such local coordinates when

$$g_2(dx^2)^2 + g_3(dx^3)^2 = e^{\psi(x^{\hat{i}})} \left[\epsilon_2(dx^{\hat{2}})^2 + \epsilon_3(dx^{\hat{3}})^2 \right],$$

for signatures $\epsilon_{\hat{k}} = \pm 1$, in (33).

Summarizing the results of this section, we can write down the generic off–diagonal metric (it is a 4D dimensional reduction of (26))

$$\begin{aligned} \circ\mathbf{g} &= e^{\psi(x^2, x^3)} [\epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3] \\ &+ \epsilon_4 h_0^2 [b^*(x^i, v)]^2 \delta v \otimes \delta v \\ &+ \epsilon_5 [b(x^2, x^3, v) - b_0(x^2, x^3)]^2 \delta y^5 \otimes \delta y^5, \\ \delta v &= dv + w_2(x^2, x^3, v) dx^2 + w_3(x^2, x^3, v) dx^3, \\ \delta y^5 &= dy^5 + n_2(x^2, x^3) dx^2 + n_3(x^2, x^3) dx^3, \end{aligned} \quad (44)$$

defining vacuum exact solutions in general relativity if the coefficients are restricted to solve the equations

$$\begin{aligned} \epsilon_2 \psi^{\bullet\bullet} + \epsilon_3 \psi'' &= 0, \\ w'_2 - w_3^\bullet + w_3 w_2^* - w_2 w_3^* &= 0, \\ n'_2 - n_3^\bullet &= 0, \end{aligned} \quad (45)$$

for $w_2 = (b^*)^{-1} (b + b_0)^\bullet$ and $w_3 = (b^*)^{-1} (b + b_0)'$, where, for instance, $n_3^\bullet = \partial_2 n_3$ and $n'_2 = \partial_3 n_2$.

We can generalize (44) similarly to (26) in order to generate solutions for nontrivial sources (36). In general, they will contain nontrivial anholonomically induced torsions. Such configurations may be restricted to the case of Levi Civita connection by solving the constraints (37)–(39) in order to be compatible with the equations (21) and (22) for the coefficients $\alpha_{\hat{k}}$ and β computed for $h_5^* \neq 0$ and $\ln |h_5^*/\sqrt{|h_4 h_5|}| = \phi(x^2, x^3, v) \neq const$, see formula (25), resulting in more general conditions than (40) and (41). Roughly speaking, all such coefficients are generated by any h_4 (or h_5) defined from (22) for prescribed values h_5 (or h_5) and $\Upsilon_2(x^{\hat{k}}, v)$. The existence of a nontrivial matter source of type (36) does not change the condition $n_{\hat{k}}^* = 0$, see (39), necessary for extracting torsionless configurations. This mean that we have to consider only trivial solutions of (23) when two functions $n_{\hat{k}} = n_{\hat{k}}(x^2, x^3)$ are subjected to the condition (42). We conclude that this class of exact solutions of the Einstein equations with nontrivial sources (36), in general relativity, is defined by the ansatz

$$\begin{aligned} \circ\mathbf{g} &= e^{\psi(x^2, x^3)} [\epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3] + \\ &h_4(x^2, x^3, v) \delta v \otimes \delta v + h_5(x^2, x^3, v) \delta y^5 \otimes \delta y^5, \\ \delta v &= dv + w_2(x^2, x^3, v) dx^2 + w_3(x^2, x^3, v) dx^3, \\ \delta y^5 &= dy^5 + n_2(x^2, x^3) dx^2 + n_3(x^2, x^3) dx^3, \end{aligned} \quad (46)$$

where the coefficients satisfy the conditions

$$\begin{aligned} \epsilon_2 \psi^{\bullet\bullet} + \epsilon_3 \psi'' &= \Upsilon_2 \\ h_5^* \phi / h_4 h_5 &= \Upsilon_2, \\ w'_2 - w_3^\bullet + w_3 w_2^* - w_2 w_3^* &= 0, \\ n'_2 - n_3^\bullet &= 0, \end{aligned} \quad (47)$$

for $w_i = \partial_i \phi / \phi^*$, see (25), being compatible with (37) and (38), for given sources $\Upsilon_4(x^{\hat{k}})$ and $\Upsilon_2(x^k, v)$. We emphasize that the second equation in (47) relates two functions h_4 and h_5 . In references [20, 21, 22, 23, 18], we investigated a number of configurations with nontrivial two and three dimensional solitons, reductions to the Riccati or Abbel equation, defining off–diagonal deformations of the black hole, wormhole or Taub NUT spacetimes. Those solutions were constructed to be with trivial or nontrivial torsions but if the coefficients of the ansatz (46) are restricted to satisfy the conditions (47) in a compatible form with (37) and (38), for sure, such metrics will solve the Einstein equations for the Levi Civita connection. We emphasize that the ansatz (46) defines Einstein spaces with a cosmological constant λ if we put $\Upsilon_2 = \Upsilon_4 = \lambda$ in (47).

Let us formulate the conditions when families of metrics (46) subjected to the conditions (47) will define exact solutions of the Ricci flows of usual Einstein spaces (for the Levi Civita connection). We consider the ansatz

$$\begin{aligned} \lambda \mathbf{g}(\chi) &= e^{\psi(x^2, x^3, \chi)} [\epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3] + \\ &h_4(x^2, x^3, v, \chi) \delta v \otimes \delta v + h_5(x^2, x^3, v, \chi) \delta y^5 \otimes \delta y^5, \\ \delta v &= dv + w_2(x^2, x^3, v) dx^2 + w_3(x^2, x^3, v) dx^3, \\ \delta y^5 &= dy^5 + n_2(x^2, x^3, \chi) [dx^2 + dx^3], \end{aligned} \quad (48)$$

which is a subfamily of (32), when $\varpi = e^{\psi(x^2, x^3, \chi)}$ and $n_2 = n_3$ does not depend on variable v and the coefficients satisfy the conditions (29) and (30), when $n_{[2]} = 0$ but $n_{[1]}$ can be nontrivial in (31), and (additionally)

$$\begin{aligned} \epsilon_2 \psi^{\bullet\bullet}(\chi) + \epsilon_3 \psi''(\chi) &= \lambda, \\ h_5^* \phi / h_4 h_5 &= \lambda, \\ w_2' - w_3^\bullet + w_3 w_2^* - w_2 w_3^* &= 0, \\ n_2'(\chi) - n_2^\bullet(\chi) &= 0, \end{aligned} \quad (49)$$

for $w_i = \partial_i \phi / \phi^*$, see (25), being compatible with (37) and (38), for given sources $\Upsilon_4 = \lambda$ and $\Upsilon_2 = \lambda$. The family of metrics (48) define a self–consistent evolution as a class of general solutions of the Ricci flow equations (12)–(14) transformed equivalently in the form (20)–(23). The additional constraints (49) define an integral subvariety (foliation) of (32) when the evolution is selected for the Levi Civita connection.

4. Nonholonomic and Parametric Transforms

Anholonomic deformations can be defined for any primary metric and frame structures on a spacetime \mathbf{V} (as a matter of principle, the primary metric can be not a solution of the gravitational field equations). Such deformations always result in a target spacetime possessing one Killing vector symmetry if the last one is constrained to satisfy the vacuum Einstein equations for the canonical d–connection, or for the Levi Civita connection. For such target spacetimes, we can always apply a parametric transform and generate a set of generic off–diagonal solutions labelled by a parameter θ (B.2). There are

possible constructions when the anholonomic frame transforms are applied to a family of metrics generated by the parametric method as new exact solutions of the vacuum Einstein equations, but such primary metrics have to be parametrized by certain type ansatz admitting anholonomic transforms to other classes of exact solutions. Additional constraints and parametrizations are necessary for generating exact solutions of holonomic or nonholonomic Ricci flow equations.

4.1 Deformations and Frame Parametrizations

Let us consider a $(n + m)$ -dimensional manifold (spacetime) \mathbf{V} , $n \geq 2, m \geq 1$, enabled with a metric structure $\check{\mathbf{g}} = \check{g} \oplus_N \check{h}$ distinguished in the form

$$\begin{aligned} \check{\mathbf{g}} &= \check{g}_i(u)(dx^i)^2 + \check{h}_a(u)(\check{\mathbf{c}}^a)^2, \\ \check{\mathbf{c}}^a &= dy^a + \check{N}_i^a(u)dx^i. \end{aligned} \quad (50)$$

The local coordinates are parametrized $u = (x, y) = \{u^\alpha = (x^i, y^a)\}$, for the indices of type $i, j, k, \dots = 1, 2, \dots, n$ (in brief, horizontal, or h-indices/ components) and $a, b, c, \dots = n + 1, n + 2, \dots, n + m$ (vertical, or v-indices/ components). We suppose that, in general, the metric (50) is not a solution of the Einstein equations but can be nonholonomically deformed in order to generate exact solutions. The coefficients $\check{N}_i^a(u)$ from (50) state a conventional $(n + m)$ -splitting \oplus_N in any point $u \in \mathbf{V}$ and define a class of 'N-adapted' local bases

$$\check{\mathbf{e}}_\alpha = \left(\check{\mathbf{e}}_i = \frac{\partial}{\partial x^i} - \check{N}_i^a(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right) \quad (51)$$

and local dual bases (co-frames) $\check{\mathbf{c}} = (c, \check{c})$, when

$$\check{\mathbf{c}}^\alpha = (c^j = dx^j, \check{\mathbf{c}}^b = dy^b + \check{N}_i^b(u) dx^i), \quad (52)$$

for $\check{\mathbf{c}} \rfloor \check{\mathbf{e}} = \mathbf{I}$, i.e. $\check{\mathbf{e}}_\alpha \rfloor \check{\mathbf{c}}^\beta = \delta_\alpha^\beta$, where the inner product is denoted by ' \rfloor ' and the Kronecker symbol is written δ_α^β . The frames (51) satisfy the nonholonomy (equivalently, anholonomy) relations

$$\check{\mathbf{e}}_\alpha \check{\mathbf{e}}_\beta - \check{\mathbf{e}}_\beta \check{\mathbf{e}}_\alpha = \check{\mathbf{w}}_{\alpha\beta}^\gamma \check{\mathbf{e}}_\gamma$$

with nontrivial anholonomy coefficients

$$\begin{aligned} \check{\mathbf{w}}_{ji}^a &= -\check{\mathbf{w}}_{ij}^a = \check{\Omega}_{ij}^a \doteq \check{\mathbf{e}}_j (\check{N}_i^a) - \check{\mathbf{e}}_i (\check{N}_j^a), \\ \check{\mathbf{w}}_{ia}^b &= -\check{\mathbf{w}}_{ai}^b = e_a(\check{N}_j^b). \end{aligned} \quad (53)$$

A metric $\mathbf{g} = g \oplus_N h$ parametrized in the form

$$\begin{aligned} \mathbf{g} &= g_i(u)(c^i)^2 + g_a(u)(\mathbf{c}^a), \\ \mathbf{c}^a &= dy^a + N_i^a(u)dx^i \end{aligned} \quad (54)$$

is a nonholonomic transform (deformation), preserving the $(n + m)$ -splitting, of the metric, $\check{\mathbf{g}} = \check{g} \oplus_N \check{h}$ if the coefficients of (50) and (54) are related by formulas

$$g_i = \eta_i(u) \check{g}_i, \quad h_a = \eta_a(u) \check{h}_a \quad \text{and} \quad N_i^a = \eta_i^a(u) \check{N}_i^a, \quad (55)$$

where the summation rule is not considered for the indices of gravitational 'polarizations' $\eta_\alpha = (\eta_i, \eta_a)$ and η_i^a in (55). For nontrivial values of $\eta_i^a(u)$, the nonholonomic frames (51) and (52) transform correspondingly into

$$\mathbf{e}_\alpha = \left(\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right) \quad (56)$$

and

$$\mathbf{c}^\alpha = (c^j = dx^j, \mathbf{c}^a = dy^a + N_i^a(u) dx^i) \quad (57)$$

with the anholonomy coefficients $\mathbf{W}_{\alpha\beta}^\gamma$ defined by N_i^a (A.7).

We emphasize that in order to generate exact solutions, the gravitational 'polarizations' $\eta_\alpha = (\eta_i, \eta_a)$ and η_i^a in (55) are not arbitrary functions but restricted in a such form that the values

$$\begin{aligned} \pm 1 &= \eta_1(u^\alpha) \check{g}_1(u^\alpha), \\ g_2(x^2, x^3) &= \eta_2(u^\alpha) \check{g}_2(u^\alpha), \quad g_3(x^2, x^3) = \eta_3(u^\alpha) \check{g}_3(u^\alpha), \\ h_4(x^i, v) &= \eta_4(u^\alpha) \check{h}_4(u^\alpha), \quad h_5(x^i, v) = \eta_5(u^\alpha) \check{h}_5(u^\alpha), \\ w_i(x^i, v) &= \eta_i^4(u^\alpha) \check{N}_i^4(u^\alpha), \quad n_i(x^i, v) = \eta_i^5(u^\alpha) \check{N}_i^5(u^\alpha), \end{aligned} \quad (58)$$

define an ansatz of type (26), or (44) (for vacuum configurations) and (46) for nontrivial matter sources $\Upsilon_2(x^2, x^3, v)$ and $\Upsilon_4(x^2, x^3)$.

Any nonholonomic deformation

$$\check{\mathbf{g}} = \check{g} \oplus_{\check{N}} \check{h} \longrightarrow \mathbf{g} = g \oplus_N h \quad (59)$$

can be described by two frame matrices of type (A.1),

$$\check{\mathbf{A}}_\alpha^{\underline{\alpha}}(u) = \begin{bmatrix} \delta_i^{\underline{i}} - \check{N}_j^b \delta_b^{\underline{a}} \\ 0 & \delta_a^{\underline{a}} \end{bmatrix}, \quad (60)$$

generating the d-metric $\check{\mathbf{g}}_{\alpha\beta} = \check{\mathbf{A}}_\alpha^{\underline{\alpha}} \check{\mathbf{A}}_\beta^{\underline{\beta}} \check{g}_{\underline{\alpha}\underline{\beta}}$, see formula (A.11), and

$$\mathbf{A}_\alpha^{\underline{\alpha}}(u) = \begin{bmatrix} \sqrt{|\eta_i|} \delta_i^{\underline{i}} - \eta_i^a \check{N}_j^b \delta_b^{\underline{a}} \\ 0 & \sqrt{|\eta_a|} \delta_a^{\underline{a}} \end{bmatrix}, \quad (61)$$

generating the d-metric $\mathbf{g}_{\alpha\beta} = \mathbf{A}_\alpha^{\underline{\alpha}} \mathbf{A}_\beta^{\underline{\beta}} \check{g}_{\underline{\alpha}\underline{\beta}}$ (58).

If the metric and N-connection coefficients (55) are stated to be those from an ansatz (26) (or (44)), we should write ${}^\circ\mathbf{g} = g \oplus_N h$ (or ${}^\circ\mathbf{g} = g \oplus_N h$) and say that the metric $\check{\mathbf{g}} = \check{g} \oplus_{\check{N}} \check{h}$ (50) was nonholonomically deformed in order to generate an exact solution of the Einstein equations for the canonical d-connection (or, in a restricted case, for the Levi Civita connection). In general, such metrics have very different geometrical and physical properties. Nevertheless, at least for some classes of 'small' nonsingular nonholonomic deformations, it is possible to preserve a similar physical interpretation by introducing small polarizations of metric coefficients and deformations of existing horizons, not changing the singular structure of curvature tensors. Explicit examples are constructed in Ref. [28].

4.2 The Geroch Transforms as Parametric Nonholonomic Deformations

We note that any metric ${}^{\circ}g_{\alpha\beta}$ defining an exact solution of the vacuum Einstein equations can be represented in the form (50). Then, any metric ${}^{\circ}\tilde{g}_{\alpha\beta}(\theta)$ (B.2) from a family of new solutions generated by the first type parametric transform can be written as (54) and related via certain polarization functions of type (55), in the parametric case depending on parameter θ , i.e. $\eta_{\alpha}(\theta) = (\eta_i(\theta), \eta_a(\theta))$ and $\eta_i^a(\theta)$. Roughly speaking, any parametric transform can be represented as a generalized class of anholonomic frame transforms additionally parametrized by θ and adapted to preserve the $(n+m)$ -splitting structure.¹⁰ The corresponding frame transforms (B.5) and (B.6) are parametrized, respectively, by matrices of type (60) and (61), also "labelled" by θ . Such nonholonomic parametric deformations

$${}^{\circ}\mathbf{g} = {}^{\circ}g \oplus_{\tilde{N}} {}^{\circ}h \longrightarrow {}^{\circ}\tilde{\mathbf{g}}(\theta) = {}^{\circ}\tilde{g}(\theta) \oplus_{N(\theta)} {}^{\circ}\tilde{h}(\theta) \quad (62)$$

are defined by the frame matrices,

$${}^{\circ}\mathbf{A}_{\alpha}^{\underline{\alpha}}(u) = \begin{bmatrix} \delta_i^{\underline{i}} - {}^{\circ}N_j^b(u)\delta_b^{\underline{a}} \\ 0 & \delta_a^{\underline{a}} \end{bmatrix}, \quad (63)$$

generating the d-metric ${}^{\circ}\mathbf{g}_{\alpha\beta} = {}^{\circ}\mathbf{A}_{\alpha}^{\underline{\alpha}} {}^{\circ}\mathbf{A}_{\beta}^{\underline{\beta}} {}^{\circ}g_{\underline{\alpha}\underline{\beta}}$ and

$$\tilde{\mathbf{A}}_{\alpha}^{\underline{\alpha}}(u, \theta) = \begin{bmatrix} \sqrt{|\eta_i(u, \theta)|}\delta_i^{\underline{i}} - \eta_i^a(u, \theta) {}^{\circ}N_j^b(u)\delta_b^{\underline{a}} \\ 0 & \sqrt{|\eta_a(u, \theta)|}\delta_a^{\underline{a}} \end{bmatrix}, \quad (64)$$

generating the d-metric ${}^{\circ}\tilde{\mathbf{g}}_{\alpha\beta}(\theta) = \tilde{\mathbf{A}}_{\alpha}^{\underline{\alpha}} \tilde{\mathbf{A}}_{\beta}^{\underline{\beta}} {}^{\circ}g_{\underline{\alpha}\underline{\beta}}$. Using the matrices (63) and (64), we can compute the matrix of parametric transforms

$$\tilde{\mathbf{B}}_{\alpha}^{\alpha'} = \tilde{\mathbf{A}}_{\alpha}^{\underline{\alpha}} {}^{\circ}\mathbf{A}_{\underline{\alpha}}^{\alpha'}, \quad (65)$$

like in (B.7), but for "boldfaced" objects, where ${}^{\circ}\mathbf{A}_{\underline{\alpha}}^{\alpha'}$ is inverse to ${}^{\circ}\mathbf{A}_{\alpha'}^{\underline{\alpha}}$,¹¹ and define the target set of metrics in the form

$${}^{\circ}\tilde{\mathbf{g}}_{\alpha\beta} = \tilde{\mathbf{B}}_{\alpha}^{\alpha'}(u, \theta) \tilde{\mathbf{B}}_{\beta}^{\beta'}(u, \theta) {}^{\circ}\mathbf{g}_{\alpha'\beta'}.$$

There are two substantial differences from the case of usual anholonomic frame transforms (59) and the case of parametric deformations (62). The first one is that the metric $\check{\mathbf{g}}$ was not constrained to be an exact solution of the Einstein equations like it was required for ${}^{\circ}\mathbf{g}$. The second one is that even \mathbf{g} can be restricted to be an exact vacuum solution, generated by a special type of deformations (58), in order to get an ansatz of type (44), an arbitrary metric from a family of solutions ${}^{\circ}\tilde{\mathbf{g}}_{\alpha\beta}(\theta)$ will not be parametrized in a form

¹⁰ It should be emphasized that such constructions are not trivial, for usual coordinate transforms, if at least one of the primary or target metrics is generic off-diagonal.

¹¹ we use a "boldface" symbol because in this case the constructions are adapted to a $(n+m)$ -splitting

that the coefficients will satisfy the conditions (45). Nevertheless, even in such cases, we can consider additional nonholonomic frame transforms when $\check{\mathbf{g}}$ is transformed into an exact solution and any particular metric from the set $\{ {}^\circ\check{\mathbf{g}}_{\alpha\beta}(\theta) \}$ will be deformed into an exact solution defined by an ansatz (44) with additional dependence on θ .

By superpositions of nonholonomic deformations, we can parametrize a solution formally constructed following by the parametric method (from a primary solution depending on variables x^2, x^3) in the form

$$\begin{aligned} {}^\circ\check{\mathbf{g}}(\theta) &= e^{\psi(x^2, x^3, \theta)} [\epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3] \\ &\quad + \epsilon_4 h_0^2 [b^*(x^i, v, \theta)]^2 \delta v \otimes \delta v \\ &\quad + \epsilon_5 [b(x^2, x^3, v, \theta) - b_0(x^2, x^3, \theta)]^2 \delta y^5 \otimes \delta y^5, \\ \delta v &= dv + w_2(x^2, x^3, v, \theta) dx^2 + w_3(x^2, x^3, v, \theta) dx^3, \\ \delta y^5 &= dy^5 + n_2(x^2, x^3, \theta) dx^2 + n_3(x^2, x^3, \theta) dx^3, \end{aligned} \quad (66)$$

with the coefficients restricted to solve the equations (45) but depending additionally on parameter θ ,

$$\begin{aligned} \epsilon_2 \psi^{\bullet\bullet}(\theta) + \epsilon_3 \psi''(\theta) &= 0, \\ w_2'(\theta) - w_3^\bullet(\theta) + w_3 w_2^*(\theta) - w_2(\theta) w_3^*(\theta) &= 0, \\ n_2'(\theta) - n_3^\bullet(\theta) &= 0, \end{aligned} \quad (67)$$

for $w_2(\theta) = (b^*(\theta))^{-1} (b(\theta) + b_0(\theta))^\bullet$ and $w_3 = (b^*(\theta))^{-1} (b(\theta) + b_0(\theta))'$, where, for instance, $n_3^\bullet(\theta) = \partial_2 n_3(\theta)$ and $n_2' = \partial_3 n_2(\theta)$.

One should be noted that even, in general, any primary solution ${}^\circ\check{\mathbf{g}}$ can not be parametrized as an ansatz (44), it is possible to define nonholonomic deformations to a such type generic off-diagonal ansatz ${}^\circ\check{\mathbf{g}}$ or any $\check{\mathbf{g}}$, defined by an ansatz (50), which in its turn can be transformed into a metric of type (66) without dependence on θ .¹²

Finally, we emphasize that in spite of the fact that both the parametric and anholonomic frame transforms can be parametrized in very similar forms by using frame transforms there is a criteria distinguishing one from another: For a "pure" parametric transform, the matrix $\tilde{\mathbf{B}}_\alpha^{\alpha'}(u, \theta)$ and related $\tilde{\mathbf{A}}_\alpha^{\alpha'}$ and ${}^\circ\mathbf{A}_\alpha^{\alpha'}$ are generated by a solution of the Geroch equations (B.4). If the "pure" nonholonomic deformations, or their superposition with a parametric transform, are introduced into consideration, the matrix $\mathbf{A}_\alpha^{\alpha'}(u)$ (61), or its generalization to a matrix $\tilde{\mathbf{A}}_\alpha^{\alpha'}$ (64), can be not derived only from solutions of (B.4). Such transforms define certain, in general, nonintegrable distributions related to new classes of Einstein equations.

4.3 Two Parameter Transforms of Nonholonomic Solutions

As a matter of principle, any first type parameter transform can be represented as a generalized anholonomic frame transform labelled by an additional parameter. It should

¹²in our formulas we shall not point dependencies on coordinate variables if that will not result in ambiguities

be also noted that there are two possibilities to define superpositions of the parameter transforms and anholonomic frame deformations both resulting in new classes of exact solutions of the vacuum Einstein equations. In the first case, we start with a parameter transform and, in the second case, the anholonomic deformations are considered from the very beginning. The aim of this section is to examine such possibilities.

Firstly, let us consider an exact vacuum solution ${}^{\circ}\mathbf{g}$ (44) in Einstein gravity generated following the anholonomic frame method. Even it is generic off–diagonal and depends on various types of integration functions and constants, it is obvious that it possess at least a Killing vector symmetry because the metric does not depend on variable y^5 . We can apply the first type parameter transform to a such metric generated by anholonomic deforms (59). If we work in a coordinate base with the coefficients of ${}^{\circ}\mathbf{g}$ defined in the form ${}^{\circ}\underline{g}_{\alpha\beta} = {}^{\circ}g_{\underline{\alpha}\underline{\beta}}$, we generate a set of exact solutions

$${}^{\circ}\tilde{g}_{\alpha\beta}(\theta') = \tilde{B}_{\alpha}^{\alpha'}(\theta') \tilde{B}_{\alpha}^{\beta'}(\theta') {}^{\circ}\underline{g}_{\alpha'\beta'},$$

see (B.2), were the transforms (B.7), labelled by a parameter θ' , are not adapted to a nonholonomic $(n+m)$ –splitting. We can elaborate N–adapted constructions starting with an exact solution parametrized in the form (54), for instance, like ${}^{\circ}\mathbf{g}_{\alpha'\beta'} = \mathbf{A}_{\alpha'}^{\alpha} \mathbf{A}_{\beta'}^{\beta} \check{g}_{\underline{\alpha}\underline{\beta}}$, with $\mathbf{A}_{\alpha'}^{\alpha}$ being of type (61) with coefficients satisfying the conditions (58). The target 'boldface' solutions are generated as transforms

$${}^{\circ}\tilde{\mathbf{g}}_{\alpha\beta}(\theta') = \tilde{\mathbf{B}}_{\alpha}^{\alpha'}(\theta') \tilde{\mathbf{B}}_{\alpha}^{\beta'}(\theta') {}^{\circ}\mathbf{g}_{\alpha'\beta'}, \quad (68)$$

where

$$\tilde{\mathbf{B}}_{\alpha}^{\alpha'} = \tilde{\mathbf{A}}_{\alpha}^{\alpha'} {}^{\circ}\mathbf{A}_{\alpha'}^{\alpha'},$$

like in (B.7), but for "boldfaced" objects, the matrix ${}^{\circ}\mathbf{A}_{\alpha'}^{\alpha'}$ is inverse to

$${}^{\circ}\mathbf{A}_{\alpha'}^{\alpha'}(u) = \begin{bmatrix} \sqrt{|\eta_{i'}|} \delta_{i'}^i - \eta_{i'}^{b'} \tilde{N}_{j'}^{b'} \delta_{b'}^a & \\ 0 & \sqrt{|\eta_{a'}|} \delta_{a'}^a \end{bmatrix}$$

and there is considered the matrix

$$\tilde{\mathbf{A}}_{\alpha}^{\alpha'}(u, \theta') = \begin{bmatrix} \sqrt{|\eta_i \tilde{\eta}_i(\theta')|} \delta_{i'}^i - \eta_i^b \tilde{\eta}_i^b(\theta') \tilde{N}_j^b \delta_b^a & \\ 0 & \sqrt{|\eta_a \tilde{\eta}_a(\theta')|} \delta_a^a \end{bmatrix},$$

where $\tilde{\eta}_i(u, \theta')$, $\tilde{\eta}_a(u, \theta')$ and $\tilde{\eta}_i^a(u, \theta')$ are gravitational polarizations of type (55).¹³ Here it should be emphasized that even ${}^{\circ}\tilde{\mathbf{g}}_{\alpha\beta}(\theta')$ are exact solutions of the vacuum Einstein equations they can not be represented by ansatz of type (66), with $\theta \rightarrow \theta'$, because the mentioned polarizations were not constrained to be of type (58) and satisfy any conditions of type (67).¹⁴

¹³ we do not summarize on repeating two indices if they both are of lower/ upper type

¹⁴ As a matter of principle, we can deform nonholonomically any solution from the family ${}^{\circ}\tilde{\mathbf{g}}_{\alpha\beta}(\theta')$ to an ansatz of type (66).

Now we prove that by using superpositions of nonholonomic and parameter transforms we can generate two parameter families of solutions. This is possible, for instance, if the metric ${}^{\circ}\mathbf{g}_{\alpha'\beta'}$ form (68), in its turn, was generated as an ansatz of type (66), from another exact solution ${}^{\circ}\mathbf{g}_{\alpha''\beta''}$. We write

$${}^{\circ}\mathbf{g}_{\alpha'\beta'}(\theta) = \tilde{\mathbf{B}}_{\alpha'}^{\alpha''}(u, \theta) \tilde{\mathbf{B}}_{\beta'}^{\beta''}(u, \theta) {}^{\circ}\mathbf{g}_{\alpha''\beta''}$$

and define the superposition of transforms

$${}^{\circ}\tilde{\mathbf{g}}_{\alpha\beta}(\theta', \theta) = \tilde{\mathbf{B}}_{\alpha}^{\alpha'}(\theta') \tilde{\mathbf{B}}_{\alpha}^{\beta'}(\theta') \tilde{\mathbf{B}}_{\alpha'}^{\alpha''}(\theta) \tilde{\mathbf{B}}_{\beta'}^{\beta''}(\theta) {}^{\circ}\mathbf{g}_{\alpha''\beta''}. \quad (69)$$

It can be considered an iteration procedure of nonholonomic parameter transforms of type (69) when an exact vacuum solution of the Einstein equations is related via a multi θ -parameters frame map with another prescribed vacuum solution. Using anholonomic deformations, one introduces (into chains of such transforms) certain classes of metrics which are not exact solutions but nonholonomically deformed from, or to, some exact solutions.

4.4 Multi-parametric Einstein Spaces and Ricci Flows

Let us denote by $\overleftarrow{\theta} = ({}^k\theta = \theta', {}^2\theta, \dots, \theta = {}^1\theta,)$ a chain of nonholonomic parametric transforms (it can be more general as (69), beginning with an arbitrary metric $\mathbf{g}_{\alpha''\beta''}$) resulting in a metric $\tilde{\mathbf{g}}_{\alpha\beta}(\overleftarrow{\theta})$. Any step of nonholonomic parametric and/ or frame transforms are parametrize matrices of type (64), (65) or (68). Here, for simplicity, we consider two important examples when $\tilde{\mathbf{g}}_{\alpha\beta}(\overleftarrow{\theta})$ will generate solutions of the nonholonomic Einstein equations or Ricci flow equations.

4.4.1 Example 1:

We get a multi-parametric ansatz of type (15) with $h_4^* \neq 0$ and $h_5^* \neq 0$ if $\tilde{\mathbf{g}}_{\alpha\beta}(\overleftarrow{\theta})$ is of type

$$\begin{aligned} {}^{\circ}\mathbf{g}(\overleftarrow{\theta}) &= \epsilon_1 dx^1 \otimes dx^1 + \epsilon_2 g_2(\overleftarrow{\theta}, \hat{x}^i) dx^2 \otimes dx^2 \\ &\quad + \epsilon_3 g_3(\overleftarrow{\theta}, \hat{x}^i) dx^3 \otimes dx^3 \\ &\quad + \epsilon_4 h_0^2(\overleftarrow{\theta}, x^i) \left[f^* \left(\overleftarrow{\theta}, x^i, v \right) \right]^2 |_{\varsigma} \left(\overleftarrow{\theta}, x^i, v \right) | \delta v \otimes \delta v \\ &\quad + \epsilon_5 \left[f \left(\overleftarrow{\theta}, x^i, v \right) - f_0(\overleftarrow{\theta}, x^i) \right]^2 \delta y^5 \otimes \delta y^5, \quad (70) \\ \delta v &= dv + w_k \left(\overleftarrow{\theta}, x^i, v \right) dx^k, \quad \delta y^5 = dy^5 + n_k \left(\overleftarrow{\theta}, x^i, v \right) dx^k, \end{aligned}$$

the indices with "hat" running the values $\hat{i}, \hat{j}, \dots = 2, 3$, where $g_{\hat{k}}(\overleftarrow{\theta}, \hat{x}^i)$ are multi-parametric families of solutions of the 2D equation (20) for given sources $\Upsilon_4(\overleftarrow{\theta}, \hat{x}^i)$,

$$\begin{aligned} \varsigma \left(\overleftarrow{\theta}, x^i, v \right) &= \varsigma_{[0]} \left(\overleftarrow{\theta}, x^i \right) - \frac{\epsilon_4}{8} h_0^2(\overleftarrow{\theta}, x^i) \times \\ &\int \Upsilon_2(\overleftarrow{\theta}, x^{\hat{k}}, v) f^* \left(\overleftarrow{\theta}, x^i, v \right) \left[f \left(\overleftarrow{\theta}, x^i, v \right) - f_0(\overleftarrow{\theta}, x^i) \right] dv, \end{aligned}$$

and the N-connection $N_i^4 = w_i(\overleftarrow{\theta}, x^k, v)$, $N_i^5 = n_i(\overleftarrow{\theta}, x^k, v)$ computed

$$w_i(\overleftarrow{\theta}, x^k, v) = -\frac{\partial_i \varsigma(\overleftarrow{\theta}, x^k, v)}{\varsigma^*(\overleftarrow{\theta}, x^k, v)}, \quad (71)$$

$$n_k(\overleftarrow{\theta}, x^k, v) = n_{k[1]}(\overleftarrow{\theta}, x^i) + n_{k[2]}(\overleftarrow{\theta}, x^i) \times \int \frac{[f^*(\overleftarrow{\theta}, x^i, v)]^2}{[f(\overleftarrow{\theta}, x^i, v) - f_0(x^i)]^3} \varsigma(\overleftarrow{\theta}, x^i, v) dv, \quad (72)$$

define an exact solution of the Einstein equations (7). We emphasize that such solutions depend on an arbitrary nontrivial function $f(\overleftarrow{\theta}, x^i, v)$, with $f^* \neq 0$, integration functions $f_0(\overleftarrow{\theta}, x^i)$, $h_0^2(\overleftarrow{\theta}, x^i)$, $\varsigma_{[0]}(\overleftarrow{\theta}, x^i)$, $n_{k[1]}(\overleftarrow{\theta}, x^i)$, $n_{k[2]}(\overleftarrow{\theta}, x^i)$ and sources $\Upsilon_2(\overleftarrow{\theta}, x^{\hat{k}}, v)$, $\Upsilon_4(\overleftarrow{\theta}, x^{\hat{i}})$. Such values for the corresponding signatures $\epsilon_\alpha = \pm 1$ have to be defined by certain boundary conditions and physical considerations. We note that formulas (71) and (72) state symbolically that at any intermediary step from the chain $\overleftarrow{\theta}$ one construct the solution following the respective formulas (27) and (28). The final aim, is to get a set of metrics (70), parametrized by $\overleftarrow{\theta}$, when for fixed values of θ -parameters, we get solutions of type (26), for the vacuum Einstein equations for the canonical d-connection.

4.4.2 Example 2:

We consider a family of ansatz, labelled by a set of parameters $\overleftarrow{\theta}$ and χ (as a matter of principle, we can identify the Ricci flow parameter χ with any θ from the set $\overleftarrow{\theta}$ considering that the evolution parameter is also related to the invariance of Killing equations, see Appendix B),

$$\begin{aligned} \lambda \mathbf{g}(\overleftarrow{\theta}, \chi) &= e^{\psi(\overleftarrow{\theta}, x^2, x^3, \chi)} [\epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3] + \\ &h_4(\overleftarrow{\theta}, x^2, x^3, v, \chi) \delta v \otimes \delta v \\ &+ h_5(\overleftarrow{\theta}, x^2, x^3, v, \chi) \chi \delta y^5 \otimes \chi \delta y^5, \\ \delta v &= dv + w_2(\overleftarrow{\theta}, x^2, x^3, v) dx^2 + w_3(\overleftarrow{\theta}, x^2, x^3, v) dx^3, \\ \chi \delta y^5 &= dy^5 + n_2(\overleftarrow{\theta}, x^2, x^3, \chi) [dx^2 + dx^3], \end{aligned} \quad (73)$$

which for any fixed set $\overleftarrow{\theta}$ is of type (48) with the coefficients are subjected to the conditions (49), in our case generalized in the form

$$\begin{aligned} \epsilon_2 \psi^{\bullet\bullet}(\overleftarrow{\theta}, \chi) + \epsilon_3 \psi''(\overleftarrow{\theta}, \chi) &= \lambda, \\ h_5^*(\overleftarrow{\theta}) \phi(\overleftarrow{\theta}) / h_4(\overleftarrow{\theta}) h_5(\overleftarrow{\theta}) &= \lambda, \\ w_2'(\overleftarrow{\theta}) - w_3^\bullet(\overleftarrow{\theta}) + w_3(\overleftarrow{\theta}) w_2^*(\overleftarrow{\theta}) - w_2(\overleftarrow{\theta}) w_3^*(\overleftarrow{\theta}) &= 0, \\ n_2'(\overleftarrow{\theta}, \chi) - n_2^\bullet(\overleftarrow{\theta}, \chi) &= 0, \end{aligned} \quad (74)$$

for $w_{\hat{i}} = \partial_{\hat{i}}\phi/\phi^*$, see (25), being compatible with (37) and (38) and considered that finally on solve the Einstein equations for given surces $\Upsilon_4 = \lambda$ and $\Upsilon_2 = \lambda$. The metrics (73) define self-consistent evolutions of a multi-parametric class of general solutions of the Ricci flow equations (12)–(14) transformed equivalently in the form (20)–(23). The additional constraints (74) define multi-parametric integral subvarieties (foliations) when the evolutions are selected for the Levi Civita connections.

5. Summary and Discussion

In this work, we have developed an unified geometric approach to constructing exact solutions in gravity and Ricci flow theories following superpositions of anholonomic frame deformations and multi-parametric transforms with Killing symmetries.

The anholonomic frame method, proposed for generalized Finsler and Lagrange theories and restricted to the Einstein and string gravity, applies the formalism of nonholonomic frame deformations [20, 21, 23, 18] (see outline of results in [19] and references therein) when the gravitational field equations transform into systems of nonlinear partial differential equations which can be integrated in general form. The new classes of solutions are defined by generic off-diagonal metrics depending on integration functions on one, two and three/ four variables (if we consider four or five dimensional, in brief, 5D or 4D, spacetimes). The important property of such solutions is that they can be generalized for effective cosmological constants induced by certain locally anisotropic matter field interactions, quantum fluctuations and/or string corrections and from Ricci flow theory.

In general relativity, there is also a method elaborated in Refs. [31, 32] as a general scheme when one (two) parameter families of exact solutions are defined by any source-free solutions of Einstein's equations with one (two) Killing vector field(s) (for nonholonomic manifolds, we call such transforms to be one-, two- or multi-parameter nonholonomic deformations/ transforms). A successive iteration procedure results in a class of solutions characterized by an infinite number of parameters for a non-Abelian group involving arbitrary functions on one variable.

Both the parametric deformation techniques combined with nonholonomic transforms state a number of possibilities to construct "target" families of exact solutions and evolution scenarios starting with primary metrics not subjected to the conditions to solve the Einstein equations. The new classes of solutions depend on group like and flow parameters and on sets of integration functions and constants resulting from the procedure of integrating systems of partial differential equations to which the field equations are reduced for certain off-diagonal metric ansatz and generalized connections. Constraining the integral varieties, for a corresponding subset of integration functions, the target solutions are determined to define Einstein spacetimes and their Ricci flow evolutions. In general, such configurations are nonholonomic but can constrained to define geometric evolutions for the Levi Civita connections.

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A The Anholonomic Frame Method

We outline the geometry of nonholonomic frame deformations and nonlinear connection (N–connection) structures [18, 19].

Let us consider a $(n + m)$ –dimensional manifold \mathbf{V} enabled with a prescribed frame structure (1) when frame transforms are linear on $N_i^b(u)$,

$$\mathbf{A}_{\underline{\alpha}}^{\underline{a}}(u) = \begin{bmatrix} e_i^{\underline{a}}(u) - N_i^b(u)e_b^{\underline{a}}(u) \\ 0 \quad e_a^{\underline{a}}(u) \end{bmatrix}, \quad (\text{A.1})$$

$$\mathbf{A}_{\underline{\beta}}^{\underline{b}}(u) = \begin{bmatrix} e^i_{\underline{\beta}}(u) \quad N_k^b(u)e^k_{\underline{\beta}}(u) \\ 0 \quad e^a_{\underline{\beta}}(u) \end{bmatrix}, \quad (\text{A.2})$$

where $i, j, \dots = 1, 2, \dots, n$ and $a, b, \dots = n + 1, n + 2, \dots, n + m$ and $u = \{u^\alpha = (x^i, y^a)\}$ are local coordinates. The geometric constructions will be adapted to a conventional $n + m$ splitting stated by a set of coefficients $\mathbf{N} = \{N_i^a(u)\}$ defining a nonlinear connection (N–connection) structure as a nonintegrable distribution

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \quad (\text{A.3})$$

with a conventional horizontal (h) subspace, $h\mathbf{V}$, (with geometric objects labelled by "horizontal" indices i, j, \dots) and vertical (v) subspace $v\mathbf{V}$ (with geometric objects labelled by indices a, b, \dots). The "boldfaced" symbols will be used to emphasize that certain spaces (geometric objects) are provided (adapted) with (to) a N–connection structure \mathbf{N} .

The transforms (A.1) and (A.2) define a N–adapted frame structure

$$\mathbf{e}_\nu = (\mathbf{e}_i = \partial_i - N_i^a(u)\partial_a, \mathbf{e}_a = \partial_a), \quad (\text{A.4})$$

and the dual frame (coframe) structure

$$\mathbf{e}^\mu = (e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a(u)dx^i). \quad (\text{A.5})$$

The frames (A.5) satisfy the certain nonholonomy (equivalently, anholonomy) relations of type (2),

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \quad (\text{A.6})$$

with anholonomy coefficients

$$W_{ia}^b = \partial_a N_i^b \quad \text{and} \quad W_{ji}^a = \Omega_{ij}^a = \mathbf{e}_j(N_i^a) - \mathbf{e}_i(N_j^a). \quad (\text{A.7})$$

A distribution (A.3) is integrable, i.e. \mathbf{V} is a foliation, if and only if the coefficients defined by $\mathbf{N} = \{N_i^a(u)\}$ satisfy the condition $\Omega_{ij}^a = 0$. In general, a spacetime with prescribed

nonholonomic splitting into h- and v-subspaces can be considered as a nonholonomic manifold [18, 17, 34].

Let us consider a metric structure on \mathbf{V} ,

$$\check{g} = \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta \quad (\text{A.8})$$

defined by coefficients

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \quad (\text{A.9})$$

This metric is generic off-diagonal, i.e. it can not be diagonalized by any coordinate transforms if $N_i^a(u)$ are any general functions. We can adapt the metric (A.8) to a N-connection structure $\mathbf{N} = \{N_i^a(u)\}$ induced by the off-diagonal coefficients in (A.9) if we impose that the conditions

$$\check{g}(e_i, e_a) = 0, \text{ equivalently, } \underline{g}_{ia} - N_i^b h_{ab} = 0,$$

where $\underline{g}_{ia} \doteq g(\partial/\partial x^i, \partial/\partial y^a)$, are satisfied for the corresponding local basis (A.4). In this case $N_i^b = h^{ab} \underline{g}_{ia}$, where h^{ab} is inverse to h_{ab} , and we can write the metric \check{g} (A.9) in equivalent form, as a distinguished metric (d-metric) adapted to a N-connection structure,

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(u) \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}(u) e^i \otimes e^j + h_{ab}(u) \mathbf{e}^a \otimes \mathbf{e}^b, \quad (\text{A.10})$$

where $g_{ij} \doteq \mathbf{g}(e_i, e_j)$ and $h_{ab} \doteq \mathbf{g}(e_a, e_b)$. The coefficients $\mathbf{g}_{\alpha\beta}$ and $\underline{g}_{\alpha\beta} = g_{\underline{\alpha}\underline{\beta}}$ are related by formulas

$$\mathbf{g}_{\alpha\beta} = \mathbf{A}_\alpha^{\underline{\alpha}} \mathbf{A}_\beta^{\underline{\beta}} \underline{g}_{\underline{\alpha}\underline{\beta}}, \quad (\text{A.11})$$

or

$$g_{ij} = e_i^{\underline{i}} e_j^{\underline{j}} \underline{g}_{\underline{i}\underline{j}} \text{ and } h_{ab} = e_a^{\underline{a}} e_b^{\underline{b}} \underline{g}_{\underline{a}\underline{b}},$$

where the frame transform is given by matrices (A.1) with $e_i^{\underline{i}} = \delta_i^{\underline{i}}$ and $e_a^{\underline{a}} = \delta_a^{\underline{a}}$. We shall call some geometric objects, for instance, tensors, connections,..., to be distinguished by a N-connection structure, in brief, d-tensors, d-connections,... if they are stated by components computed with respect to N-adapted frames (A.4) and (A.5). In this case, the geometric constructions are elaborated in N-adapted form, i.e. they are adapted to the nonholonomic distribution (A.3).

Any vector field $\mathbf{X} = (hX, vX)$ on $T\mathbf{V}$ can be written in N-adapted form as a d-vector

$$\mathbf{X} = X^\alpha \mathbf{e}_\alpha = (hX = X^i e_i, vX = X^a e_a).$$

In a similar form, we can 'N-adapt' any tensor object and get a d-tensor.

By definition, a d-connection is adapted to the distribution (A.3) and splits into h- and v-covariant derivatives, $\mathbf{D} = hD + vD$, where $hD = \{\mathbf{D}_k = (L_{jk}^i, L_{bk}^a)\}$ and $vD = \{\mathbf{D}_c = (C_{jk}^i, C_{bc}^a)\}$ are correspondingly introduced as h- and v-parametrizations of the coefficients

$$L_{jk}^i = (\mathbf{D}_k e_j)] e^i, \quad L_{bk}^a = (\mathbf{D}_k e_b)] e^a, \quad C_{jc}^i = (\mathbf{D}_c e_j)] e^i, \quad C_{bc}^a = (\mathbf{D}_c e_b)] e^a.$$

The components $\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$, with the coefficients defined with respect to (A.5) and (A.4), completely define a d-connection \mathbf{D} on a N-anholonomic manifold \mathbf{V} .

The simplest way to perform a local covariant calculus by applying d-connections is to use N-adapted differential forms and to introduce the d-connection 1-form $\Gamma^\alpha_\beta = \Gamma^\alpha_{\beta\gamma} \mathbf{e}^\gamma$, when the N-adapted components of d-connection $\mathbf{D}_\alpha = (\mathbf{e}_\alpha \rfloor \mathbf{D})$ are computed following formulas

$$\Gamma^\gamma_{\alpha\beta} (u) = (\mathbf{D}_\alpha \mathbf{e}_\beta) \rfloor \mathbf{e}^\gamma, \quad (\text{A.12})$$

where " \rfloor " denotes the interior product. We define in N-adapted form the torsion $\mathbf{T} = \{\mathcal{T}^\alpha\}$ (3),

$$\mathcal{T}^\alpha \doteq \mathbf{D} \mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma^\alpha_\beta \wedge \mathbf{e}_\beta, \quad (\text{A.13})$$

and curvature $\mathbf{R} = \{\mathcal{R}^\alpha_\beta\}$ (4),

$$\mathcal{R}^\alpha_\beta \doteq \mathbf{D} \Gamma^\alpha_\beta = d\Gamma^\alpha_\beta - \Gamma^\gamma_\beta \wedge \Gamma^\alpha_\gamma. \quad (\text{A.14})$$

The coefficients of torsion \mathbf{T} (A.13) of a d-connection \mathbf{D} (in brief, d-torsion) are computed with respect to N-adapted frames (A.5) and (A.4),

$$\begin{aligned} T^i_{jk} &= L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji}, \\ T^a_{bi} &= T^a_{ib} = \frac{\partial N^a_i}{\partial y^b} - L^a_{bi}, \quad T^a_{bc} = C^a_{bc} - C^a_{cb}, \end{aligned} \quad (\text{A.15})$$

where, for instance, T^i_{jk} and T^a_{bc} are respectively the coefficients of the $h(hh)$ -torsion $hT(hX, hY)$ and $v(vv)$ -torsion $vT(vX, vY)$. In a similar form, we can compute the coefficients of a curvature \mathbf{R} , d-curvatures.

There is a preferred, canonical d-connection structure, $\widehat{\mathbf{D}}$, on a N-anholonomic manifold \mathbf{V} constructed only from the metric and N-connection coefficients $[g_{ij}, h_{ab}, N^a_i]$ and satisfying the conditions $\widehat{\mathbf{D}}\mathbf{g} = 0$ and $\widehat{T}^i_{jk} = 0$ and $\widehat{T}^a_{bc} = 0$. It should be noted that, in general, the components \widehat{T}^i_{ja} , \widehat{T}^a_{ji} and \widehat{T}^a_{bi} are not zero. This is an anholonomic frame (equivalently, off-diagonal metric) effect. Hereafter, we consider only geometric constructions with the canonical d-connection which allow, for simplicity, to omit "hats" on d-objects. We can verify by straightforward calculations that the linear connection $\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$ with the coefficients defined

$$\mathbf{D}_{\mathbf{e}_k}(\mathbf{e}_j) = L^i_{jk} \mathbf{e}_i, \quad \mathbf{D}_{\mathbf{e}_k}(e_b) = L^a_{bk} e_a, \quad \mathbf{D}_{e_b}(\mathbf{e}_j) = C^i_{jb} \mathbf{e}_i, \quad \mathbf{D}_{e_c}(e_b) = C^a_{bc} e_a,$$

where

$$\begin{aligned} L^i_{jk} &= \frac{1}{2} g^{ir} (\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}), \\ L^a_{bk} &= e_b(N^a_k) + \frac{1}{2} h^{ac} (\mathbf{e}_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k), \\ C^i_{jc} &= \frac{1}{2} g^{ik} e_c g_{jk}, \quad C^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_c h_{cd} - e_d h_{bc}), \end{aligned} \quad (\text{A.16})$$

uniquely solve the conditions stated for the canonical d-connection.

The Levi Civita linear connection $\nabla = \{ \text{,}\Gamma_{\beta\gamma}^\alpha \}$, uniquely defined by the conditions $\nabla T = 0$ and $\nabla \check{g} = 0$, is not adapted to the distribution (A.3). Denoting $\text{,}\Gamma_{\beta\gamma}^\alpha = (\text{,}L_{jk}^i, \text{,}L_{jk}^a, \text{,}L_{bk}^i, \text{,}L_{bk}^a, \text{,}C_{jb}^i, \text{,}C_{jb}^a, \text{,}C_{bc}^i, \text{,}C_{bc}^a)$, for

$$\begin{aligned} \nabla_{\mathbf{e}_k}(\mathbf{e}_j) &= \text{,}L_{jk}^i \mathbf{e}_i + \text{,}L_{jk}^a e_a, \quad \nabla_{\mathbf{e}_k}(e_b) = \text{,}L_{bk}^i \mathbf{e}_i + \text{,}L_{bk}^a e_a, \\ \nabla_{e_b}(\mathbf{e}_j) &= \text{,}C_{jb}^i \mathbf{e}_i + \text{,}C_{jb}^a e_a, \quad \nabla_{e_c}(e_b) = \text{,}C_{bc}^i \mathbf{e}_i + \text{,}C_{bc}^a e_a, \end{aligned}$$

after a straightforward calculus we get

$$\begin{aligned} \text{,}L_{jk}^i &= L_{jk}^i, \quad \text{,}L_{jk}^a = -C_{jb}^i g_{ik} h^{ab} - \frac{1}{2} \Omega_{jk}^a, \\ \text{,}L_{bk}^i &= \frac{1}{2} \Omega_{jk}^c h_{cb} g^{ji} - \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C_{hb}^j, \\ \text{,}L_{bk}^a &= L_{bk}^a + \frac{1}{2} (\delta_c^a \delta_d^b + h_{cd} h^{ab}) [L_{bk}^c - e_b(N_k^c)], \\ \text{,}C_{kb}^i &= C_{kb}^i + \frac{1}{2} \Omega_{jk}^a h_{cb} g^{ji} + \frac{1}{2} (\delta_j^i \delta_k^h - g_{jk} g^{ih}) C_{hb}^j, \\ \text{,}C_{jb}^a &= -\frac{1}{2} (\delta_c^a \delta_b^d - h_{cb} h^{ad}) [L_{dj}^c - e_d(N_j^c)], \quad \text{,}C_{bc}^a = C_{bc}^a, \\ \text{,}C_{ab}^i &= -\frac{g^{ij}}{2} \{ [L_{aj}^c - e_a(N_j^c)] h_{cb} + [L_{bj}^c - e_b(N_j^c)] h_{ca} \}, \end{aligned} \tag{A.17}$$

where Ω_{jk}^a are computed as in the second formula in (A.7).

For our purposes, it is important to state the conditions when both the Levi Civita connection and the canonical d-connection may be defined by the same set of coefficients with respect to a fixed frame of reference. Following formulas (A.16) and (A.17), we obtain equality $\text{,}\Gamma_{\beta\gamma}^\alpha = \mathbf{\Gamma}_{\alpha\beta}^\gamma$ if

$$\Omega_{jk}^c = 0 \tag{A.18}$$

(there are satisfied the integrability conditions and our manifold admits a foliation structure),

$$\text{,}C_{kb}^i = C_{kb}^i = 0 \tag{A.19}$$

and $L_{aj}^c - e_a(N_j^c) = 0$, which, following the second formula in (A.16), is equivalent to

$$\mathbf{e}_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d = 0. \tag{A.20}$$

We conclude this section with the remark that if the conditions (A.18), (A.19) and (A.20) hold true for the metric (A.8), equivalently (A.10), the torsion coefficients (A.15) vanish. This results in respective equalities of the coefficients of the Riemann, Ricci and Einstein tensors (the conditions (6) being satisfied) for two different linear connections.

B The Killing Vectors Formalism

The first parametric method (on holonomic (pseudo) Riemannian spaces, it is also called the Geroch method [31]) proposes a scheme of constructing a one-parameter family of vacuum exact solutions (labelled by tilde "̃" and depending on a real parameter θ)

$$\text{,}\tilde{g}(\theta) = \text{,}\tilde{g}_{\alpha\beta} e^\alpha \otimes e^\beta \tag{B.1}$$

beginning with any source-free solution ${}^{\circ}g = \{{}^{\circ}g_{\alpha\beta}\}$ with Killing vector $\xi = \{\xi_{\alpha}\}$ symmetry satisfying the conditions ${}_{,}\mathcal{E} = 0$ (Einstein equations) and $\nabla_{\xi}({}^{\circ}g) = 0$ (Killing equations). We denote this 'primary' spacetime $(V, {}^{\circ}g, \xi_{\alpha})$ and follow the conventions: The class of metrics ${}^{\circ}\tilde{g}$ is generated by the transforms

$${}^{\circ}\tilde{g}_{\alpha\beta} = \tilde{B}_{\alpha}{}^{\alpha'}(u, \theta) \tilde{B}_{\beta}{}^{\beta'}(u, \theta) {}^{\circ}g_{\alpha'\beta'} \quad (\text{B.2})$$

where the matrix $\tilde{B}_{\alpha}{}^{\alpha'}$ is parametrized in the form when

$${}^{\circ}\tilde{g}_{\alpha\beta} = \lambda \tilde{\lambda}^{-1} ({}^{\circ}g_{\alpha\beta} - \lambda^{-1} \xi_{\alpha} \xi_{\beta}) + \tilde{\lambda} \mu_{\alpha} \mu_{\beta} \quad (\text{B.3})$$

for

$$\begin{aligned} \tilde{\lambda} &= \lambda [(\cos \theta - \omega \sin \theta)^2 + \lambda^2 \sin^2 \theta]^{-1} \\ \mu_{\tau} &= \tilde{\lambda}^{-1} \xi_{\tau} + \alpha_{\tau} \sin 2\theta - \beta_{\tau} \sin^2 \theta. \end{aligned}$$

A rigorous proof [31] states that the metrics (B.1) define also exact vacuum solutions with ${}_{,}\tilde{\mathcal{E}} = 0$ if and only if the values $\xi_{\alpha}, \alpha_{\tau}, \mu_{\tau}$ from (B.3), subjected to the conditions $\lambda = \xi_{\alpha} \xi_{\beta} {}^{\circ}g^{\alpha\beta}$, $\omega = \xi^{\gamma} \alpha_{\gamma}, \xi^{\gamma} \mu_{\gamma} = \lambda^2 + \omega^2 - 1$, solve the equations

$$\begin{aligned} \nabla_{\alpha} \omega &= \epsilon_{\alpha\beta\gamma\tau} \xi^{\beta} \nabla^{\gamma} \xi^{\tau}, \quad \nabla_{[\alpha} \alpha_{\beta]} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\tau} \nabla^{\gamma} \xi^{\tau}, \\ \nabla_{[\alpha} \mu_{\beta]} &= 2\lambda \nabla_{\alpha} \xi_{\beta} + \omega \epsilon_{\alpha\beta\gamma\tau} \nabla^{\gamma} \xi^{\tau}, \end{aligned} \quad (\text{B.4})$$

where the Levi Civita connection ∇ is defined by ${}^{\circ}g$ and $\epsilon_{\alpha\beta\gamma\tau}$ is the absolutely antisymmetric tensor. The existence of solutions for (B.4) (Geroch's equations) is guaranteed by the Einstein's and Killing equations.

The first type of parametric transforms (B.2) can be parametrized by a matrix $\tilde{B}_{\alpha}{}^{\alpha'}$ with the coefficients depending functionally on solutions for (B.4). Fixing a signature $g_{\underline{\alpha}\underline{\beta}} = \text{diag}[\pm 1, \pm 1, \dots, \pm 1]$ and a local coordinate system on $(V, {}^{\circ}g, \xi_{\alpha})$, we define a local frame of reference $e_{\alpha'} = A_{\alpha'}^{\underline{\alpha}}(u) \partial_{\underline{\alpha}}$, like in (1), for which

$${}^{\circ}g_{\alpha'\beta'} = A_{\alpha'}^{\underline{\alpha}} A_{\beta'}^{\underline{\beta}} g_{\underline{\alpha}\underline{\beta}}. \quad (\text{B.5})$$

We note that $A_{\alpha'}^{\underline{\alpha}}$ have to be constructed as a solution of a system of quadratic algebraic equations (B.5) for given values $g_{\underline{\alpha}\underline{\beta}}$ and ${}^{\circ}g_{\alpha'\beta'}$. In a similar form, we can write $\tilde{e}_{\alpha} = \tilde{A}_{\alpha}^{\underline{\alpha}}(\theta, u) \partial_{\underline{\alpha}}$ and

$${}^{\circ}\tilde{g}_{\alpha\beta} = \tilde{A}_{\alpha}^{\underline{\alpha}} \tilde{A}_{\beta}^{\underline{\beta}} g_{\underline{\alpha}\underline{\beta}}. \quad (\text{B.6})$$

The method guarantees that the family of spacetimes $(V, {}^{\circ}\tilde{g})$ is also vacuum Einstein but for the corresponding families of Levi Civita connections $\tilde{\nabla}$. In explicit form, the matrix $\tilde{B}_{\alpha}{}^{\alpha'}(u, \theta)$ of parametric transforms can be computed by introducing the relations (B.5), (B.6) into (B.2),

$$\tilde{B}_{\alpha}{}^{\alpha'} = \tilde{A}_{\alpha}^{\underline{\alpha}} A_{\underline{\alpha}}^{\alpha'} \quad (\text{B.7})$$

where $A_{\underline{\alpha}}^{\alpha'}$ is inverse to $A_{\alpha'}^{\underline{\alpha}}$.

The second parametric method [32] was similarly developed which yields a family of new exact solutions involving two arbitrary functions on one variables, beginning with any two commuting Killing fields for which a certain pair of constants vanish (for instance, the exterior field of a rotating star). By successive iterating such parametric transforms, one generates a class of exact solutions characterized by an infinite number of parameters and involving arbitrary functions. For simplicity, in this work we shall consider only a nonholonomic version of the first parametric method.

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Killing Symmetries of Deformed Relativity in Five Dimensions

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Abstract: This is the first of two papers devoted to investigating the main mathematical aspects of the Kaluza-Klein-like scheme known as Deformed Relativity in five dimensions (DR5). It is based on a five-dimensional Riemannian space in which the four-dimensional space-time metric is deformed (*i.e.* it depends on the energy) and energy plays the role of the fifth dimension. After a brief survey of the physical and mathematical foundations of DR5, we discuss in detail the Killing symmetries of the theory. In particular, we consider the case of physical relevance in which the metric coefficients are power functions of the energy (Power Ansatz). In order to solve the related Killing equations, we introduce a simplifying hypothesis of functional independence (Υ hypothesis). The explicit expressions of the Killing vectors for the energy-dependent metrics corresponding to the four fundamental interactions (electromagnetic, weak, strong and gravitational) are derived. A preliminary discussion of the infinitesimal-algebraic structure of the Killing symmetries of DR5 is also given.

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1. Introduction

1.1 How Many Dimensions?

The problem of the ultimate geometrical structure of the physical world — both at a large and a small scale — is an old-debated one. After Einstein, the generally accepted view is that physical phenomena do occur in a four-dimensional manifold, with three spatial and one time dimensions, and that space-time possesses a *global* Riemannian structure, whereas it is *locally* flat (i.e. endowed with a Minkowskian geometry). Moreover, Einstein taught us that introducing extra dimensions (time, in this case) can provide a better description of (and even simplify) the laws of nature (electromagnetism for Special Relativity and gravity for General Relativity).

This latter teaching by Einstein was followed by Kaluza [1] and Klein [2], who added a fifth dimension to the four space-time ones in order to unify electromagnetism and gravitation (let us recall however that Nordström [3] was the first to realize that in a five-dimensional spacetime the field equations do split naturally into Einstein's and Maxwell's equations). Although unsuccessful, the Kaluza-Klein (KK) theory constituted the first attempt to unification of fundamental interactions within a multidimensional space-time. In this spirit, it was later generalized and extended to higher dimensions, in the hope of achieving unification of all interactions, including weak and strong forces [4]. However, a true revival of multidimensional theories starting from 1970 was due to the advent of string theory [5] and supersymmetry [6]. As is well known, their combination, superstring theory [7], provides a framework for gravity quantization. Modern generalizations [8,9] of the Kaluza-Klein scheme require a minimum number of 11 dimensions in order to accommodate the Standard Model of electroweak and strong interactions; let us recall that 11 is also the maximum number of dimensions required by supergravity theories [10]. For an exhaustive review of Kaluza-Klein theories we refer the reader to ref.[11].

A basic problem in any multidimensional scheme is the hidden nature of the extra dimensions, namely to explain why the Universe looks four-dimensional. A possible solution (first proposed by Klein) is assuming that each extra dimension is *compactified*, namely it is curled up in a circle, which from a mathematical standpoint is a compact set, whence the name (*cylindricity condition*)². In such a view, therefore, space-time is endowed with a cylindrical geometrical structure. The radius R of the circle is taken to be so small (roughly of the order of the Planck length) as to make the extra dimension unobservable at distances exceeding the compactification scale.

In the last decade of the past century the hypothesis of non-compactified extra dimensions began to be taken into serious consideration. Five-dimensional theories of the Kaluza-Klein type, but with no cylindricity condition on the extra dimension, were built up *e.g.* by Wesson (the so-called "Space-Time-Mass" (STM) theory, in which the fifth dimension is the rest mass [9]) and by Fukui ("Space-Time-Mass-(Electric) Charge"

² Compactification of the extra dimensions can be achieved also by assuming compact spaces different from (and more complex than) a circle.

(S.T.M.C.) theory, with charge as extra dimension [12]). In such a kind of theories, the (non-compactified) extra dimension is a physical quantity, but not a spatial one in its strict sense. More recently, instead, the idea of a true hyperspace, with large *space* dimensions, was put forward (starting from the 1998 pioneering ADD model [13]). The basic assumption is that gravity is the only force aware of the extra space dimensions. This leads to view the physical world as a multidimensional space ("*bulk*") in which the ordinary space is represented by a three-dimensional surface ("*3-d. brane*"). Electromagnetic, weak and strong forces are trapped within the brane, and do not "feel" the extra space dimensions. On the contrary, gravitons escape the brane and spread out the whole hyperspace. In this picture, the reason the gravitational force appears to be so weak is because it is diluted by the extra dimensions. In the ADD scheme (and similar), the size L of the extra dimensions is related to their number. Only one extra dimension would have a size greater than the solar system, and therefore would have been already discovered. For two extra dimensions, it is possible to show that $L \sim 0.2mm$, whereas for three it is $L \sim 1nm$. In this framework, therefore, the extra dimensions are still finite (although not microscopic as in the KK theories).

The ADD model has been also modified in order to allow for infinite (in the sense of unlimited) extra dimensions. In the "*warped-geometric*" model [14], it is still assumed that the ordinary space is confined to a 3-d. brane embedded in a bulk. Only the gravitons can escape the brane along the extra dimension, but they feel the gravitational field of the brane and therefore do not venture out of it to long distances. This amounts to say that the geometry of the extra dimension is warped (the five-dimensional metric of the hyperspace contains an exponentially decreasing "*warp factor*", namely the probability of finding a graviton decays outside the brane along the extra dimension). An analogous result can be obtained by hypothesizing the existence of two branes, put a distance L apart along the extra dimension, one trapping gravity and the other not. If the two branes have opposite tension, the geometry of space between the branes is warped too. Models with both an infinite, warped extra dimension and a finite compact one have been also considered [15].

Needless to say, it is impossible to get direct evidence of extra dimensions. However, one can probe them in an indirect way, because the existence of extra dimensions has several observational consequences, both in astrophysics and cosmology and in particle physics, depending on the model considered. In compactified theories, new excited states appear within the extra dimensions (*Kaluza-Klein towers*), with energies $E_n = nhc/R$. They affect the carriers of the electromagnetic, weak and strong forces by turning them into a family of increasingly massive clones of the original particle, thus magnifying the strengths of the nongravitational forces. Such effects can therefore be detected in particle accelerators. A research carried at CERN's Large Electron-Positron Collider (LEP) provided no evidence of such extradimensional influences at up to an energy of 4 *TeV*. This result puts a limit of $0.5 \times 10^{-19}m$ to the size L of the extra dimensions [16]. The ADD model foresees deviations from the Newton's inverse-square law of gravity for objects closer together than the size of the extra dimension. Such a stronger gravitational

attraction ($\sim r^{-4}$) could be observed in tabletop experiments (of the Cavendish type). In general, in models with a 3-d. brane, gravitons leaving the brane into, or entering it from, the extra dimensions could provide signatures for the hyperspace in accelerator experiments. In the former case, one has to look for missing energy in a collision process, due to the disappearing of the gravitons into the extra dimensions. In the latter, since gravitons can decay into pairs of photons, electrons, or muons, detecting an excess of these particles at specific energy and mass levels would indirectly provide evidence for the existence of dimensions beyond our own.

It must also be noted that other fields besides the gravitational one are expected to be present in the bulk. Bulk gauge fields are associated with "new forces", the strength of which is predicted to be roughly a million times stronger than gravity. These stronger forces can manifest themselves in different ways, and could be detectable also in tabletop experiments. For example, they can simulate antigravity effects on submillimeter distance scales, since gauge forces between like-charged objects are naturally repulsive.

The hypothesis of a multidimensional space-time allows one not only to unify interactions and quantize gravity, but to solve or at least to address from a more basic viewpoint a number of fundamental problems still open in particle physics and astrophysics. Let us mention for instance the weakness of the gravitational force, the abundance of matter over antimatter, the extraordinarily large number of elementary particles, the nature of dark matter and the smallness of the cosmological constant.

1.2 Deforming Space-Time

The introduction of extra dimensions is not the only way to generalizing the four-dimensional Einsteinian picture. Another road which was followed is preserving the four-dimensional spacetime manifold, by however equipping it with a global and/or local geometry different from the Minkowskian or the Riemannian one. These generalized 4-d. metrics are mainly of the Finsler type [17], like the Bogoslawski [18] and the isotopic ones [19]. Such a kind of approach points essentially at accounting for possible violations of standard relativity and Lorentz invariance.

In this connection, two of us (F.C. and R.M.) introduced a generalization of Special Relativity, called *Deformed Special Relativity (DSR)* [20, 21]. It was essentially aimed, in origin, at dealing in a phenomenological way with a possible breakdown of local Lorentz invariance (LLI). Actually the experimental data of some physical processes seem to provide some intriguing evidence of a (local) breakdown of Lorentz invariance. All the phenomena considered show indeed an inadequacy of the Minkowski metric in describing them, at different energy scales and for the four fundamental interactions involved (electromagnetic, weak, gravitational and strong). On the contrary, they apparently admit a consistent interpretation in terms of a *deformed Minkowski space-time, with metric coefficients depending on the energy exchanged in the process considered* [20].

DSR is therefore a (four-dimensional) generalization of the (local) space-time structure based on an energy-dependent deformation of the usual Minkowski geometry. What's

more, the corresponding deformed metrics obtained from the experimental data provide an *effective dynamical description of the interactions* ruling the phenomena considered (at least at the energy scale and in the energy range considered). Then DSR implements, for all four interactions, the so-called “*solidarity principle*“, between space-time and interaction (so that the peculiar features of every interaction determine — locally — its own space-time structure), which — following B. Finzi [21] — can be stated as follows: “*Space-time is solid with interactions, so that their respective properties affect mutually*”.

Moreover, it was shown that the deformed Minkowski space with energy-dependent metric admits a natural embedding in a five-dimensional space-time, with energy as extra dimension [20, 22]. Namely, the four-dimensional, deformed, energy-dependent space-time is only a manifestation (a “shadow”, to use the famous word of Minkowski) of a larger, five-dimensional space, in which energy plays the role of the fifth dimension. The new formalism one gets in this way (*Deformed Relativity in Five Dimensions, DR5*) is a *Kaluza-Klein-like one*, the main points of departure from a standard *KK* scheme being the deformation of the Minkowski space-time and the use of energy as extra dimension (this last feature entails, among the others, that the DR5 formalism is *noncompactified*).

DR5 is therefore a generalization of Einstein’s Relativity sharing both characteristics of a change of the 4-d Minkowski metric and the presence of extra dimensions.

The purpose of the present paper is to illustrate the DR5 formalism and to give new results on the isometries of the five-dimensional space of the theory.

The paper is organized as follows. Sect. 2 contains a brief review of the formalism of the four-dimensional deformed Minkowski space and gives the explicit expressions of the deformed metrics obtained, for the fundamental interactions, by the phenomenological analysis of the experimental data. In Sect. 3 we illustrate the main features of the DR5 scheme, namely its geometrical structure — based on a five-dimensional space \mathfrak{R}_5 in which the four-dimensional space-time is deformed and the energy E plays the role of fifth dimension — and the solution of the related five-dimensional Einstein equations. In particular, we consider the special case of physical relevance in which the metric coefficients are powers of the energy (Power Ansatz). In Sects.4-5 the problem of the Killing isometries of \mathfrak{R}_5 is discussed in detail. In order to solve the five-dimensional Killing equations, we introduce an hypothesis of functional independence, which allows us to get explicit solutions (in the Power Ansatz) for the four phenomenological metrics of fundamental interactions. Sect.6 contains a preliminary discussion of the related Killing algebras. Concluding remarks are put forward in Sect. 7.

2. Deformed Special Relativity: A Survey

2.1 Deformed Minkowski Space-Time

Let us briefly review the main features of the formalism of the (four-dimensional) deformed Minkowski space [20].

If $M(x, g_{SR}, R)$ is the usual Minkowski space of the standard Special Relativity (SR)

(where x is a fixed Cartesian frame), endowed with the metric tensor

$$g_{SR} = \text{diag}(1, -1, -1, -1), \quad (1)$$

the deformed Minkowski space $\widetilde{M}(x, g_{DSR})$ is the same vector space on the real field as M , with the same frame x , but with metric g_{DSR} given by³

$$g_{DSR}(E) = \text{diag}(b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E))$$

$$\stackrel{\text{ESC}}{=} \delta_{\mu\nu}^{\text{off}} [b_0^2(E)\delta_{\mu 0} - b_1^2(E)\delta_{\mu 1} - b_2^2(E)\delta_{\mu 2} - b_3^2(E)\delta_{\mu 3}], \quad (2)$$

where the metric coefficients $\{b_\mu^2(E)\}$ ($\mu = 0, 1, 2, 3$) are (dimensionless) positive functions of the energy E of the process considered⁴: $b_\mu^2 = b_\mu^2(E)$. The generalized interval in \widetilde{M} reads therefore

$$\begin{aligned} ds^2 &= b_0^2(E)c^2dt^2 - b_1^2(E)dx^2 - b_2^2(E)dy^2 - b_3^2(E)dz^2 = \\ &= g_{\mu\nu,DSR}dx^\mu dx^\nu = dx * dx \end{aligned} \quad (3)$$

with $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, c being the usual light speed in vacuum. The last equality in (3) defines the scalar product $*$ in the deformed Minkowski space \widetilde{M} . The relativity theory based on \widetilde{M} is called *Deformed Special Relativity*, *DSR*[20].

We want to stress that — although uncommon — the use of an energy-dependent space-time metric is not new. Indeed, it can be traced back to Einstein himself. In order to account for the modified rate of a clock in presence of a gravitational field, Einstein first generalized the expression of the special-relativistic interval with metric (1), by introducing a "time curvature" as follows:

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2dt^2 - dx^2 - dy^2 - dz^2, \quad (4)$$

where ϕ is the Newtonian gravitational potential. In the present scheme, the reason whereby one considers energy as the variable upon which the metric coefficients depend is twofold. On one side, it has a phenomenological basis in the fact that we want to exploit this formalism in order to derive the deformed metrics corresponding to physical processes, whose experimental data are just expressed in terms of the energy of the process considered. On the other hand, one expects on physical grounds that a possible deformation of the space-time to be intimately related to the energy of the concerned phenomenon (in analogy to the gravitational case, where space-time curvature is determined by the energy-matter distribution).

³ In the following, Greek and Latin indices will label space-time ($\mu = 0, 1, 2, 3$) and space coordinates ($i = 1, 2, 3$), respectively. Moreover, we shall employ the notation "ESCon" ("ESCOff") to mean that the Einstein sum convention on repeated indices is (is not) used.

⁴ Quantity E is to be understood as the energy measured by the detectors via their electromagnetic interaction in the usual Minkowski space.

Let us recall that the metric (2) is supposed to hold locally, *i.e.* in the space-time region where the process occurs. Moreover, it is supposed to play a *dynamical* role, thus providing a geometric description of the interaction considered, especially as far as nonlocal, nonpotential forces are concerned. In other words, each interaction produces its own metric, formally expressed by the metric tensor g_{DSR} , but realized via different choices of the set of parameters $b_\mu(E)$. We refer the reader to ref. [20] for a more detailed discussion.

It is also worth to notice that the space-time described by the interval (3) actually has zero curvature, and therefore *it is not* a "true" Riemannian space (whence the term "deformation" used to describe such a situation). Therefore, on this respect, the geometrical description of the fundamental interactions based on the metric (2) is different from that adopted in General Relativity to describe gravitation. Moreover, for each interaction the corresponding metric reduces to the Minkowskian one, $g_{\mu\nu,SR}$, for a suitable value of the energy, E_0 , characteristic of the interaction considered (see below). But the energy of the process is fixed, and cannot be changed at will. Thus, although it would be in principle possible to recover the Minkowski space by a suitable change of coordinates (e.g. by a rescaling), this would amount to a mere mathematical operation, devoid of any physical meaning.

Inside the deformed space-time, a maximal causal speed u can be defined, whose role is analogous to that of the light speed in vacuum for the usual Minkowski space-time. It can be shown that, for an isotropic 3-dimensional space ($b_1 = b_2 = b_3 = b$), its expression is

$$u = \frac{b_0}{b}c. \quad (5)$$

This speed u can be considered as the speed of the interaction ruling the process described by the deformation of the metric. It is easily seen that there may be maximal causal speeds which are *superluminal*, depending on the interaction considered, because

$$u \begin{matrix} \geq \\ < \end{matrix} c \iff \frac{b_0}{b} \begin{matrix} \geq \\ < \end{matrix} 1. \quad (6)$$

Starting from the deformed space-time \widetilde{M} , one can develop the Deformed Special Relativity in a straightforward way. For instance, the generalized Lorentz transformations, *i.e.* those transformations which preserve the interval (3), for an isotropic three-space and for a boost, say, along the x -axis, read as follows [20]

$$\begin{cases} x' = \widetilde{\gamma}(x - vt); \\ y' = y; \\ z' = z; \\ t' = \widetilde{\gamma} \left(t - \widetilde{\beta}^2 \frac{x}{v} \right), \end{cases} \quad (7)$$

where v is the relative speed of the reference frames, and

$$\widetilde{\beta} = \frac{v}{u}; \quad (8)$$

$$\tilde{\gamma} = \left(1 - \tilde{\beta}^2\right)^{-1/2}. \quad (9)$$

We refer the reader to [20] for a thorough discussion of the generalized Lorentz transformations (*i.e.* the Killing isometries of \tilde{M}).

It must be carefully noted that, like the metric, the generalized Lorentz transformations, too, depend on the energy and on the interaction considered (through the deformed rapidity parameter $\tilde{\beta}$: see the expression (5) of the maximal speed u). This means that one gets different transformation laws for different interactions and for different values of E , but still with the same functional dependence on the energy, so that the invariance of the deformed interval (3) is always ensured (provided the process considered does always occur via the same interaction).

The existence, in DSR, of several coordinate transformations connecting inertial observers and of more maximal causal speeds, *a priori* different for each interaction, establish a connection of this scheme with Lorentzian Relativity (LR) (rather than with the Einsteinian one, ER), *i.e.* the version of Special Relativity due to Lorentz and Poincaré [23]⁵. These features, rather than constituting drawbacks of LR with respect to the ER unifying principles of uniqueness and invariance, actually testify the more flexible mathematical structure of Lorentzian relativity, thus able to fit the diversified nature of the different physical forces. We can therefore state that *Deformed Special Relativity inherits the legacy of Lorentzian Relativity* (see second ref. [20] for an in-depth discussion of this point).

From the knowledge of the generalized Lorentz transformations in the deformed Minkowski space \tilde{M} , it is easy to derive the main kinematical and dynamical laws valid in DSR. For this topic and further features of DSR the interested reader is referred to ref. [20].

We want also to stress that Deformed Special Relativity is a theory different from Doubly Special Relativity (although both of them have the same acronym). The latter is a generalization of SR in which, besides the speed of light, the Planck length is an invariant quantity too [24]. In spite of their differences, the two formalisms share some common points, like *e.g.*: the existence of a generalized energy-momentum dispersion law; generalized boost transformations; generalized commutation relations for the generators of the Poincaré algebra; an energy-dependent metric for the Minkowski space-time; a light speed depending on energy [24-25].

2.2 Description of Interactions by Energy-Dependent Metrics

Let us briefly review the results obtained for the deformed metrics, describing the four fundamental interactions — electromagnetic, weak, strong and gravitational —, from the phenomenological analysis of the experimental data [20]. First of all, let us stress that, in

⁵ Let us however stress that the strict dependence of both the coordinate transformations and the maximal causal speed on the class of the physical phenomena, to which the (Galileian) Principle of Relativity applies, is already present in the very foundations of Special Relativity (see refs.[20,21]).

all the cases considered, *one gets evidence for a departure of the space-time metric from the Minkowskian one* (at least in the energy range examined).

The explicit functional form of the DSR metric (2) for the four interactions is as follows.

1) **Electromagnetic interaction.** The experiments considered are those on the superluminal propagation of e.m. waves in conducting waveguides with variable section (first observed at Cologne in 1992) [26]. The introduction, in this framework, of a deformed Minkowski space is motivated by ascribing the superluminal speed of the signals to some nonlocal e.m. effect, inside the narrower part of the waveguide, which can be described in terms of an effective deformation of space-time inside the barrier region [20]. Since we are dealing with electromagnetic forces (which are usually described by the Minkowskian metric), we can assume $b_0^2 = 1$ (this is also justified by the fact that all the relevant deformed quantities depend actually on the ratio b/b_0). Assuming moreover an isotropically deformed three-space ($b_1 = b_2 = b_3 = b$)⁶, one gets [20]

$$g_{DSR,e.m.}(E) = \text{diag} (1, -b_{e.m.}^2(E), -b_{e.m.}^2(E), -b_{e.m.}^2(E)); \quad (10)$$

$$b_{e.m.}^2(E) = \begin{cases} (E/E_{0e.m.})^{1/3}, & 0 \leq E \leq E_{0e.m.} \\ 1, & E_{0e.m.} \leq E \end{cases} = \\ = 1 + \Theta(E_{0e.m.} - E) \left[\left(\frac{E}{E_{0e.m.}} \right)^{1/3} - 1 \right], E > 0. \quad (11)$$

(where $\Theta(x)$ is the Heaviside theta function, stressing the piecewise structure of the metric). The threshold energy $E_{0e.m.}$ is the energy value at which the metric parameters are constant, i.e. the metric becomes Minkowskian. The fit to the experimental data yields

$$E_{0,e.m.} = (4.5 \pm 0.2) \mu eV. \quad (12)$$

Notice that the value obtained for E_0 is of the order of the energy corresponding to the coherence length of a photon for radio-optical waves ($E_{coh} \simeq 1 \mu eV$).

2) **Weak interaction.** The experimental input was provided by the data on the pure leptonic decay of the meson K_s^0 , whose lifetime τ is known in a wide energy range ($30 \div 350 GeV$) [29] (an almost unique case). Use has been made of the deformed law of time dilation as a function of the energy, which reads [20]

$$\tau = \frac{\tau_0}{\left[1 - \left(\frac{b}{b_0} \right)^2 + \left(\frac{b}{b_0} \right)^2 \left(\frac{m}{E} \right)^2 \right]^{1/2}}. \quad (13)$$

⁶ Notice that the assumption of spatial isotropy for the electromagnetic interaction in the waveguide propagation is only a matter of convenience, since waveguide experiments do not provide any physical information on space directions different from the propagation one (the axis of the waveguide). An analogous consideration holds true for the weak case, too (see below).

As in the electromagnetic case, an isotropic three-space was assumed, whereas the isochrony with the usual Minkowski metric (i.e. $b_0^2 = 1$) was *derived* by the fit of (13) to the experimental data. The corresponding metric is therefore given by

$$g_{DSR,weak}(E) = \text{diag} \left(1, -b_{weak.}^2(E), -b_{weak.}^2(E), -b_{weak.}^2(E) \right); \quad (14)$$

$$b_{weak.}^2(E) = \begin{cases} (E/E_{0weak})^{1/3}, & 0 \leq E \leq E_{0weak} \\ 1 & E_{0weak} \leq E \end{cases} = . \\ = 1 + \Theta(E_{0weak.} - E) \left[\left(\frac{E}{E_{0weak}} \right)^{1/3} - 1 \right], E > 0. \quad (15)$$

with

$$E_{0,weak} = (80.4 \pm 0.2) \text{ GeV}. \quad (16)$$

Two points are worth stressing. First, the value of E_{0weak} — i.e. the energy value at which the weak metric becomes Minkowskian — corresponds to the mass of the W -boson, through which the K_s^0 -decay occurs. Moreover, the leptonic metric (14)-(15) has the same form of the electromagnetic metric (10)-(11). Therefore, one recovers, by the DSR formalism, the well-known result of the Glashow-Weinberg-Salam model that, at the energy scale E_{0weak} , the weak and the electromagnetic interactions are mixed. We want also to notice that, in both the electromagnetic and the weak case, the metric parameter exhibits a "sub-Minkowskian" behavior, i.e. $b(E)$ approaches 1 from below as energy increases.

3) **Strong interaction.** The phenomenon considered is the so-called Bose-Einstein (BE) effect in the strong production of identical bosons in high-energy collisions, which consists in an enhancement of their correlation probability [28]. The DSR formalism permits to derive a generalized BE correlation function, depending on all the four metric parameters $b_\mu(E)$ [20]. By using the experimental data on pion pair production, obtained in 1984 by the UA1 group at CERN [29], one gets the following expression of the strong metric for the two-pion BE phenomenon [20]:

$$g_{DSR,strong}(E) = \\ = \text{diag} \left(b_{strong}^2(E), -b_{1,strong}^2(E), -b_{2,strong}^2(E), -b_{strong}^2(E) \right); \quad (17)$$

$$b_{strong}^2(E) = \begin{cases} 1, & 0 \leq E \leq E_{0strong} \\ (E/E_{0strong})^2, & E_{0strong} \leq E \end{cases} = \\ = 1 + \Theta(E - E_{0,strong}) \left[\left(\frac{E}{E_{0,strong}} \right)^2 - 1 \right], E > 0; \quad (18)$$

$$b_{1,strong}^2(E) = \left(\sqrt{2}/5\right)^2; \quad (19)$$

$$b_{2,strong}^2 = (2/5)^2. \quad (20)$$

with

$$E_{0,strong} = (367.5 \pm 0.4) \text{ GeV}. \quad (21)$$

The threshold energy $E_{0,strong}$ is still the value at which the metric becomes Minkowskian. Let us stress that, in this case, contrarily to the electromagnetic and the weak ones, *a deformation of the time coordinate occurs*; moreover, *the three-space is anisotropic*, with two spatial parameters constant (but different in value) and the third one variable with energy in an "over-Minkowskian" way. It is also worth to recall that the strong metric parameters b_μ admit of a sensible physical interpretation: the spatial parameters are (related to) the spatial sizes of the interaction region ("fireball") where pions are produced, whereas the time parameter is essentially the mean life of the process. We refer the reader to ref.[20] for further details.

4) **Gravitation.** It is possible to show that the gravitational interaction, too (at least on a *local* scale, i.e. in a neighborhood of Earth) can be described in terms of an energy-dependent metric, whose time coefficient was derived by fitting the experimental results on the relative rates of clocks at different heights in the gravitational field of Earth [30]. No information can be derived from the experimental data about the space parameters. Physical considerations — for whose details the reader is referred to ref.[20] — lead to assume a gravitational metric of the same type of the strong one, *i.e.* spatially anisotropic and with one spatial parameter (say, b_3) equal to the time one: $b_0(E) = b_3(E) = b(E)$. The energy-dependent gravitational metric has therefore the form

$$g_{DSR,grav}(E) = \text{diag} \left(b_{grav}^2(E), -b_{1,grav}^2(E), -b_{2,grav}^2(E), -b_{grav}^2(E) \right); \quad (22)$$

$$b_{grav}^2(E) = \begin{cases} 1, & 0 \leq E \leq E_{0grav} \\ \frac{1}{4}(1 + E/E_{0grav})^2, & E_{0grav} \leq E \end{cases} = \\ = 1 + \Theta(E - E_{0,grav.}) \left[\frac{1}{4} \left(1 + \frac{E}{E_{0,grav.}} \right)^2 - 1 \right], E > 0 \quad (23)$$

(as already said, the coefficients $b_{1,grav}^2(E)$ and $b_{2,grav}^2(E)$ are presently undetermined at phenomenological level), with

$$E_{0,grav} = (20.2 \pm 0.1) \mu eV. \quad (24)$$

The gravitational metric (22),(23) is *over-Minkowskian*, asymptotically Minkowskian with decreasing energy, like the strong one. Intriguingly enough, the value of the threshold energy for the gravitational case E_{0grav} is approximately of the same order of magnitude of the thermal energy corresponding to the $2.7^\circ K$ cosmic background radiation in the Universe.

Moreover, the comparison of the values of the threshold energies for the four fundamental interactions yields

$$E_{0e.m.} < E_{0grav} < E_{0weak} < E_{0strong} \quad (25)$$

i.e. an increasing arrangement of E_0 from the electromagnetic to the strong interaction. Moreover

$$\frac{E_{0grav}}{E_{0e.m.}} = 4.49 \pm 0.02 ; \frac{E_{0strong}}{E_{0weak}} = 4.57 \pm 0.01, \quad (26)$$

namely

$$\frac{E_{0grav}}{E_{0e.m.}} \simeq \frac{E_{0strong}}{E_{0weak}} \quad (27)$$

an intriguing result indeed.

Let us finally stress that in the last decade we designed and carried out new experiments, related to all four fundamental interactions, which test and confirm some of the predictions of the DSR formalism, thus providing a possible evidence for the deformation of space-time. Among them, let us quote the anomalous behavior of some photon systems (at variance with respect to standard electrodynamics and quantum mechanics) [31-35]; the measurement of the speed of propagation of gravitational effects [20]; the effectiveness of ultrasounds in speeding up the decay of radioactive elements and in triggering nuclear reactions in liquid solutions [36-38, 20]. Moreover, such experiments point out the need for considering the energy as a fifth coordinate, thus casting a bridge toward the five-dimensional formalism of DR5. We refer the reader to second ref. [20] for an in-depth discussion of this point.

3. Embedding Deformed Minkowski Space in a Five-Dimensional Riemann Space

3.1 From LLI Breakdown to Energy as Fifth Dimension

Both the analysis of the physical processes considered in deriving the phenomenological energy-dependent metrics for the four fundamental interactions, and the experiments [31-37], seem to provide evidence (indirect and direct, respectively) for a breakdown of LLI invariance (at least in its usual, special-relativistic sense). But it is well known that, in general, the breakdown of a symmetry is the signature of the need for a *wider, exact* symmetry. In the case of the breaking of a space-time symmetry — as the Lorentz one — this is often related to the possible occurrence of higher-dimensional schemes. This is indeed the case, and *energy does in fact represent an extra dimension*.

In the description of interactions by energy-dependent metrics, we saw that energy plays in fact a *dual* role. Indeed, on one side, it constitutes a *dynamic variable*, because it specifies the dynamical behavior of the process under consideration, and, via the metric coefficients, it provides us with a dynamical map — in the energy range of interest — of the interaction ruling the given process. On the other hand, it represents a *parameter*

characteristic of the phenomenon considered (and therefore, for a given process, it cannot be changed at will). In other words, when describing a given process, the deformed geometry of space-time (in the interaction region where the process is occurring) is “frozen” at the situation described by those values of the metric coefficients $\{b_\mu^2(E)\}_{\mu=0,1,2,3}$ corresponding to the energy value of the process considered. Namely, a fixed value of E determines the space-time structure of the interaction region at that given energy. In this respect, therefore, the energy of the process has to be considered as a *geometrical quantity* intimately related to the very geometrical structure of the physical world. In other words, from a geometrical point of view, all goes on as if were actually working on “slices” (sections) of a five-dimensional space, in which the extra dimension is just represented by the energy. In other words, a fixed value of the energy determines the space-time structure of the interaction region for the given process *at that given energy*. In this respect, therefore, E is to be regarded as a *geometrical quantity*, intimately connected to the very geometrical structure of the physical world itself. Then, the four-dimensional, deformed, energy-dependent space-time is just a manifestation (or a “shadow”, to use the famous word of Minkowski) of a larger space with energy as fifth dimension.

The simplest way to take account of (and to make explicit) the double role of energy in DSR is assuming that E represents an extra metric dimension — on the same footing of space and time — and therefore embedding the 4-d deformed Minkowski space $\widetilde{M}(E)$ of DSR in a 5-d (Riemannian) space-time-energy manifold \mathfrak{R}_5 . This leads to build up a “Kaluza-Klein-like” scheme, with energy as fifth dimension, we shall refer to in the following as *Five-Dimensional Deformed Relativity* (DR5) [20, 22].

3.2 The 5-dimensional Space-Time-Energy Manifold \mathfrak{R}_5

On the basis of the above arguments, we assume therefore that physical phenomena do occur in a world which is actually described by a 5-dimensional space-time-energy manifold \mathfrak{R}_5 endowed with the energy-dependent metric⁷:

$$\begin{aligned} g_{AB,DR5}(E) &\equiv \text{diag}(b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E), f(E)) \stackrel{\text{ESC}}{=} \text{off} \\ &= \delta_{AB} (b_0^2(E)\delta_{A0} - b_1^2(E)\delta_{A1} - b_2^2(E)\delta_{A2} - b_3^2(E)\delta_{A3} + f(E)\delta_{A5}). \end{aligned} \quad (28)$$

It follows from Eq.(28) that E , which is an independent *non-metric* variable in DSR, becomes a *metric* coordinate in \mathfrak{R}_5 . Then, whereas $g_{\mu\nu,DSR}(E)$ (given by Eq.(2)) is a deformed, Minkowskian metric tensor, $g_{AB,DR5}(E)$ is a genuine Riemannian metric tensor.

⁷ In the following, capital Latin indices take values in the range $\{0, 1, 2, 3, 5\}$, with index 5 labelling the fifth dimension. We choose to label by 5 the extra coordinate, instead of using 4, in order to avoid confusion with the notation often adopted for the (imaginary) time coordinate in a (formally) Euclidean Minkowski space.

Therefore, the infinitesimal interval of \mathfrak{R}_5 is given by:

$$\begin{aligned} ds_{DR5}^2(E) &\equiv dS^2(E) \equiv g_{AB,DR5}(E)dx^A dx^B = \\ &= b_0^2(E) (dx^0)^2 - b_1^2(E) (dx^1)^2 - b_2^2(E) (dx^2)^2 - b_3^2(E) (dx^3)^2 + f(E) (dx^5)^2 = \\ &= b_0^2(E)c^2 (dt)^2 - b_1^2(E) (dx^1)^2 - b_2^2(E) (dx^2)^2 - b_3^2(E) (dx^3)^2 + f(E)l_0^2 (dE)^2, \end{aligned} \quad (29)$$

where we have put

$$x^5 \equiv l_0 E, \quad l_0 > 0. \quad (30)$$

Since the space-time metric coefficients are dimensionless, it can be assumed that they are functions of the ratio E/E_0 , where E_0 is an energy scale characteristic of the interaction (and the process) considered (for instance, the energy threshold in the phenomenological metrics of Subsect.2.2). The coefficients $\{b_\mu^2(E)\}$ of the metric of $\widetilde{M}(E)$ can be therefore expressed as

$$\left\{ b_\mu \left(\frac{E}{E_0} \right) \right\} \equiv \left\{ b_\mu \left(\frac{x^5}{x_0^5} \right) \right\} = \{ b_\mu(x^5) \}, \quad \forall \mu = 0, 1, 2, 3 \quad (31)$$

where we put

$$x_0^5 \equiv l_0 E_0. \quad (32)$$

As to the fifth metric coefficient, one assumes that it too is a function of the energy only: $f = f(E) \equiv f(x^5)$ (although, in principle, nothing prevents from assuming that, in general, f may depend also on space-time coordinates $\{x^\mu\}$, $f = f(\{x^\mu\}, x^5)$). Unlike the other metric coefficients, it may be $f(E) \leq 0$. Therefore, *a priori*, the energy dimension may have either a timelike or a spacelike signature in \mathfrak{R}_5 , depending on $\text{sgn}(f(E)) = \pm 1$. In the following, it will be sometimes convenient assuming $f(E) \in R_0^+$ and explicitly introducing the double sign in front of the fifth coefficient.

In terms of x^5 , the (*covariant*) metric tensor can be written as

$$\begin{aligned} g_{AB,DR5}(x^5) &= \text{diag}(b_0^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5), \pm f(x^5)) = \\ &\stackrel{ESC \text{ off}}{=} \delta_{AB} [b_0^2(x^5)\delta_{A0} - b_1^2(x^5)\delta_{A1} - b_2^2(x^5)\delta_{A2} - b_3^2(x^5)\delta_{A3} \pm f(x^5)\delta_{A5}]. \end{aligned} \quad (33)$$

On account of the relation

$$g_{DR5}^{AB}(x^5)g_{BC,DR5}(x^5) = \delta_C^A, \quad (34)$$

the *contravariant* metric tensor reads

$$\begin{aligned} g_{DR5}^{AB}(x^5) &= \text{diag}(b_0^{-2}(x^5), -b_1^{-2}(x^5), -b_2^{-2}(x^5), -b_3^{-2}(x^5), \pm (f(x^5))^{-1}) = \\ &\stackrel{ESC \text{ off}}{=} \delta_{AB} \left[\begin{array}{c} b_0^{-2}(x^5)\delta_{A0} - b_1^{-2}(x^5)\delta_{A1} - b_2^{-2}(x^5)\delta_{A2} - b_3^{-2}(x^5)\delta_{A3} + \\ \pm (f(x^5))^{-1} \delta_{A5} \end{array} \right]. \end{aligned} \quad (35)$$

The space \mathfrak{R}_5 has the following "slicing property"

$$\mathfrak{R}_5|_{dx^5=0 \Leftrightarrow x^5=\bar{x}^5} = \widetilde{M}(\bar{x}^5) = \left\{ \widetilde{M}(x^5) \right\}_{x^5=\bar{x}^5} \quad (36)$$

(where \bar{x}^5 is a fixed value of the fifth coordinate) or, at the level of the metric tensor:

$$\begin{aligned} g_{AB,DR5}(x^5)|_{dx^5=0 \Leftrightarrow x^5=\bar{x}^5 \in R_0^+} &= \\ &= \text{diag} \left(b_0^2(\bar{x}^5), -b_1^2(\bar{x}^5), -b_2^2(\bar{x}^5), -b_3^2(\bar{x}^5), \pm f(\bar{x}^5) \right) = g_{AB,DSR}(\bar{x}^5). \end{aligned} \quad (37)$$

We recall that in general, in the framework of 5-d Kaluza-Klein (KK) theories, the fifth dimension must be necessarily spacelike, since, in order to avoid the occurrence of causal (loop) anomalies, the number of timelike dimensions cannot be greater than one. But it is worth to stress that the present theory is not a Kaluza-Klein one. In "true" KK theories, due to the lack of observability of the extra dimensions, it is necessary to impose to them the cylindrical condition. This is not required in the framework of DR5, since the fifth dimension (energy) is a physically observable quantity (think to the Minkowski space of standard SR: There is no need to hide the fourth dimension, since time is an observable quantity). Actually, in DR5 not only the cylindrical condition is not implemented, but it is even reversed. In fact, the metric tensor $g_{AB,DR5}(x^5)$ depends only on the fifth coordinate x^5 . Therefore, one does not assume the compactification of the extra coordinate (one of the main methods of implementing the cylindrical condition in modern hyperdimensional KK theories, as discussed in Subsect.1.1), which remains therefore extended (*i.e.* with infinite compactification radius). DR5 belongs therefore to the class of noncompactified KK theories.

The problem of the possible occurrence of causal anomalies in presence of more time-like dimensions is then left open in the "pseudo-Kaluza-Klein" context of DR5. This is reflected in the uncertainty in the sign of the energy metric coefficient $f(x^5)$. In particular, it cannot be excluded *a priori* that the signature of x^5 can change. This occurs whenever the function $f(x^5)$ does vanish for some energy values. As a consequence, in correspondence to the energy values which are zeros of $f(x^5)$, the metric $g_{AB,DR5}(x^5)$ is *degenerate*.

3.3 DR5 and Warped Geometry

We have seen in Subsect.1.1 that in some multidimensional spacetime theories the geometry of the extra (spatial) dimension(s) is warped, in order to avoid recourse to compactification.

If ζ is the extra coordinate, a typical 5-d. warped interval reads

$$dS^2 = a(\zeta)g_{\mu\nu}dx^\mu dx^\nu - d\zeta^2, \quad (38)$$

where $g_{\mu\nu}$ is the Minkowski metric and $a(\zeta)$ the warp factor, given by

$$a(\zeta) = e^{-k|\zeta|}. \quad (39)$$

The decay constant $k > 0$ is proportional to σ , the energy density (per unit three-volume) of the brane. It is assumed of course that the brane is located at $\zeta = 0$. As a consequence, the metric induced on the brane is Minkowskian.

It can be shown that metric (38) is a solution of the five-dimensional Einstein equations with a cosmological constant proportional to the square of the energy density of the brane: $\Lambda_{(5)} \sim \sigma^2$.

A generalization of the above warped interval is

$$\begin{aligned} dS^2 &= a(\zeta)c^2dt^2 - b(\zeta)d\mathbf{x}^2 - d\zeta^2; \\ a(0) &= b(0) = 1, \end{aligned} \tag{40}$$

where the warp factors of time and 3-d. space, $a(\zeta)$ and $b(\zeta)$, are different. This metric accounts for Lorentz-violating effects, provided the wave function of the particles spreads in the fifth dimension (this may be the case for gravitons).

We want to remark that intervals (38), (40) are just special cases of the DR5 interval (29). In fact either metric is, for the space-time part, a spatially isotropic one with $f(E) = -1$.

Of course, the warped metrics (38), (40) and the DR5 metrics have a profound physical difference. In the former ones, ζ is an hidden space dimension: the (exponentially decaying) warp factors are just introduced in order to make this extra dimension unobservable. In the DR5 framework, the fifth dimension is the energy, and therefore an observable physical quantity. This entails, among the others, that the metric coefficients may have any functional form. Moreover — as repeatedly stressed — DSR and DR5 metrics are assumed to provide a *local* description of the physical processes ruled by one of the fundamental interactions; on the contrary, warped metrics do describe the physical world at a *large, global* scale.

However, the similar structure of the two types of metrics entails that the mathematical study of the formal properties of the space \mathfrak{R}_5 of DR5 can be of some utility for warped geometry, too. Results obtained *e.g.* for DR5 isometries may hold in some cases for the warped models (or be adapted with suitable changes). This provides a further reason to exploring the mathematical features of DR5.

3.4 5-d. Metrics of Fundamental Interactions

3.4.1 Phenomenological Metrics

Let us now consider the 4-d. metrics of the deformed Minkowski spaces $\widetilde{M}(x^5)$ for the four fundamental interactions (electromagnetic, weak, strong and gravitational) (see Subsect.2.2). In passing from the deformed, special-relativistic 4-d. framework of DSR to the general-relativistic 5-d. one of DR5 — geometrically corresponding to the embedding of the deformed 4-d. Minkowski spaces $\left\{ \widetilde{M}(x^5) \right\}_{x^5 \in R_0^+}$ (where x^5 is a parameter) in the 5-d. Riemann space \mathfrak{R}_5 (where x^5 is a metric coordinate), in general the phenomenological metrics (10)-(11), (14)-(15), (17)-(20), (22)-(23) take the following 5-d.form ($f(x^5) \in R_0^+$

$\forall x^5 \in R_0^+$:

$$\begin{aligned}
& g_{AB,DR5,e.m.}(x^5) = \\
& = \text{diag} \left(1, - \left\{ 1 + \widehat{\Theta}(x_{0,e.m.}^5 - x^5) \left[\left(\frac{x^5}{x_{0,e.m.}^5} \right)^{1/3} - 1 \right] \right\}, \right. \\
& \quad \left. - \left\{ 1 + \widehat{\Theta}(x_{0,e.m.}^5 - x^5) \left[\left(\frac{x^5}{x_{0,e.m.}^5} \right)^{1/3} - 1 \right] \right\}, \right. \\
& \quad \left. - \left\{ 1 + \widehat{\Theta}(x_{0,e.m.}^5 - x^5) \left[\left(\frac{x^5}{x_{0,e.m.}^5} \right)^{1/3} - 1 \right] \right\}, \pm f(x^5) \right); \tag{41}
\end{aligned}$$

$$\begin{aligned}
& g_{AB,DR5,weak}(x^5) = \\
& = \text{diag} \left(1, - \left\{ 1 + \widehat{\Theta}(x_{0,weak}^5 - x^5) \left[\left(\frac{x^5}{x_{0,weak}^5} \right)^{1/3} - 1 \right] \right\}, \right. \\
& \quad \left. - \left\{ 1 + \widehat{\Theta}(x_{0,weak}^5 - x^5) \left[\left(\frac{x^5}{x_{0,weak}^5} \right)^{1/3} - 1 \right] \right\}, \right. \\
& \quad \left. - \left\{ 1 + \widehat{\Theta}(x_{0,weak}^5 - x^5) \left[\left(\frac{x^5}{x_{0,weak}^5} \right)^{1/3} - 1 \right] \right\}, \pm f(x^5) \right); \tag{42}
\end{aligned}$$

$$\begin{aligned}
& g_{AB,DR5,strong}(x^5) = \\
& = \text{diag} \left(1 + \widehat{\Theta}(x^5 - x_{0,strong}^5) \left[\left(\frac{x^5}{x_{0,strong}^5} \right)^2 - 1 \right], - \left(\frac{\sqrt{2}}{5} \right)^2, \right. \\
& \quad \left. - \left(\frac{2}{5} \right)^2, - \left\{ 1 + \widehat{\Theta}(x^5 - x_{0,strong}^5) \left[\left(\frac{x^5}{x_{0,strong}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right); \tag{43}
\end{aligned}$$

$$\begin{aligned}
& g_{AB,DR5,grav.}(x^5) = \\
& = \text{diag} \left(1 + \widehat{\Theta}(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], -b_{1,grav.}^2(x^5), \right. \\
& \quad \left. -b_{2,grav.}^2(x^5), - \left\{ 1 + \widehat{\Theta}(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right). \tag{44}
\end{aligned}$$

As we are going to show, all the above metrics — derived on a mere phenomenological basis, from the experimental data on some physical phenomena ruled by the four fundamental interactions, at least as far as their space-time part is concerned — can be recovered as solutions of the vacuum Einstein equations in the five-dimensional space \mathfrak{R}_5 , natural covering of the deformed Minkowski space $\widetilde{M}(x^5)$.

3.4.2 Power Ansatz

We have seen in Subsect.3.3 that the space-time metric coefficients can be considered functions of the ratio E/E_0 (see Eq.(31)). Therefore, for the metric g_{DR5} written in the form (29), we can put, following also the hints from phenomenology (see Subsect.2.2):

$$\left\{ \begin{array}{l} b_0^2(E) = (E/E_0)^{q_0} ; \\ b_1^2(E) = (E/E_0)^{q_1} ; \\ b_2^2(E) = (E/E_0)^{q_2} ; \\ b_3^2(E) = (E/E_0)^{q_3} ; \\ f(E) = (E/E_0)^r , \end{array} \right. \quad (45)$$

$r, q_\mu \in \mathbb{R} \quad \forall \mu = 0, 1, 2, 3.$

The corresponding 5-d. metric reads therefore

$$g_{AB,DR5power}(\tilde{\mathbf{q}}, x^5) = \text{diag} \left(\left(\frac{x^5}{x_0^5} \right)^{q_0}, - \left(\frac{x^5}{x_0^5} \right)^{q_1}, - \left(\frac{x^5}{x_0^5} \right)^{q_2}, - \left(\frac{x^5}{x_0^5} \right)^{q_3}, \pm \left(\frac{x^5}{x_0^5} \right)^r \right) \quad (46)$$

($q_0, q_1, q_2, q_3, r \in \mathbb{Q}$, $A, B = 0, 1, 2, 3, 5$), in which the double sign of the energy coefficient has been made clear and the (fake) 5-vector $\tilde{\mathbf{q}} \equiv (q_0, q_1, q_2, q_3, r)$ introduced⁸. In the following we shall refer to the form (45) as the "Power Ansatz".

We have seen in 3.5.1 that embedding the DSR phenomenological metrics for the four interactions in \mathfrak{R}_5 leads to expressions (41)-(44). In the context of the Power Ansatz, and

⁸ In the following, we shall use the tilded-bold notation $\tilde{\mathbf{v}}$ for a (true or fake) vector in \mathfrak{R}_5 , in order to distinguish it from a vector \mathbf{v} in the usual 3-d.space.

by making their piecewise structure explicit, they can be written in the form

$$\begin{aligned}
 & g_{AB,DR5,e.m.,weak}(x^5) = \\
 & = \left\{ \begin{array}{l}
 \text{diag} \left(1, - \left(\frac{x^5}{x_{0,e.m.,weak}^5} \right)^{1/3}, - \left(\frac{x^5}{x_{0,e.m.,weak}^5} \right)^{1/3}, \right. \\
 \left. - \left(\frac{x^5}{x_{0,e.m.,weak}^5} \right)^{1/3}, \pm \left(\frac{x^5}{x_{0,e.m.,weak}^5} \right)^r \right), \\
 0 < x^5 < x_{0,e.m.,weak}^5; \\
 \text{diag} \left(1, -1, -1, -1, \pm \left(\frac{x^5}{x_{0,e.m.,weak}^5} \right)^r \right), \\
 x^5 \geq x_{0,e.m.,weak}^5;
 \end{array} \right.
 \end{aligned} \tag{47}$$

$$\begin{aligned}
 & g_{AB,DR5,strong}(x^5) = \\
 & = \left\{ \begin{array}{l}
 \text{diag} \left(\left(\frac{x^5}{x_{0,strong}^5} \right)^2, - \left(\frac{\sqrt{2}}{5} \right)^2, - \left(\frac{2}{5} \right)^2, \right. \\
 \left. - \left(\frac{x^5}{x_{0,strong}^5} \right)^2, \pm \left(\frac{x^5}{x_{0,strong}^5} \right)^r \right), \\
 x^5 > x_{0,strong}^5; \\
 \text{diag} \left(1, - \left(\frac{\sqrt{2}}{5} \right)^2, - \left(\frac{2}{5} \right)^2, -1, \pm \left(\frac{x^5}{x_{0,strong}^5} \right)^r \right), \\
 0 < x^5 \leq x_{0,strong}^5;
 \end{array} \right.
 \end{aligned} \tag{48}$$

$$\begin{aligned}
& g_{AB,DR5,grav.}(x^5) = \\
& = \begin{cases} \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -b_{1,grav.}^2(x^5), -b_{2,grav.}^2(x^5), \right. \\ \left. \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, \pm \left(\frac{x^5}{x_{0,grav.}^5} \right)^r \right), \\ x^5 > x_{0,grav.}^5; \\ \text{diag} \left(1, -b_{1,grav.}^2(x^5), -b_{2,grav.}^2(x^5), -1, \pm \left(\frac{x^5}{x_{0,grav.}^5} \right)^r \right), \\ 0 < x^5 \leq x_{0,grav.}^5. \end{cases} \tag{49}
\end{aligned}$$

In the gravitational metric $g_{AB,DR5,grav.}(x^5)$ the expressions of the two space coefficients $b_{1,grav.}^2(x^5)$ and $b_{2,grav.}^2(x^5)$ have not been made explicit, due to their indeterminacy at experimental level.

The phenomenological 5-d. metrics in the Power Ansatz are therefore characterized by the parameter sets

$$\tilde{\mathbf{q}}_{e.m./weak} = \begin{cases} (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, r), & 0 < x^5 < x_{0,e.m./weak}^5; \\ (0, 0, 0, 0, r), & x^5 \geq x_{0,e.m./weak}^5; \end{cases} \tag{50}$$

$$\tilde{\mathbf{q}}_{strong} = \begin{cases} (2, (0, 0), 2, r), & x^5 > x_{0,strong}^5; \\ (0, (0, 0), 0, r), & 0 < x^5 \leq x_{0,strong}^5; \end{cases} \tag{51}$$

$$\tilde{\mathbf{q}}_{grav.} = \begin{cases} (\underline{2}, ?, ?, \underline{2}, r), & x^5 > x_{0,grav.}^5; \\ (0, ?, ?, 0, r), & 0 < x^5 \leq x_{0,grav.}^5. \end{cases} \tag{52}$$

where the question marks "?" reflect the unknown nature of the two gravitational spatial coefficients.

Let us clarify the notation adopted for $\tilde{\mathbf{q}}_{strong}$ and $\tilde{\mathbf{q}}_{grav.}$. The zeros in brackets in $\tilde{\mathbf{q}}_{strong}$ reflect the fact that such exponents do not refer to the metric tensor $g_{AB,DR5power}(x^5)$ (Eq.(45)), but to the more general tensor

$$\begin{aligned}
& g_{AB,DR5power-conform}(x^5) = \\
& = \text{diag} \left(\vartheta_0 \left(\frac{x^5}{x_0^5} \right)^{q_0}, -\vartheta_1 \left(\frac{x^5}{x_0^5} \right)^{q_1}, -\vartheta_2 \left(\frac{x^5}{x_0^5} \right)^{q_2}, -\vartheta_3 \left(\frac{x^5}{x_0^5} \right)^{q_3}, \pm \vartheta_5 \left(\frac{x^5}{x_0^5} \right)^r \right), \tag{53}
\end{aligned}$$

with $\tilde{\vartheta} = (\vartheta_A)$ being a constant 5-vector. Eq.(52) can be written in matrix form as

$$\mathbf{g}_{DR5power-conform}(x^5) = \mathbf{g}_{DR5power}(x^5) \tilde{\vartheta} \tag{54}$$

where $\tilde{\vartheta}$ is meant to be a column vector. The passage from the metric tensor $g_{AB,DR5power}(x^5)$ to $g_{AB,DR5power-conform}(x^5)$ is obtained by means of the tensor transformation law in \mathfrak{R}_5 (ESC on)

$$g_{AB,DR5power-conform}(x^5) = \frac{\partial x^K}{\partial x'^A} \frac{\partial x^L}{\partial x'^B} g_{KL,DR5power}(x^5) \quad (55)$$

induced by the following 5-d. *anisotropic rescaling* of the coordinates of \mathfrak{R}_5 :

$$dx^A = \sqrt{\vartheta_A} dx'_A \leftrightarrow x^A = \sqrt{\vartheta_A} x'_A. \quad (56)$$

Such a transformation allows one to get, in the Power Ansatz, metric coefficients *constant* (i.e. independent of the energy) *but different*. This is just the case of the two constant space coefficients $b_1^2(x^5)$, $b_2^2(x^5)$ in the strong metric. In this case, the vector ϑ explicitly reads

$$\tilde{\vartheta}_{strong} = \left(0, \left(\frac{\sqrt{2}}{5} \right)^2, \left(\frac{2}{5} \right)^2, 0, ? \right), \quad (57)$$

where the question mark "?" reflects again the unknown nature of $\pm f(x^5)$.

The underlined 2, $\underline{2}$, in $\tilde{\mathbf{q}}_{grav.}$ are due to the fact that actually the functional form of the related metric coefficients is not $\left(\frac{x^5}{x_{0,grav.}^5} \right)^2$ but $\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2$. Again, it is possible to recover the phenomenological 5-d. metric $g_{AB,DR5,grav.}(x^5)$ from the Power Ansatz form $g_{AB,DR5power}(x^5)$ by a rescaling and a translation of the energy. In fact, one has

$$x^5 \longrightarrow x^{5'} = x^5 - \overline{x_0^5} \Leftrightarrow dx^{5'} = dx^5. \quad (58)$$

Such a translation in energy is allowed because we are just working in the framework of DR5. Therefore

$$b_0^2(x^5) = b_0^2(x^{5'} + \overline{x_0^5}) = b_{0,new}^2(x^{5'}) = \left(\frac{x^{5'} + \overline{x_0^5}}{x_0^5} \right)^2 = \left(\frac{x^{5'}}{x_0^5} + \frac{\overline{x_0^5}}{x_0^5} \right)^2. \quad (59)$$

By rescaling the threshold energy (in a physically consistent way, because it amounts to a redefinition of the scale of measure of energy)

$$x_0^5 \longrightarrow x_0^{5'} = x_0^5 \left(\frac{\tilde{x_0^5}}{x_0^5} \right), \quad (60)$$

one gets

$$b_{0,new}^2(x^{5'}) = \left(\frac{x^{5'} \tilde{x_0^5}}{x_0^{5'} x_0^5} + \frac{\tilde{x_0^5}}{x_0^5} \right)^2 = \left(\frac{\tilde{x_0^5}}{x_0^{5'}} \right)^2 \left(1 + \frac{x^{5'}}{x_0^5} \right)^2. \quad (61)$$

This metric time coefficient is of the gravitational type, except for the factor $\left(\frac{\tilde{x_0^5}}{x_0^{5'}} \right)^2$, which however can be got rid of by the following rescaling of the time coordinate:

$$x^0 \longrightarrow x'^0 = \frac{\tilde{x_0^5}}{x_0^{5'}} x^0. \quad (62)$$

This is a conformal transformation corresponding to a redefinition of the scale of measure of time. Notice that the above rescaling procedure of energy and time does not account for the factor $1/4$ in front of $\left(1 + \frac{x^5}{x_{0,grav.}^5}\right)^2$. This can be dealt with by the method followed for $\tilde{\mathbf{q}}_{strong}$, namely by considering the generalized metric $g_{AB,DR5power-conform}(x^5)$, where now the vector $\tilde{\vartheta}$ is given by $\tilde{\vartheta} = (\frac{1}{4}, ?, ?, \frac{1}{4}, ?)$ (as before, the question marks reflect the unknown nature of the related metric coefficients).

Notice that both in Eqs.(46)-(48) and in Eqs.(49)-(51) it was *assumed* that

$$q_{\mu.int.}(x_{0,int.}^5) = 0, \quad \mu = 0, 1, 2, 3, \text{ int.} = e.m., weak, strong, grav., \quad (63)$$

for simplicity reasons, since — as already stressed in Subject.2.2 — nothing can be said on the behavior of the metrics at the energy thresholds.

Let us introduce the *left and right specifications* $\hat{\Theta}_L(x)$, $\hat{\Theta}_R(x)$ of the Heaviside theta function, defined respectively by

$$\hat{\Theta}_L(x) \equiv \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}, \quad (64)$$

$$\hat{\Theta}_R(x) \equiv \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad (65)$$

and satisfying the complementarity relation

$$1 - \hat{\Theta}_R(x) = \hat{\Theta}_L(x). \quad (66)$$

Then, the exponent sets (49)-(51) can be written in compact form as

$$\begin{aligned} \tilde{\mathbf{q}}_{e.m./weak} &= \left(0, \frac{1}{3}\hat{\Theta}_L(x_{0,e.m./weak}^5 - x^5), \right. \\ &\left. \frac{1}{3}\hat{\Theta}_L(x_{0,e.m./weak}^5 - x^5), \frac{1}{3}\hat{\Theta}_L(x_{0,e.m./weak}^5 - x^5), r\right), \end{aligned} \quad (67)$$

$$\tilde{\mathbf{q}}_{strong} = \left(2\hat{\Theta}_L(x^5 - x_{0,strong}^5), (0, 0), 2\hat{\Theta}_L(x^5 - x_{0,strong}^5), r\right), \quad (68)$$

$$\tilde{\mathbf{q}}_{grav.} = \left(2\hat{\Theta}_L(x^5 - x_{0,grav.}^5), ?, ?, 2\hat{\Theta}_L(x^5 - x_{0,grav.}^5), r\right), \quad (69)$$

where the underlining and the question marks have the same meaning as above.

3.5 Einstein's Field Equations in \mathfrak{R}_5 and Their Solutions

3.5.1 Riemannian Structure of \mathfrak{R}_5

We have seen that, unlike $\widetilde{M}(x^5)$, which is a flat pseudoeuclidean space, \mathfrak{R}_5 is a genuine Riemann one. Its affine structure is determined by the five-dimensional affine connection

$\Gamma_{BC}^A(x^5)$ (which rules the parallel transport of vectors in \mathfrak{R}_5), defined by

$$\Gamma_{BC}^A(x^5) \equiv \frac{\partial x^A}{\partial \xi^D} \frac{\partial^2 \xi^D}{\partial x^B \partial x^C}, \quad (70)$$

where $\{\xi^A\}$, $\{x^A\}$ are the coordinates in a locally inertial (Lorentzian) frame and in a generic frame, respectively. Let us recall that Γ_{BC}^A is *not* a true tensor, since it vanishes in a locally inertial frame (namely, in absence of a gravitational field).

Due to the compatibility between affine geometry and metric geometry in Riemann spaces (characterized by the vanishing of the covariant derivative of the metric, and therefore torsion-free), it is possible to express the connection components in terms of the metric tensor as

$$\Gamma_{AB}^I(x^5) = \frac{1}{2} g_{DR5}^{IK} (\partial_B g_{KA,DR5} + \partial_A g_{KB,DR5} - \partial_K g_{AB,DR5}) = \left\{ \begin{array}{c} I \\ AB \end{array} \right\}, \quad (71)$$

where the quantities $\left\{ \begin{array}{c} I \\ AB \end{array} \right\}$ are the second-kind Christoffel symbols.

The Riemann-Christoffel (curvature) tensor in \mathfrak{R}_5 is given by

$$R_{BCD}^A(x^5) = \partial_C \Gamma_{BD}^A - \partial_D \Gamma_{BC}^A + \Gamma_{KC}^A \Gamma_{BD}^K - \Gamma_{KD}^A \Gamma_{BC}^K. \quad (72)$$

By contraction of R_{BCD}^A on two and four indices, respectively, we get as usual the five-dimensional Ricci tensor $R_{AB}(x^5)$, given explicitly by

$$R_{AB}(x^5) = \partial_I \Gamma_{AB}^I - \partial_B \Gamma_{AI}^I + \Gamma_{AB}^I \Gamma_{IK}^K - \Gamma_{AI}^K \Gamma_{BK}^I, \quad (73)$$

and the scalar curvature $R(x^5) = R_A^A(x^5)$.

The explicit expressions of $\Gamma_{BC}^A(x^5)$, $R_{ABCD}(x^5)$, $R_{AB}(x^5)$ and $R(x^5)$ are given in App. A⁹.

From the knowledge of the Riemann-Christoffel tensor and of its contractions it is possible to derive the Einstein equations in the space \mathfrak{R}_5 by exploiting the Hamilton principle. The five-dimensional Hilbert-Einstein action in \mathfrak{R}_5 reads

$$S_{DR5} = -\frac{1}{16\pi\tilde{G}} \int d^5x \sqrt{\pm\tilde{g}(x^5)} R(x^5) - \Lambda_{(5)} \int d^5x \sqrt{\pm\tilde{g}(x^5)}, \quad (74)$$

where $\tilde{g}(x^5) = \det g_{DR5}(x^5)$, \tilde{G} is the five-dimensional "gravitational" constant, and $\Lambda_{(5)}$ is the "cosmological" constant in \mathfrak{R}_5 . The form of the second term of the action (73) clearly shows that $\Lambda_{(5)}$ is assumed to be a genuine constant, although it might also, in principle, depend on both the fifth coordinate (namely, on the energy E) and the space-time coordinates x : $\Lambda_{(5)} = \Lambda_{(5)}(x, x^5)$. The double sign in the square root accords to that

⁹ Henceforth, in order to simplify the notation, we adopt units such that $c=(\text{velocity of light})=1$ and $\ell_0=1$.

in front of the fifth metric coefficient $f(x^5)$. Among the problems concerning S_{DR5} , let us quote its physical meaning (as well as that of \tilde{G}) and the meaning of those energy values $\overline{x^5}$ such that $S_{DR5}(\overline{x^5}) = 0$ (due to a possible degeneracy of the metric).

Then, a straightforward use of the variational methods yields the (vacuum) Einstein equations in \mathfrak{R}_5 in the form

$$R_{AB}(x^5) - \frac{1}{2}g_{AB,DR5}(x^5)R(x^5) = \Lambda_{(5)}g_{AB,DR5}(x^5). \quad (75)$$

In the following, we shall confine ourselves to the case $\Lambda_{(5)} = 0$. Notice that assuming a vanishing cosmological constant has the physical motivation (at least as far as gravitation is concerned and one is not interested into quantum effects) that $\Lambda_{(5)}$ is related to the vacuum energy; experimental evidence shows that, at least in our four-dimensional space, $\Lambda \simeq 3 \cdot 10^{-52}m^{-2}$.

3.5.2 Solving Vacuum Einstein's Equations in the Power Ansatz

Solving Einstein's equations in the five-dimensional, deformed space \mathfrak{R}_5 in the general case is quite an impossible task. However, solutions of Eqs.(74) with $\Lambda_{(5)} = 0$ can be obtained in some cases of special physical relevance, namely for a spatially isotropic deformed metric ($b_1(E) = b_2(E) = b_3(E) = b(E)$) and in the Power Ansatz discussed in Subsect.3.5.2.

Let us consider this last case, which is of special interest to our present aims, namely a metric of the type (45). In order to simplify the solution of the Einstein equations, we assume here simply, for the dimensional coefficient $f(E)$:

$$f(E) = E^r \quad (76)$$

($r \in \mathbb{R}$), being understood that the characteristic parameter E_0 is possibly contained in ℓ_0 . Of course, the Einstein equations (74) reduce now (for $\Lambda_{(5)} = 0$) to the following algebraic equations in the five exponents q_0, q_1, q_2, q_3, r :

$$\left\{ \begin{array}{l} (2+r)(q_3 + q_1 + q_2) - q_1^2 - q_2^2 - q_3^2 - q_1q_2 - q_1q_3 - q_2q_3 = 0 ; \\ (2+r)(q_3 + q_0 + q_2) - q_2^2 - q_3^2 - q_0^2 - q_2q_3 - q_2q_0 - q_3q_0 = 0; \\ (2+r)(q_3 + q_0 + q_1) - q_1^2 - q_3^2 - q_0^2 - q_1q_3 - q_1q_0 - q_3q_0 = 0; \\ (2+r)(q_0 + q_1 + q_2) - q_1^2 - q_2^2 - q_0^2 - q_1q_2 - q_1q_0 - q_2q_0 = 0; \\ q_1q_2 + q_1q_3 + q_1q_0 + q_2q_3 + q_2q_0 + q_3q_0 = 0. \end{array} \right. \quad (77)$$

It can be shown [20] that Eqs.(76) admit twelve possible classes of solutions, which can be classified according to the values of the five-dimensional set $\tilde{\mathbf{q}} \equiv (q_0, q_1, q_2, q_3, r)$. Explicitly one has:

- **Class (I):** $\tilde{\mathbf{q}}_I = \left(q_2, -q_2 \left(\frac{2q_3 + q_2}{2q_2 + q_3} \right), q_2, q_3, \frac{q_3^2 - 2q_3 + 2q_2q_3 - 4q_2 + 3q_2^2}{2q_2 + q_3} \right);$
- **Class (II):** $\tilde{\mathbf{q}}_{II} = (0, q_1, 0, 0, q_1 - 2);$

- **Class (III):** $\tilde{\mathbf{q}}_{III} = (q_2, -q_2, q_2, q_2, -2(1 - q_2));$

- **Class (IV):** $\tilde{\mathbf{q}}_{IV} = (0, 0, 0, q_3, q_3 - 2);$
- **Class (V):** $\tilde{\mathbf{q}}_V = (-q_3, -q_3, -q_3, q_3, -(1 + q_3));$
- **Class (VI):** $\tilde{\mathbf{q}}_{VI} = (q_0, 0, 0, 0, q_0 - 2);$
- **Class (VII):** $\tilde{\mathbf{q}}_{VII} = (q_0, -q_0, -q_0, -q_0, -2 - q_0);$

- **Class (VIII):** $\tilde{\mathbf{q}}_{VIII} = (0, 0, 0, 0, r);$

- **Class (IX):** $\tilde{\mathbf{q}}_{IX} = (0, 0, q_2, 0, -2 + q_2);$

- **Class (X):** $\tilde{\mathbf{q}}_X = \left(q_0, -\frac{q_3q_0 + q_2q_3 + q_2q_0}{q_2 + q_3 + q_0}, q_2, q_3, r_X \right),$ with

$$r_X = \frac{q_3^2 + q_3q_0 - 2q_3 + q_2q_3 - 2q_2 + q_2q_0 + q_2^2 - 2q_0 + q_0^2}{q_2 + q_3 + q_0};$$

- **Class (XI):** $\tilde{\mathbf{q}}_{XI} = \left(q_0, -\frac{q_2(2q_0 + q_2)}{2q_2 + q_0}, q_2, q_2, \frac{3q_2^2 - 4q_2 + 2q_2q_0 - 2q_0 + q_0^2}{2q_2 + q_0} \right);$
- **Class (XII):** $\tilde{\mathbf{q}}_{XII} = \left(q_0, q_2, q_2, -\frac{q_2(2q_0 + q_2)}{2q_2 + q_0}, r_{XII} \right),$ with

$$r_{XII} = \frac{q_3^2 + q_3q_0 - 2q_3 + q_2q_3 - 2q_2 + q_2q_0 + q_2^2 - 2q_0 + q_0^2}{q_2 + q_3 + q_0}.$$

3.5.3 Discussion of Solutions

The twelve classes of solutions found in the Power Ansatz allow one to recover, as special cases, *all* the phenomenological metrics discussed in Subsect.3.5.1. Let us write explicitly the interval in \mathfrak{R}_5 in such a case:

$$dS^2 = \left(\frac{E}{E_0} \right)^{q_0} dt^2 - \left(\frac{E}{E_0} \right)^{q_1} dx^2 - \left(\frac{E}{E_0} \right)^{q_2} dy^2 - \left(\frac{E}{E_0} \right)^{q_3} dz^2 + E^r dE^2. \quad (78)$$

Then, it is easily seen that the Minkowski metric is recovered from *all* classes of solutions. Solution (VIII) corresponds directly to a Minkowskian space-time, with the exponent r of the fifth coefficient undetermined. In the other cases, we have to put: $q_1 = 0$ for class (II); $q_2 = 0$ for classes (III) and (IX); $q_3 = 0$ for (IV) and (V); $q_0 = 0$ for (VI) and (VII) (for all the previous solutions, it is $r = -2$); $q_2 = q_3 = 0$ for class (I); $q_2 = q_3 = q_0 = 0$ for class (X); $q_2 = q_0 = 0$ for class (XI); $q_2 = q_0 = 0$ for class (XII). The latter four solutions have $r = 0$, and therefore correspond to a five-dimensional Minkowskian flat space.

If we set $q_1 = 1/3$ in class (II), $q_3 = 1/3$ in class (IV), or $q_2 = 1/3$ in class (IX) (corresponding in all three cases to the value $r = 5/3$ for the exponent of the fifth

metric coefficient), we get a metric of the "electroweak type" (see Eqs.(19.12),(19.13)), i.e. with unit time coefficient and one space coefficient behaving as $(E/E_0)^{1/3}$, *but spatially anisotropic*, since two of the space metric coefficients are constant and Minkowskian (precisely, the y, z coefficients for class (II); the x, y coefficients for class (IV); and the x, z ones for class (IX)). Notice that such an anisotropy does not disagree with the phenomenological results; indeed, in the analysis of the experimental data one was forced to assume spatial isotropy in the electromagnetic and in the weak cases, simply because of the lack of experimental information on two of the space dimensions.

Putting $q_0 = 1$ in class (VI), we find a metric which is spatially Minkowskian, with a time coefficient linear in E , i.e. a (gravitational) metric of the Einstein type (4).

Class (I) allows one to get as a special case a metric of the strong type (see Eqs.(17)-(20)). This is achieved by setting $q_2 = 2$, whence $q_1 = -4(q_3 + 1)/(q_3 + 4)$; $r = (q_3^2 + 2q_3 + 4)/(q_3 + 4)$. Moreover, for $q_3 = 0$, it is $q_1 = -1$; $r = 1$. In other words, one finds a solution corresponding to $b_0(E) = b(E) = (E/E_0)$ and spatially anisotropic, *i.e.* a metric of the type (44).

Finally, the three classes (X)-(XII) admit as special case the gravitational metric (45), which is recovered by putting $q_0 = 2$ and $q_1 = q_2 = q_3 = 0$ (whence also $r = 0$) and by a rescaling and a translation of the energy (see Subsect.3.5.2).

In conclusion, we can state that *the formalism of DR5 permits to recover, as solutions of the vacuum Einstein equations, all the phenomenological energy-dependent metrics of the electromagnetic, weak, strong and gravitational type* (and also the gravitational one of the Einstein kind, Eq.(4)).

4. Killing Equations in the Space \mathfrak{R}_5

In the present and in the following Sections, we shall deal with the problem of the isometries of the space \mathfrak{R}_5 of DR5. This will allow us to determine the symmetry properties of DR5, by getting also preliminary informations on the infinitesimal structure of the related algebras.

4.1 General Case

Let us discuss the Killing symmetries of the space \mathfrak{R}_5 [20].

In \mathfrak{R}_5 , the Lie derivative \mathcal{L} of a rank-2 covariant tensor field T_{AB} along the 5-vector field $\tilde{\xi} = \{\xi_A(x, x^5) \equiv \xi_A(x^B)\}$ is given as usual by

$$\mathcal{L}_{\tilde{\xi}} T_{AB} = T_{AC}^C \xi_{C;B} + T_{BC}^C \xi_{C;A} + T_{AB;C} \xi^C, \quad (79)$$

with ";" A " denoting covariant derivative with respect to x^A . If the tensor coincides with the metric tensor g_{AB} (whose covariant derivative vanishes), its Lie derivative becomes

$$\mathcal{L}_{\tilde{\xi}} g_{AB} = \xi_{A;B} + \xi_{B;A} = \xi_{[A;B]}, \quad (80)$$

where the bracket $[\cdot; \cdot]_{\sim}$ means symmetrization with respect to the enclosed indices.

Then, a 5-vector ξ is a Killing vector if the Lie derivative of the metric tensor with respect to ξ vanishes, *i.e.*

$$\mathcal{L}_{\xi} g_{AB} = 0 \Leftrightarrow \xi_{[A;B]} = 0 \Leftrightarrow \xi_{A;B} + \xi_{B;A} = 0 \quad (81)$$

are the Killing equations in \mathfrak{R}_5 . Since the Lie derivative is nothing but the generalization of directional derivative, this means that the Killing vectors correspond to isometric directions. The integrability conditions of Eqs.(80) are given by

$$\xi_{A;BC} = \xi_{C;[BA]} = R^D{}_{CBA}\xi_D \Leftrightarrow \mathcal{L}_{\xi}\Gamma^A{}_{BC} = 0, \quad (82)$$

where R_{ABCD} and $\Gamma^A{}_{BC}$ are the 5-d Riemann-Christoffel tensor and affine connection (71), (70), respectively. In turn, Eqs.(81) are integrable under the conditions

$$\mathcal{L}_{\xi} R_{ABCD} = 0. \quad (83)$$

For metric (33), from the Christoffel symbols $\Gamma^A{}_{BC}$ of the metric $g_{AB,DR_5}(x^5)$, Eqs.(80) take the form of the following system of 15 coupled, partial derivative differential equations in \mathfrak{R}_5 for the Killing vector $\xi_A(x^B)$:

$$f(x^5)\xi_{0,0}(x^A) \pm b_0(x^5)b'_0(x^5)\xi_5(x^A) = 0; \quad (84)$$

$$\left. \begin{aligned} \xi_{0,1}(x^A) + \xi_{1,0}(x^A) &= 0 \\ \xi_{0,2}(x^A) + \xi_{2,0}(x^A) &= 0 \\ \xi_{0,3}(x^A) + \xi_{3,0}(x^A) &= 0 \end{aligned} \right\} \text{type I conditions}; \quad (85)$$

$$b_0(x^5)(\xi_{0,5}(x^A) + \xi_{5,0}(x^A)) - 2b'_0(x^5)\xi_0(x^A) = 0 \left. \vphantom{b_0(x^5)} \right\} \text{type II condition}; \quad (86)$$

$$f(x^5)\xi_{1,1}(x^A) \mp b_1(x^5)b'_1(x^5)\xi_5(x^A) = 0; \quad (87)$$

$$\left. \begin{aligned} \xi_{1,2}(x^A) + \xi_{2,1}(x^A) &= 0 \\ \xi_{1,3}(x^A) + \xi_{3,1}(x^A) &= 0 \end{aligned} \right\} \text{type I conditions}; \quad (88)$$

$$b_1(x^5)(\xi_{1,5}(x^A) + \xi_{5,1}(x^A)) - 2b'_1(x^5)\xi_1(x^A) = 0 \left. \vphantom{b_1(x^5)} \right\} \text{type II condition}; \quad (89)$$

$$f(x^5)\xi_{2,2}(x^A) \mp b_2(x^5)b'_2(x^5)\xi_5(x^A) = 0; \quad (90)$$

$$\xi_{2,3}(x^A) + \xi_{3,2}(x^A) = 0 \left. \vphantom{\xi_{2,3}(x^A)} \right\} \text{type I condition}; \quad (91)$$

$$b_2(x^5)(\xi_{2,5}(x^A) + \xi_{5,2}(x^A)) - 2b'_2(x^5)\xi_2(x^A) = 0 \left. \vphantom{b_2(x^5)} \right\} \text{type II condition}; \quad (92)$$

$$f(x^5)\xi_{3,3}(x^A) \mp b_3(x^5)b'_3(x^5)\xi_5(x^A) = 0; \quad (93)$$

$$b_3(x^5)(\xi_{3,5}(x^A) + \xi_{5,3}(x^A)) - 2b'_3(x^5)\xi_3(x^A) = 0 \left. \vphantom{b_3(x^5)} \right\} \text{type II condition}; \quad (94)$$

$$2f(x^5)\xi_{5,5}(x^A) - f'(x^5)\xi_5(x^A) = 0, \quad (95)$$

where now ", A" denotes ordinary derivative with respect to x^A .

Equations (83)-(94) can be divided in "fundamental" equations and "constraint" equations (of type I and II). The above system is in general overdetermined, *i.e.* its solutions will contain numerical coefficients satisfying a given algebraic system. Explicitly solving it yields

$$\xi_\mu(x^A) = F_\mu(x^{A \neq \mu}) + \pm(-\delta_{\mu 0} + \delta_{\mu 1} + \delta_{\mu 2} + \delta_{\mu 3})b_\mu(x^5)b'_\mu(x^5)(f(x^5))^{-1/2} \int dx^\mu F_5(x); \quad (96)$$

$$\xi_5(x^A) = (f(x^5))^{1/2}F_5(x). \quad (97)$$

The five unknown functions $F_A(x^{B \neq A})$ are restricted by the two following types of conditions:

$$\begin{aligned} I) \text{ Type I (Cardinality 4, } \mu \neq \nu \neq \rho \neq \sigma): \\ \pm A_\mu(x^5)G_{,\nu\rho\sigma}(x) + B_\mu(x^5)G_{,\mu\mu\nu\rho\sigma}(x) + \\ + b_\mu(x^5)F_{\mu,5}(x^{A \neq \mu}) - 2b'_\mu(x^5)F_\mu(x^{A \neq \mu}) = 0 \end{aligned} \quad (98)$$

$$\begin{aligned} II) \text{ Type II (Cardinality 6, symm. in } \mu, \nu, \mu \neq \nu \neq \rho \neq \sigma): \\ F_{\mu,\nu}(x^{A \neq \mu}) + F_{\nu,\mu}(x^{A \neq \nu}) + \\ \pm(-\delta_{\mu 0} + \delta_{\mu 1} + \delta_{\mu 2} + \delta_{\mu 3})b_\mu(x^5)b'_\mu(x^5)(f(x^5))^{-1/2}G_{,\nu\nu\rho\sigma}(x) + \\ \pm(-\delta_{\nu 0} + \delta_{\nu 1} + \delta_{\nu 2} + \delta_{\nu 3})b_\nu(x^5)b'_\nu(x^5)(f(x^5))^{-1/2}G_{,\mu\mu\rho\sigma}(x) = 0 \end{aligned} \quad (99)$$

where we introduced the fake 4-vectors¹⁰ (ESC off)

$$\begin{aligned} A_\mu(x^5) \equiv (-\delta_{\mu 0} + \delta_{\mu 1} + \delta_{\mu 2} + \delta_{\mu 3})b_\mu(x^5)(f(x^5))^{-1/2} \cdot \\ \cdot \left[- (b'_\mu(x^5))^2 + b_\mu(x^5)b''_\mu(x^5) - \frac{1}{2}b_\mu(x^5)b'_\mu(x^5)f'(x^5)(f(x^5))^{-1} \right]; \end{aligned} \quad (100)$$

$$B_\mu(x^5) \equiv b_\mu(x^5)(f(x^5))^{1/2}; \quad (101)$$

and defined the function

$$G(x) \equiv \int d^4x F_5(x). \quad (102)$$

¹⁰ Indeed, it is in general (ESC off) ($i = 1, 2, 3$)

$$b_\mu(x^5) = g_{\mu\mu,DR5}(x^5)(\delta_{\mu 0} - \delta_{\mu i});$$

$$f_\mu(x^5) = \pm g_{55,DR5}(x^5),$$

which clearly show the non-vector nature of $A_\mu(x^5)$ and $B_\mu(x^5)$ in the 4-d. subspaces of \mathfrak{R}_5 .

4.2 The Hypothesis Υ of Functional Independence

In order to get the explicit forms of the functions $F_A(x^{B \neq A})$ (and therefore of the Killing vector (95), (96)), it is necessary to analyze conditions I and make suitable simplifying hypotheses.

To this aim, consider the following equation in the (suitably regular) functions $\alpha_1(x^5)$, $\alpha_2(x^5)$ and $\beta_1(x^\mu)$, $\beta_2(x^\mu)$:

$$\alpha_1(x^5)\beta_1(x^\mu) + \alpha_2(x^5)\beta_2(x^\mu) = 0. \quad (103)$$

If $\alpha_1(x^5) \neq 0$, $\alpha_2(x^5) \neq 0$, the solutions of Eq.(102) are given by the following two cases:

1) $\exists \gamma \in R_0 : \alpha_1(x^5) = \gamma \alpha_2(x^5) (\forall x^5 \in R_0)$ (*functional linear dependence* between $\alpha_1(x^5)$ and $\alpha_2(x^5)$). Then $\beta_2(x^\mu) = -\gamma \beta_1(x^\mu) (\forall x^\mu \in R, \mu = 0, 1, 2, 3)$.

2) $\nexists \gamma \in R_0 : \alpha_1(x^5) = \gamma \alpha_2(x^5) (\forall x^5 \in R_0)$ (*functional linear independence* between $\alpha_1(x^5)$ and $\alpha_2(x^5)$). Then $\beta_2(x^\mu) = 0 = \beta_1(x^\mu) (\forall x^\mu \in R, \mu = 0, 1, 2, 3)$.

Let us now consider type I conditions for $\mu = 0$. By taking their derivative with respect to x^0 one gets:

$$\begin{aligned} & \partial_0 \left(I \Big|_{\mu=0} \right) : \\ & \pm A_0(x^5) G_{,0123}(x) + B_0(x^5) G_{,000123}(x) = 0 \Leftrightarrow \\ & \Leftrightarrow \pm A_0(x^5) F_5(x) + B_0(x^5) F_{5,00}(x) = 0. \end{aligned} \quad (104)$$

If $A_0(x^5) \neq 0$ and $B_0(x^5) \neq 0$, we have the following two possibilities:

1) $\exists c_0 \in R_0 : \pm A_0(x^5) = c_0 B_0(x^5) (\forall x^5 \in R_0^+)$. From (103) one gets ($\forall x^0, x^1, x^2, x^3 \in R$):

$$G_{,000123}(x) = \mp c_0 G_{,0123}(x) \Leftrightarrow F_{5,00}(x) = \mp c_0 F_5(x); \quad (105)$$

2) $\nexists c_0 \in R_0 : \pm A_0(x^5) = c_0 B_0(x^5) (\forall x^5 \in R_0^+)$. It follows from (103) ($\forall x^0, x^1, x^2, x^3 \in R$):

$$G_{,000123}(x) = 0 = G_{,0123}(x) \Leftrightarrow F_{5,00}(x) = 0 = F_5(x). \quad (106)$$

In general, let us take the derivative with respect to x^μ of type I conditions (ESC off):

$$\begin{aligned} & \partial_\mu I : \\ & \pm A_\mu(x^5) G_{,\mu\nu\rho\sigma}(x) + B_\mu(x^5) G_{,\mu\mu\mu\nu\rho\sigma}(x) = 0 \Leftrightarrow \\ & \Leftrightarrow \pm A_\mu(x^5) F_5(x) + B_\mu(x^5) F_{5,\mu\mu}(x) = 0. \end{aligned} \quad (107)$$

If $G(x)$ satisfies the Schwarz lemma at any order, since $\mu \neq \nu \neq \rho \neq \sigma$, one gets

$$G_{,\mu\nu\rho\sigma}(x) = G_{,0123}(x) (= F_5(x)), \quad (108)$$

namely the function $F_5(x)$ is present in $\partial_\mu I \quad \forall \mu = 0, 1, 2, 3$. It is therefore *sufficient* to assume that there exists at least a special index

$$\bar{\mu} \in \{0, 1, 2, 3\} : \begin{cases} \exists c_{\bar{\mu}} \in R_0 : \pm A_{\bar{\mu}}(x^5) = c_{\bar{\mu}} B_{\bar{\mu}}(x^5) (\forall x^5 \in R_0^+) \\ A_{\bar{\mu}}(x^5) \neq 0, B_{\bar{\mu}}(x^5) \neq 0 \end{cases} \quad (109)$$

in order that $(\forall x^0, x^1, x^2, x^3 \in R)$

$$\begin{aligned} (F_5(x) =) G_{,0123}(x) = 0 &= G_{,\bar{\mu}\bar{\mu}\bar{\mu}123}(x) (= F_{5,\bar{\mu}\bar{\mu}}(x)) \xRightarrow{\text{(in gen.)}} \\ &\xRightarrow{\text{(in gen.)}} G_{,\mu\mu\mu123}(x) (= F_{5,\mu\mu}(x)) = 0, \quad \forall \mu = 0, 1, 2, 3. \end{aligned} \quad (110)$$

In the following the existence hypothesis

$$\begin{aligned} &\exists \text{ (at least one) } \bar{\mu} \in \{0, 1, 2, 3\} : \\ &\left\{ \begin{array}{l} \exists c_{\bar{\mu}} \in R_0 : \pm A_{\bar{\mu}}(x^5) = c_{\bar{\mu}} B_{\bar{\mu}}(x^5) (\forall x^5 \in R_0^+) \\ A_{\bar{\mu}}(x^5) \neq 0, B_{\bar{\mu}}(x^5) \neq 0 \end{array} \right. \end{aligned} \quad (111)$$

will be called "*Υ hypothesis*" of functional independence.

The above reasoning can be therefore summarized as

$$\begin{aligned} &\exists \text{ (at least one) } \bar{\mu} \in \{0, 1, 2, 3\} : \\ Hp. \Upsilon : &\underbrace{\left\{ \begin{array}{l} \exists c_{\bar{\mu}} \in R_0 : \pm A_{\bar{\mu}}(x^5) = c_{\bar{\mu}} B_{\bar{\mu}}(x^5) (\forall x^5 \in R_0^+); \\ A_{\bar{\mu}}(x^5) \neq 0, B_{\bar{\mu}}(x^5) \neq 0, \end{array} \right.} \\ &\quad \downarrow \\ &\underbrace{(F_5(x) =) G_{,0123}(x) = 0, \forall x^0, x^1, x^2, x^3 \in R,} \\ &\quad \downarrow \\ (F_{5,\mu\mu}(x) =) &G_{,\mu\mu\mu123}(x) = 0, \quad \forall x^0, x^1, x^2, x^3 \in R, \forall \mu = 0, 1, 2, 3. \end{aligned} \quad (112)$$

4.3 Solving Killing equations in \mathfrak{R}_5 in the Υ -Hypothesis

Then, by assuming the hypothesis Υ of functional independence to hold, and replacing Eq.(110) in (95) and (96), one gets for the covariant Killing 5-vector $\xi_A(x, x^5)$:

$$\left. \begin{aligned} \xi_\mu(x^A) &= F_\mu(x^{A \neq \mu}), \quad \forall \mu = 0, 1, 2, 3, \\ \xi_5(x^A) &= 0 \end{aligned} \right\} \Rightarrow \xi_A(x^B) = (F_\mu(x^{A \neq \mu}), 0). \quad (113)$$

The conditions to be satisfied now by the 4 unknown functions $F_\mu(x^{A \neq \mu})$ ($\forall \mu, \nu = 0, 1, 2, 3$) are obtained by substituting Eq.(110) in Eqs.(97) and (98) and read

$$\begin{aligned} &I) \text{ Type I (Cardinality 4):} \\ &b_\mu(x^5)F_{\mu,5}(x^{A \neq \mu}) - 2b'_\mu(x^5)F_\mu(x^{A \neq \mu}) = 0; \end{aligned} \quad (114)$$

$$\begin{aligned} &II) \text{ Type II (Cardinality 6, symmetry in } \mu, \nu, \mu \neq \nu): \\ &F_{\mu,\nu}(x^{A \neq \mu}) + F_{\nu,\mu}(x^{A \neq \nu}) = 0. \end{aligned}$$

Solving the equations of type I yields

$$F_\mu(x^{A \neq \mu}) = b_\mu^2(x^5)\widetilde{F}_\mu(x^{\nu \neq \mu}), \quad \forall \mu = 0, 1, 2, 3 \quad (115)$$

and eventually ($\forall \mu, \nu = 0, 1, 2, 3, \mu \neq \nu$)

$$\left. \begin{aligned} F_{\mu,\nu}(x^{A \neq \mu}) + F_{\nu,\mu}(x^{A \neq \nu}) &= 0 \\ F_\mu(x^{A \neq \mu}) &= b_\mu^2(x^5)\widetilde{F}_\mu(x^{\nu \neq \mu}) \end{aligned} \right\} \begin{array}{l} \Rightarrow \\ \text{(in gen.)} \\ \Leftarrow \end{array}$$

$$\Rightarrow \text{(in gen.)} \left\{ b_\mu^2(x^5)\frac{\partial \widetilde{F}_\mu(x^{\rho \neq \mu})}{\partial x^\nu} + b_\nu^2(x^5)\frac{\partial \widetilde{F}_\nu(x^{\rho \neq \nu})}{\partial x^\mu} = 0. \right. \quad (116)$$

Summarizing, we can state that, in the hypothesis Υ of functional independence, the covariant Killing 5-vector $\xi_A(x, x^5)$ has the form (ESC off)

$$\xi_A(x^B) = \left(b_\mu^2(x^5)\widetilde{F}_\mu(x^{\nu \neq \mu}), 0 \right) \quad (117)$$

where the 4 unknown real functions of 3 real variables $\{\widetilde{F}_\mu(x^{\rho \neq \mu})\}$ are solutions of the following system of 6 (due to the symmetry in μ and ν) non-linear, partial derivative equations

$$b_\mu^2(x^5)\frac{\partial \widetilde{F}_\mu(x^{\rho \neq \mu})}{\partial x^\nu} + b_\nu^2(x^5)\frac{\partial \widetilde{F}_\nu(x^{\rho \neq \nu})}{\partial x^\mu} = 0, \quad \mu, \nu = 0, 1, 2, 3, \mu \neq \nu, \quad (118)$$

which is in general overdetermined, *i.e.* its explicit solutions will depend on numerical coefficients obeying a given system.

Solving system (117) is quite easy (although cumbersome: for details, see ref.[39]). The final solution yields the following expressions for the components of the *contravariant* Killing 5-vector $\xi^A(x, x^5)$ satisfying the 15 Killing equations (83)-(94) in the hypothesis Υ of functional independence (110):

$$\begin{aligned}\xi^0(x^1, x^2, x^3) &= \widetilde{F}_0(x^1, x^2, x^3) = \\ &= d_8 x^1 x^2 x^3 + d_7 x^1 x^2 + d_6 x^1 x^3 + d_4 x^2 x^3 + \\ &+ (d_5 + a_2) x^1 + d_3 x^2 + d_2 x^3 + (a_1 + d_1 + K_0); \end{aligned} \quad (119)$$

$$\begin{aligned}\xi^1(x^0, x^2, x^3) &= -\widetilde{F}_1(x^0, x^2, x^3) = \\ &= -h_2 x^0 x^2 x^3 - h_1 x^0 x^2 - h_8 x^0 x^3 - h_4 x^2 x^3 - \\ &- (h_7 + e_2) x^0 - h_3 x^2 - h_6 x^3 - (K_1 + h_5 + e_1); \end{aligned} \quad (120)$$

$$\begin{aligned}\xi^2(x^0, x^1, x^3) &= -\widetilde{F}_2(x^0, x^1, x^3) = \\ &= -l_2 x^0 x^1 x^3 - l_1 x^0 x^1 - l_6 x^0 x^3 - l_4 x^1 x^3 - \\ &- (l_5 + e_4) x^0 - l_3 x^1 - l_8 x^3 - (l_7 + K_2 + e_3); \end{aligned} \quad (121)$$

$$\begin{aligned}\xi^3(x^0, x^1, x^2) &= -\widetilde{F}_3(x^0, x^1, x^2) = \\ &= -m_8 x^0 x^1 x^2 - m_7 x^0 x^1 - m_6 x^0 x^2 - m_4 x^1 x^2 - \\ &- (m_5 + g_2) x^0 - m_3 x^1 - m_2 x^2 - (m_1 + g_1 + c); \end{aligned} \quad (122)$$

$$\xi^5 = 0 \neq \xi^5(x, x^5). \quad (123)$$

where (some of) the real parameters d_i , h_i , l_i , m_i ($i = 1, 2, \dots, 8$), e_k ($k = 1, 2, 3, 4$), g_l , a_l ($l = 1, 2$) satisfy the following algebraic system of 6 constraints:

$$\left\{ \begin{array}{l}
(01) \left\{ \begin{array}{l} b_0^2(x^5) [d_8x^2x^3 + d_7x^2 + d_6x^3 + (d_5 + a_2)] + \\ + b_1^2(x^5) [h_2x^2x^3 + h_1x^2 + h_8x^3 + (h_7 + e_2)] = 0; \end{array} \right. \\
(02) \left\{ \begin{array}{l} b_0^2(x^5) (d_8x^1x^3 + d_7x^1 + d_4x^3 + d_3) + \\ + b_2^2(x^5) [l_2x^1x^3 + l_1x^1 + l_6x^3 + (l_5 + e_4)] = 0; \end{array} \right. \\
(03) \left\{ \begin{array}{l} b_0^2(x^5) (d_8x^1x^2 + d_6x^1 + d_4x^2 + d_2) + \\ + b_3^2(x^5) [m_8x^1x^2 + m_7x^1 + m_6x^2 + (m_5 + g_2)] = 0; \end{array} \right. \\
(12) \left\{ \begin{array}{l} b_1^2(x^5) (h_2x^0x^3 + h_1x^0 + h_4x^3 + h_3) + \\ + b_2^2(x^5) (l_2x^0x^3 + l_1x^0 + l_4x^3 + l_3) = 0; \end{array} \right. \\
(13) \left\{ \begin{array}{l} b_1^2(x^5) (h_2x^0x^2 + h_8x^0 + h_4x^2 + h_6) + \\ + b_3^2(x^5) (m_8x^0x^2 + m_7x^0 + m_4x^2 + m_3) = 0; \end{array} \right. \\
(23) \left\{ \begin{array}{l} b_2^2(x^5) (l_2x^0x^1 + l_6x^0 + l_4x^1 + l_8) + \\ + b_3^2(x^5) (m_8x^0x^1 + m_6x^0 + m_4x^1 + m_2) = 0. \end{array} \right.
\end{array} \right. \quad (124)$$

4.4 Power Ansatz and Reductivity of the Hypothesis Υ

We want now to investigate if and when the *simplifying* Υ hypothesis (110) — we exploited in order to solve the Killing equations in \mathfrak{R}_5 — is *reductive*. To this aim, one needs to consider explicit forms of the 5-d. Riemannian metric $g_{AB,DR5}(x^5)$. As it was seen in Subsect. 3.6, the "Power Ansatz" allows one to recover all the phenomenological metrics derived for the four fundamental interactions. So it is worth considering such a case, corresponding to a 5-d. metric of the form (45), with $\tilde{\mathbf{q}} \equiv (q_0, q_1, q_2, q_3, r) = (q_\mu, r)$.

In the Power Ansatz, the (fake) 4-vectors $A_\mu(x^5)$ and $B_\mu(x^5)$ (Eqs.(99), (100)) take

the following explicit forms

$$\begin{aligned} A_{\mu,power}(x^5) &= \frac{1}{(x_0^5)^2} (\delta_{\mu 0} - \delta_{\mu 1} - \delta_{\mu 2} - \delta_{\mu 3}) \frac{q_\mu}{2} \left(1 + \frac{r}{2}\right) \left(\frac{x^5}{x_0^5}\right)^{\frac{3}{2}q_\mu - \frac{1}{2}r - 2} = \\ &= A_{\mu,power}(\tilde{\mathbf{q}}; x^5); \end{aligned} \quad (125)$$

$$B_{\mu,power}(x^5) = \left(\frac{x^5}{x_0^5}\right)^{\frac{1}{2}q_\mu + \frac{1}{2}r} = B_{\mu,power}(\tilde{\mathbf{q}}; x^5). \quad (126)$$

Therefore

$$\frac{\pm A_{\mu,power}(\tilde{\mathbf{q}}; x^5)}{B_{\mu,power}(\tilde{\mathbf{q}}; x^5)} = \pm \frac{1}{(x_0^5)^2} (\delta_{\mu 0} - \delta_{\mu 1} - \delta_{\mu 2} - \delta_{\mu 3}) \frac{q_\mu}{2} \left(1 + \frac{r}{2}\right) \left(\frac{x^5}{x_0^5}\right)^{q_\mu - r - 2}. \quad (127)$$

Since $x^5 \in R_0^+$, one gets respectively

$$\begin{aligned} A_{\mu,power}(\tilde{\mathbf{q}}; x^5) \neq 0 &\Leftrightarrow \frac{q_\mu}{2} \left(1 + \frac{r}{2}\right) \neq 0 \Leftrightarrow \\ &\Leftrightarrow \begin{cases} q_\mu \neq 0 \\ 1 + \frac{r}{2} \neq 0 \Leftrightarrow 2 + r \neq 0 \end{cases}; \end{aligned} \quad (128)$$

$$B_{\mu,power}(\tilde{\mathbf{q}}; x^5) \neq 0, \forall q_\mu, r \in \mathbb{Q}. \quad (129)$$

Then

$$\frac{\pm A_{\mu,power}(\tilde{\mathbf{q}}; x^5)}{B_{\mu,power}(\tilde{\mathbf{q}}; x^5)} = c_{(\mu; q_\mu, r)} \in R_{(0)}, \forall x^5 \in R_0^+ \Leftrightarrow q_\mu - r - 2 = 0. \quad (130)$$

It follows that, if $A_{\mu,power}(\tilde{\mathbf{q}}; x^5) \neq 0$ and $B_{\mu,power}(\tilde{\mathbf{q}}; x^5) \neq 0$, assuming the Power Ansatz form for $g_{AB,DR5}(x^5)$ yields the following expression of the hypothesis Υ of functional independence (110):

$$\begin{aligned} \exists (\text{at least one}) \bar{\mu} \in \{0, 1, 2, 3\} : &\left\{ \begin{array}{l} q_{\bar{\mu}} - (r + 2) \neq 0 \\ \left\{ \begin{array}{l} q_{\bar{\mu}} \neq 0 \\ r + 2 \neq 0 \end{array} \right\} \end{array} \right\} \Leftrightarrow \\ &\Leftrightarrow q_{\bar{\mu}} \neq 0, r + 2 \neq 0, q_{\bar{\mu}} \neq r + 2. \end{aligned} \quad (131)$$

In other words, in the framework of the "Power Ansatz" for the metric tensor *the reductive nature of the Υ hypothesis depends on the value of the rational parameters q_0, q_1, q_2, q_3 and r , exponents of the components of $g_{AB,DR5power}(x^5)$.*

A similar result holds true if one assumes for the \mathfrak{R}_5 metric the generalized form $g_{AB,DR5power-conform}(x^5)$ (52), obtained by the anisotropic rescaling (55). We have, in

this case, for $A_\mu(x^5)$ and $B_\mu(x^5)$:

$$\begin{aligned} & A_{\mu, power-conform}(x^5) = \\ &= \frac{1}{(x_0^5)^2} \frac{(\vartheta_\mu)^{\frac{3}{2}}}{(\vartheta_5)^{\frac{1}{2}}} (\delta_{\mu 0} - \delta_{\mu 1} - \delta_{\mu 2} - \delta_{\mu 3}) \frac{q_\mu}{2} \left(1 + \frac{r}{2}\right) \left(\frac{x^5}{x_0^5}\right)^{\frac{3}{2}q_\mu - \frac{1}{2}r - 2} = \\ &= \frac{(\vartheta_\mu)^{\frac{3}{2}}}{(\vartheta_5)^{\frac{1}{2}}} A_{\mu, power}(x^5); \end{aligned} \quad (132)$$

$$\begin{aligned} & B_{\mu, power-conform}(x^5) = \\ &= (\vartheta_\mu \vartheta_5)^{\frac{1}{2}} \left(\frac{x^5}{x_0^5}\right)^{\frac{1}{2}q_\mu + \frac{1}{2}r} = (\vartheta_\mu \vartheta_5)^{\frac{1}{2}} B_{\mu, power}(x^5). \end{aligned} \quad (133)$$

whence

$$\begin{aligned} & \frac{\pm A_{\mu, power}(\tilde{\mathbf{q}}; x^5)}{B_{\mu, power}(\tilde{\mathbf{q}}; x^5)} = \\ &= \pm \frac{1}{(x_0^5)^2} \frac{\vartheta_\mu}{\vartheta_5} (\delta_{\mu 0} - \delta_{\mu 1} - \delta_{\mu 2} - \delta_{\mu 3}) \frac{q_\mu}{2} \left(1 + \frac{r}{2}\right) \left(\frac{x^5}{x_0^5}\right)^{q_\mu - r - 2} \end{aligned} \quad (134)$$

and

$$\frac{\pm A_{\mu, power}(\tilde{\mathbf{q}}; x^5)}{B_{\mu, power}(\tilde{\mathbf{q}}; x^5)} = c_{(\mu; q_\mu, r)} \in R_{(0)}, \forall x^5 \in R_0^+ \Leftrightarrow q_\mu - r - 2 = 0. \quad (135)$$

Therefore conditions (126)-(129), obtained in the power case, hold unchanged, together with expression (130) of the Υ hypothesis. Thus, we can conclude that, independently of a possible anisotropic, conformal rescaling of the coordinates of the type (55), the reductive nature of the Υ hypothesis depends only on the value of the parameters q_0, q_1, q_2, q_3 and r .

The discussion of the possible reductivity of the Υ hypothesis for all the 12 classes of solutions of the 5-d. Einstein equations in vacuum derived in Subsect. 3.6.2 (labelled by the 5-d. set $\tilde{\mathbf{q}} \equiv (q_0, q_1, q_2, q_3, r)$) allows one to state that in 5 general cases such hypothesis of functional independence is reductive indeed. The Killing equations can be explicitly solved in such cases. We refer the reader to Appendix B for these general cases, and go to discuss the special cases of the 5-d. phenomenological power metrics describing the four fundamental interactions (see Subsect.3.4.2).

5. Killing Symmetries for the 5-d. Metrics of Fundamental Interactions

We want now to investigate the possible reductivity of the hypothesis Υ of functional independence (110) for the 5-d. metrics (41)-(44), and solve the related Killing equations. Due to the piecewise structure of these phenomenological metrics, we shall distinguish the two energy ranges $x^5 \geq x_0^5$ (above threshold, case **a**) and $0 < x^5 < x_0^5$ (below threshold,

case **b**) for sub-Minkowskian metrics (electromagnetic and weak), and $0 < x^5 \leq x_0^5$ (below threshold, case **a'**) and $x^5 > x_0^5$ (above threshold, case **b'**) for over-Minkowskian metrics (strong and gravitational). Needless to say, cases **a**, **a'** correspond to the Minkowskian behavior of the related metrics, whereas **b**, **b'** refer to the non-Minkowskian one.

5.1 Electromagnetic and Weak Interactions

5.1.1 Validity of the Υ -Hypothesis

Case a) (Minkowskian conditions). In the energy range $x^5 \geq x_0^5$ the 5-d. metrics (41), (42) read:

$$g_{AB,DR5}(x^5) = \text{diag} (1, -1, -1, -1, \pm f(x^5)) . \quad (136)$$

This metric is a special case of

$$g_{AB,DR5}(x^5) = \text{diag} (a, -b, -c, -d, \pm f(x^5)) \quad (137)$$

($a, b, c, d, f(x^5) \in R_0^+$). From definitions (124) and (125) one gets:

$$\left. \begin{aligned} A_\mu(x^5) &= 0 \\ B_\mu(x^5) &= (f(x^5))^{\frac{1}{2}} \end{aligned} \right\} \forall \mu = 0, 1, 2, 3. \quad (138)$$

Therefore the hypothesis Υ of functional independence (110) is not satisfied $\forall \mu \in \{0, 1, 2, 3\}$. The 15 Killing equations corresponding to the e.m. and weak metrics (41), (42) are:

$$\left\{ \begin{array}{l} f(x^5)\xi_{0,0}(x^A) = 0; \\ \xi_{0,1}(x^A) + \xi_{1,0}(x^A) = 0; \\ \xi_{0,2}(x^A) + \xi_{2,0}(x^A) = 0; \\ \xi_{0,3}(x^A) + \xi_{3,0}(x^A) = 0; \\ \xi_{0,5}(x^A) + \xi_{5,0}(x^A) = 0; \\ f(x^5)\xi_{1,1}(x^A) = 0; \\ \xi_{1,2}(x^A) + \xi_{2,1}(x^A) = 0; \\ \xi_{1,3}(x^A) + \xi_{3,1}(x^A) = 0; \\ \xi_{1,5}(x^A) + \xi_{5,1}(x^A) = 0; \\ f(x^5)\xi_{2,2}(x^A) = 0; \\ \xi_{2,3}(x^A) + \xi_{3,2}(x^A) = 0; \\ \xi_{2,5}(x^A) + \xi_{5,2}(x^A) = 0; \\ f(x^5)\xi_{3,3}(x^A) = 0; \\ \xi_{3,5}(x^A) + \xi_{5,3}(x^A) = 0; \\ 2f(x^5)\xi_{5,5}(x^A) - f'(x^5)\xi_5(x^A) = 0. \end{array} \right. \quad (139)$$

Solving this system requires some cumbersome algebra but is trivial. The result for the contravariant Killing vector is

$$\xi^0(x^1, x^2, x^3, x^5) = T^0 - B^1x^1 - B^2x^2 - B^3x^3 + \Xi^0F(x^5); \quad (140)$$

$$\xi^1(x^0, x^2, x^3, x^5) = T^1 - B^1x^0 + \Theta^3x^2 - \Theta^2x^3 - \Xi^1F(x^5); \quad (141)$$

$$\xi^2(x^0, x^1, x^3, x^5) = T^2 - B^2x^0 - \Theta^3x^1 + \Theta^1x^3 - \Xi^2F(x^5); \quad (142)$$

$$\xi^3(x^0, x^1, x^2, x^5) = T^3 - B^3x^0 + \Theta^2x^1 - \Theta^1x^2 - \Xi^3F(x^5); \quad (143)$$

$$\xi^5(x, x^5) = \pm (f(x^5))^{-\frac{1}{2}} [T^5 - \Xi^0x^0 - \Xi^1x^1 - \Xi^2x^2 - \Xi^3x^3]. \quad (144)$$

Here, we have put

$$F(x^5) = \int dx^5 (f(x^5))^{\frac{1}{2}} \quad (145)$$

and omitted an unessential integration constant in (144) (it would only amount to a redefinition of T^μ , $\mu = 0, 1, 2, 3$ in Eqs.(139)-(143)). By inspection of such equations, it is easy to get the following physical interpretation of the real parameters entering into the expression of $\xi^A(x, x^5)$:

$$\begin{aligned} \Theta^1, \Theta^2, \Theta^3 \in R & \xrightarrow{\text{periodicity } T=2\pi} \Theta^1, \Theta^2, \Theta^3 \in [0, 2\pi); \\ \text{space-space angles of true rotations} & \\ B^1, B^2, B^3 \in R & \quad ; \\ \text{space-time angles of pseudorotations} & \\ T^0, T^1, T^2, T^3 \in R & \quad . \\ \text{space-time translation parameters} & \end{aligned} \quad (146)$$

As to the parameters Ξ^μ ($\mu = 0, 1, 2, 3$) and T^5 , their physical meaning (if any) depends on the signature of the fifth dimension.

The metric (136) can be dealt with along the same lines with only minor changes. In particular, the contravariant Killing vector is obtained from Eqs.(139)-(143) by a suitable rescaling of the parameters in the space-time components, namely

$$\xi^0(x^1, x^2, x^3, x^5) = \frac{1}{a} [T^0 - B^1 x^1 - B^2 x^2 - B^3 x^3 + \Xi^0 F(x^5)]; \quad (147)$$

$$\xi^1(x^0, x^2, x^3, x^5) = \frac{1}{b} [T^1 - B^1 x^0 + \Theta^3 x^2 - \Theta^2 x^3 - \Xi^1 F(x^5)]; \quad (148)$$

$$\xi^2(x^0, x^1, x^3, x^5) = \frac{1}{c} [T^2 - B^2 x^0 - \Theta^3 x^1 + \Theta^1 x^3 - \Xi^2 F(x^5)]; \quad (149)$$

$$\xi^3(x^0, x^1, x^2, x^5) = \frac{1}{d} [T^3 - B^3 x^0 + \Theta^2 x^1 - \Theta^1 x^2 - \Xi^3 F(x^5)]; \quad (150)$$

$$\xi^5(x, x^5) = \pm (f(x^5))^{-\frac{1}{2}} [T^5 - \Xi^0 x^0 - \Xi^1 x^1 - \Xi^2 x^2 - \Xi^3 x^3]. \quad (151)$$

Case b) (Non-Minkowskian conditions). In this energy range the form of the metrics (41), (42) is:

$$g_{AB,DR5}(x^5) = \text{diag} \left(1, - \left(\frac{x^5}{x_0^5} \right)^{1/3}, - \left(\frac{x^5}{x_0^5} \right)^{1/3}, - \left(\frac{x^5}{x_0^5} \right)^{1/3}, \pm f(x^5) \right). \quad (152)$$

We have, from the definitions (124) and (125) of the "vectors" $A_\mu(x^5)$ and $B_\mu(x^5)$:

$$\begin{aligned} A_0(x^5) &= 0; \\ A_i(x^5) &= \\ &= - \left(\frac{x^5}{x_0^5} \right)^{1/6} \frac{1}{6\sqrt{f(x^5)} (x^5)^{\frac{2}{3}} (x_0^5)^{\frac{1}{3}}} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \right], \\ &\quad \forall i = 1, 2, 3; \end{aligned} \quad (153)$$

$$B_0(x^5) = 0;$$

$$B_i(x^5) = \sqrt{f(x^5)} \left(\frac{x^5}{x_0^5}\right)^{1/6} \quad \forall i = 1, 2, 3. \quad (154)$$

Therefore

$$\frac{\pm A_i(x^5)}{B_i(x^5)} = \mp \frac{1}{6f(x^5) (x^5)^{\frac{2}{3}} (x_0^5)^{\frac{1}{3}}} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \right] \quad \forall i = 1, 2, 3. \quad (155)$$

Then, the Υ hypothesis (110) is not satisfied for $\mu = 0$ but it does for $\mu = i = 1, 2, 3$ under the following condition:

$$\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \neq cf(x^5) (x^5)^{\frac{2}{3}}, c \in R. \quad (156)$$

Therefore, on the basis of the results of Subsect.4.3, it is easy to get that the contravariant components of the 5-d. Killing vector $\xi^A(x, x^5)$ for metric (151) in the range $0 < x^5 < x_0^5$ are given by Eqs.(118)-(122), where (some of) the real parameters satisfy the following system (namely system (123) for metric (151)):

$$(01) \quad \left\{ \begin{array}{l} [d_8x^2x^3 + d_7x^2 + d_6x^3 + (d_5 + a_2)] + \\ + \left(\frac{x^5}{x_0^5}\right)^{1/3} [h_2x^2x^3 + h_1x^2 + h_8x^3 + (h_7 + e_2)] = 0; \end{array} \right.$$

$$(02) \quad \left\{ \begin{array}{l} (d_8x^1x^3 + d_7x^1 + d_4x^3 + d_3) + \\ + \left(\frac{x^5}{x_0^5}\right)^{1/3} [l_2x^1x^3 + l_1x^1 + l_6x^3 + (l_5 + e_4)] = 0; \end{array} \right.$$

$$(03) \quad \left\{ \begin{array}{l} (d_8x^1x^2 + d_6x^1 + d_4x^2 + d_2) + \\ + \left(\frac{x^5}{x_0^5}\right)^{1/3} [m_8x^1x^2 + m_7x^1 + m_6x^2 + (m_5 + g_2)] = 0; \end{array} \right.$$

$$(12) \quad \left\{ \begin{array}{l} \left(\frac{x^5}{x_0^5}\right)^{1/3} (h_2x^0x^3 + h_1x^0 + h_4x^3 + h_3) + \\ + \left(\frac{x^5}{x_0^5}\right)^{1/3} (l_2x^0x^3 + l_1x^0 + l_4x^3 + l_3) = 0; \end{array} \right.$$

$$(13) \quad \left\{ \begin{array}{l} \left(\frac{x^5}{x_0^5}\right)^{1/3} (h_2x^0x^2 + h_8x^0 + h_4x^2 + h_6) + \\ + \left(\frac{x^5}{x_0^5}\right)^{1/3} (m_8x^0x^2 + m_7x^0 + m_4x^2 + m_3) = 0; \end{array} \right.$$

$$(23) \quad \left\{ \begin{array}{l} \left(\frac{x^5}{x_0^5}\right)^{1/3} (l_2x^0x^1 + l_6x^0 + l_4x^1 + l_8) + \\ + \left(\frac{x^5}{x_0^5}\right)^{1/3} (m_8x^0x^1 + m_6x^0 + m_4x^1 + m_2) = 0. \end{array} \right. \quad (157)$$

Solving system (156) yields:

$$\begin{aligned}
 d_2 = d_3 = d_4 = d_6 = d_7 = d_8 = 0; \\
 m_4 = m_6 = m_7 = m_8 = 0; \quad m_5 = -g_2; \\
 h_1 = h_2 = h_4 = h_8 = 0; \quad h_3 = -l_3; \quad h_6 = -m_3; \quad h_7 = -e_2; \quad h_8 = -m_7; \\
 l_1 = l_2 = l_4 = l_6 = 0; \quad l_5 = -e_4; \quad l_8 = -m_2; \\
 a_2 = -d_5.
 \end{aligned} \tag{158}$$

Then, one gets the following expression for $\xi^A(x, x^5)$:

$$\xi^0 = \widetilde{F}_0 = (a_1 + d_1 + K_0); \tag{159}$$

$$\xi^1(x^2, x^3) = -\widetilde{F}_1(x^2, x^3) = l_3x^2 + m_3x^3 - (K_1 + h_5 + e_1); \tag{160}$$

$$\xi^2(x^1, x^3) = -\widetilde{F}_2(x^1, x^3) = -l_3x^1 + m_2x^3 - (l_7 + K_2 + e_3); \tag{161}$$

$$\xi^3(x^0, x^1, x^2) = -\widetilde{F}_3(x^0, x^1, x^2) = -m_3x^1 - m_2x^2 - (m_1 + g_1 + c); \tag{162}$$

$$\xi^5 = 0. \tag{163}$$

5.1.2 Killing Isometries for Electromagnetic and Weak Metrics

The 5-d. contravariant Killing vector $\xi^A(x, x^5)$ for the whole range of energies, Eqs.(146)-(150) and (158)-(162), can be cast in a compact form by using the distribution $\widehat{\Theta}_R(x^5 - x_0^5)$ (right specification of the Heaviside distribution: see Eq.(64)), by ridenominating ($\forall i = 1, 2, 3$)

$$\begin{aligned}
 B^i &\equiv \zeta^i; \\
 \Theta^i &\equiv \theta^i; \\
 \Xi^0 &\equiv \zeta^5,
 \end{aligned} \tag{164}$$

and putting

$$\begin{aligned}
 (a_1 + d_1 + K_0) &= T^0; \\
 -(K_1 + h_5 + e_1) &= T^1; \\
 -(l_7 + K_2 + e_3) &= T^2; \\
 -(m_1 + g_1 + c) &= T^3; \\
 l_3 &= \Theta^3; \\
 m_3 &= -\Theta^2; \\
 m_2 &= \Theta^1.
 \end{aligned} \tag{165}$$

Then, one gets

$$\begin{aligned} & \xi^0(x^1, x^2, x^3, x^5) = \\ & = \widehat{\Theta}_R(x^5 - x_0^5) [-\zeta^1 x^1 - \zeta^2 x^2 - \zeta^3 x^3 + \zeta^5 F(x^5)] + T^0; \end{aligned} \quad (166)$$

$$\begin{aligned} & \xi^1(x^0, x^2, x^3, x^5) = \\ & = \widehat{\Theta}_R(x^5 - x_0^5) [-\zeta^1 x^0 - \Xi^1 F(x^5)] + \theta^3 x^2 - \theta^2 x^3 + T^1; \end{aligned} \quad (167)$$

$$\begin{aligned} & \xi^2(x^0, x^1, x^3, x^5) = \\ & = \widehat{\Theta}_R(x^5 - x_0^5) [-\zeta^2 x^0 - \Xi^2 F(x^5)] - \theta^3 x^1 + \theta^1 x^3 + T^2; \end{aligned} \quad (168)$$

$$\begin{aligned} & \xi^3(x^0, x^1, x^2, x^5) = \\ & = \widehat{\Theta}_R(x^5 - x_0^5) [-\zeta^3 x^0 - \Xi^3 F(x^5)] + \theta^2 x^1 - \theta^1 x^2 + T^3; \end{aligned} \quad (169)$$

$$\begin{aligned} & \xi^5(x, x^5) = \\ & = \widehat{\Theta}_R(x^5 - x_0^5) \left\{ \mp (f(x^5))^{-\frac{1}{2}} [\zeta^5 x^0 + \Xi^1 x^1 + \Xi^2 x^2 + \Xi^3 x^3 - T^5] \right\}, \end{aligned} \quad (170)$$

valid for both ranges $0 < x^5 < x_0^5$ and $x^5 \geq x_0^5$.

By considering slices of \mathfrak{R}_5 at $dx^5 = 0$, one gets:

$$x^5 = \overline{x^5} \in R_0^+ \left\{ \begin{array}{l} \Leftrightarrow dx^5 = 0 \underset{\text{(in } \underline{\mathfrak{R}}^{\text{gen.}})}{\Rightarrow} F(x^5) = \int dx^5 (f(x^5))^{\frac{1}{2}} = 0; \\ \underset{\text{(in } \underline{\mathfrak{R}}^{\text{gen.}})}{\Rightarrow} \xi^5(x, x^5) = 0. \end{array} \right. \quad (171)$$

Therefore, it easily follows from the expression of the Killing vector (165)-(169) that, in the energy range $x^5 \geq x_0^5$, the 5-d. Killing group of such constant-energy sections is the standard Poincaré group $P(1, 3)$:

$$P(1, 3)_{STD.} = SO(1, 3)_{STD.} \otimes_s Tr.(1, 3)_{STD.} \quad (172)$$

(as it must be), whereas, in the energy range $0 < x^5 < x_0^5$, the Killing group is given by

$$SO(3)_{STD.} \otimes_s Tr.(1, 3)_{STD.} \quad (173)$$

5.1.3 Solution of Killing Equations below Threshold with Violated Υ -Hypothesis

In case b), the hypothesis Υ of functional independence (110) does not hold for any value of μ if the metric coefficient $f(x^5)$ satisfies the following equation

$$\frac{1}{2} \frac{f'(x^5)}{f(x^5)} - cf(x^5) (x^5)^{\frac{2}{3}} + \frac{1}{x^5} = 0, c \in R. \quad (174)$$

Such ordinary differential equation (ODE) belongs to the homogeneous class of type G and to the special rational subclass of Bernoulli's ordinary differential equations (it becomes separable for $c = 0$). The only solution of (173) is

$$f(x^5) = \frac{1}{6c(x^5)^{\frac{5}{3}} + \gamma(x^5)^2}, c, \gamma \in R. \quad (175)$$

Since $f(x^5)$ must be dimensionless, it is convenient to make this feature explicit by introducing the characteristic parameter $x_0^5 \in R_0^+$ (which, as by now familiar, is the threshold energy of the interaction considered) so that $f(x^5) \equiv f\left(\frac{x^5}{x_0^5}\right)$. Eq.(174) can be therefore rewritten as

$$f(x^5) \equiv f\left(\frac{x^5}{x_0^5}\right) = \frac{1}{6c\left(\frac{x^5}{x_0^5}\right)^{\frac{5}{3}} + \gamma\left(\frac{x^5}{x_0^5}\right)^2}, c, \gamma \in R, \quad (176)$$

where of course a rescaling of constants c and γ occurred. Moreover, because in general $f(x^5)$ has to be strictly positive $\forall x^5 \in R_0^+$, c and γ must necessarily satisfy the condition:

$$\begin{aligned} c, \gamma \in R : 6c + \gamma\left(\frac{x^5}{x_0^5}\right)^{\frac{1}{3}} > 0 \quad \forall x^5 \in R_0^+ &\Leftrightarrow \\ \Leftrightarrow c, \gamma \in R^+ \text{ (not both zero)}. &\quad (177) \end{aligned}$$

Therefore, *imposing the complete violation of the Υ hypothesis of functional independence allows one to determine the functional form of the fifth metric coefficient.* This result will be seen to hold also for the strong and the gravitational interaction above threshold (see Subsubsects.5.2.3 and 5.3.2).

Then, we get the following expression for the 5-d. metric describing e.m. and weak interactions in the energy range $0 < x^5 < x_0^5$ if the Υ hypothesis is not satisfied by any value of μ :

$$\begin{aligned} g_{AB,DR5}(x^5) = \text{diag} \left(1, -\left(\frac{x^5}{x_0^5}\right)^{1/3}, -\left(\frac{x^5}{x_0^5}\right)^{1/3}, -\left(\frac{x^5}{x_0^5}\right)^{1/3}, \right. \\ \left. \pm \left(6c\left(\frac{x^5}{x_0^5}\right)^{\frac{5}{3}} + \gamma\left(\frac{x^5}{x_0^5}\right)^2 \right)^{-1} \right). \quad (178) \end{aligned}$$

The solution of the Killing equations is cumbersome in this case, too. After some tedious and lengthy algebra, one gets the following expression for the contravariant Killing 5-vector $\xi^A(x, x^5)$ corresponding to the e.m. and weak metric (151):

$$\xi^0 = c_0; \quad (179)$$

$$\xi^1(x^2, x^3) = - (a_2x^2 + a_3x^3 + a_4) (x_0^5)^{1/3}; \quad (180)$$

$$\xi^2(x^1, x^3) = (a_2x^1b_1x^3 - b_6) (x_0^5)^{1/3}; \quad (181)$$

$$\xi^3(x^1, x^2) = (a_3x^1 - b_1x^2 - b_2) (x_0^5)^{1/3}; \quad (182)$$

$$\xi^5 = 0, \quad (183)$$

where the dimensions of the real transformation parameters are (on account of the fact that ξ has the dimension of a length)

$$[a_2] = [a_3] = [b_1] = l^{-1/3}, \quad [a_4] = [b_2] = [b_6] = l^{2/3}, \quad [c_0] = l. \quad (184)$$

The 5-d. Killing group of isometries is therefore

$$SO(3)_{STD.(E_3)} \otimes_s Tr.(1, 3)_{STD}. \quad (185)$$

with E_3 being the 3-d. manifold with metric $g_{ij} = - \left(\frac{x^5}{x_0^5} \right)^{1/3} \text{diag}(1, 1, 1)$.

5.2 Strong Interaction

5.2.1 Validity of the Υ -Hypothesis

Case a') (Minkowskian conditions). In the energy range $0 < x^5 \leq x_0^5$ the metric (43) for strong interaction reads

$$g_{AB,DR5}(x^5) = \text{diag} \left(1, -\frac{2}{25}, -\frac{4}{25}, -1, \pm f(x^5) \right). \quad (186)$$

From Eqs.(124) and (125) one finds, for the fake vectors $A_\mu(x^5)$, $B_\mu(x^5)$ in this case:

$$\begin{aligned} A_\mu(x^5) &= 0, \forall \mu = 0, 1, 2, 3; \\ B_0(x^5) &= B_3(x^5) = \frac{5}{\sqrt{2}}B_1(x^5) = \frac{5}{2}B_2(x^5) = (f(x^5))^{\frac{1}{2}}. \end{aligned} \quad (187)$$

Therefore the Υ -hypothesis (110) is not satisfied by any value of $\mu \in \{0, 1, 2, 3\}$. The 15 Killing equations corresponding to metric (185) are given by Eq.(138), *i.e.* coincide with those relevant to the 5-d. e.m. and weak metrics in the range $x^5 \geq x_0^5$. Since the contravariant metric tensor is

$$g_{DR5}^{AB}(x^5) = \text{diag} \left(1, -\frac{25}{2}, -\frac{25}{4}, -1, \pm (f(x^5))^{-1} \right), \quad (188)$$

the components of the contravariant Killing 5-vector $\xi^A(x, x^5) \stackrel{\text{ESC}}{=} \text{on } g_{DR5}^{AB}(x^5)\xi_B(x, x^5)$ are given by:

$$\xi^0(x^1, x^2, x^3, x^5) = -B^1x^1 - B^2x^2 - B^3x^3 + \Xi^0F(x^5) + T^0; \quad (189)$$

$$\xi^1(x^0, x^2, x^3, x^5) = \frac{25}{2} [-B^1x^0 + \Theta^3x^2 - \Theta^2x^3 - \Xi^1F(x^5) + T^1]; \quad (190)$$

$$\xi^2(x^0, x^1, x^3, x^5) = \frac{25}{4} [-B^2x^0 - \Theta^3x^1 + \Theta^1x^3 - \Xi^2F(x^5) + T^2]; \quad (191)$$

$$\xi^3(x^0, x^1, x^2, x^5) = -B^3x^0 + \Theta^2x^1 - \Theta^1x^2 - \Xi^3F(x^5) + T^3; \quad (192)$$

$$\xi^5(x, x^5) = \mp (f(x^5))^{-\frac{1}{2}} [\Xi^0x^0 + \Xi^1x^1 + \Xi^2x^2 + \Xi^3x^3 - T^5], \quad (193)$$

in the same notation of Eqs.(139)-(143).

Case b') (Non-Minkowskian conditions). In the energy range $x^5 > x_0^5$ the 5-d. strong metric takes the form:

$$g_{AB,DR5}(x^5) = \text{diag} \left(\left(\frac{x^5}{x_0^5} \right)^2, -\frac{2}{25}, -\frac{4}{25}, -\left(\frac{x^5}{x_0^5} \right)^2, \pm f(x^5) \right). \quad (194)$$

From Eqs.(124) and (125) one gets:

$$\begin{aligned} A_0(x^5) &= -A_3(x^5) = \frac{(x^5)^2}{(x_0^5)^3} (f(x^5))^{-\frac{1}{2}} \left(\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \right); \\ A_1(x^5) &= A_2(x^5) = 0; \end{aligned} \quad (195)$$

$$\begin{aligned} B_0(x^5) &= B_3(x^5) = \frac{x^5}{x_0^5} (f(x^5))^{\frac{1}{2}}; \\ B_1(x^5) &= \frac{1}{\sqrt{2}} B_2(x^5) = \frac{\sqrt{2}}{5} (f(x^5))^{\frac{1}{2}}; \end{aligned} \quad (196)$$

$$\frac{\pm A_0(x^5)}{B_0(x^5)} = \frac{\mp A_3(x^5)}{B_3(x^5)} = \pm \frac{x^5}{f(x^5)(x_0^5)^2} \left(\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \right). \quad (197)$$

Therefore, on account of the strict positiveness of $f(x^5)$, the hypothesis Υ of functional independence is satisfied by $\mu = 0, 3$ under the following constraints:

$$\left. \begin{aligned} \frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} &\neq 0; \\ \frac{x^5}{f(x^5)} \left(\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \right) &\neq c \Leftrightarrow \frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \neq c \frac{f(x^5)}{x^5}, c \in R_0 \end{aligned} \right\} \Leftrightarrow \begin{aligned} \Leftrightarrow \frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} &\neq c \frac{f(x^5)}{x^5}, c \in R. \end{aligned} \quad (198)$$

The case is analogous to the case b) of the e.m. and weak metrics. Thus, the components of the contravariant Killing 5-vector $\xi^A(x, x^5)$ for the phenomenological strong metric in

the range $x^5 > x_0^5$ are given by Eqs. (158)-(162), where (some of) the real parameters are constrained to obey the following system (see Eq.(156)):

$$\begin{aligned}
 (01) \quad & \left\{ \begin{aligned} & \left(\frac{x^5}{x_0^5}\right)^2 [d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2)] + \\ & + \frac{2}{25} [h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + e_2)] = 0; \end{aligned} \right. \\
 (02) \quad & \left\{ \begin{aligned} & \left(\frac{x^5}{x_0^5}\right)^2 (d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3) + \\ & + \frac{4}{25} [l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4)] = 0; \end{aligned} \right. \\
 (03) \quad & \left\{ \begin{aligned} & \left(\frac{x^5}{x_0^5}\right)^2 (d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2) + \\ & + \left(\frac{x^5}{x_0^5}\right)^2 [m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + (m_5 + g_2)] = 0; \end{aligned} \right. \\
 (12) \quad & \left\{ \begin{aligned} & \frac{2}{25} (h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3) + \\ & + \frac{4}{25} (l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3) = 0; \end{aligned} \right. \\
 (13) \quad & \left\{ \begin{aligned} & \frac{2}{25} (h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6) + \\ & + \left(\frac{x^5}{x_0^5}\right)^2 (m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3) = 0; \end{aligned} \right. \tag{199} \\
 (23) \quad & \left\{ \begin{aligned} & \frac{4}{25} (l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8) + \\ & + \left(\frac{x^5}{x_0^5}\right)^2 (m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2) = 0. \end{aligned} \right.
 \end{aligned}$$

The solutions of this system are given by

$$\begin{aligned}
 d_3 = d_4 = d_6 = d_7 = d_8 = 0; \quad d_2 = -(m_5 + g_2); \\
 m_2 = m_3 = m_4 = m_6 = m_7 = m_8 = 0; \\
 h_1 = h_2 = h_4 = h_6 = h_8 = 0; \tag{200} \\
 l_1 = l_2 = l_4 = l_6 = l_8 = 0; \quad l_5 = -e_4; \\
 h_3 = -2l_3; \quad h_7 = -e_2; \quad a_2 = -d_5.
 \end{aligned}$$

Replacing (199) into Eqs.(158)-(162) yields the explicit form of the Killing 5-vector $\xi^A(x, x^5)$:

$$\xi^0(x^3) = \widetilde{F}_0(x^3) = d_2 x^3 + (a_1 + d_1 + K_0); \tag{201}$$

$$\xi^1(x^2) = -\widetilde{F}_1(x^2) = 2l_3 x^2 - (K_1 + h_5 + e_1); \tag{202}$$

$$\xi^2(x^1) = -\widetilde{F}_2(x^1) = -l_3x^1 - (l_7 + K_2 + e_3); \quad (203)$$

$$\xi^3(x^0) = -\widetilde{F}_3(x^0) = d_2x^0 - (m_1 + g_1 + c); \quad (204)$$

$$\xi^5 = 0. \quad (205)$$

5.2.2 Killing Isometries for Strong Metric

As in the e.m. and weak case, it is possible to express the contravariant Killing vector of the phenomenological strong metric in a unique form, valid in the whole energy range. This is done by ridenominating the parameters in Eqs.(188)-(192) as follows:

$$\left\{ \begin{array}{l} B^i \equiv \zeta^i; \\ \Theta^i \equiv \theta^i; \\ \Xi^0 \equiv \zeta^5 \end{array} \right. \quad (206)$$

($\forall i = 1, 2, 3$) and putting, in Eqs.(200)-(204):

$$\begin{aligned} (a_1 + d_1 + K_0) &= T^0; \\ -(K_1 + h_5 + e_1) &= \frac{25}{2}T^1; \\ -(l_7 + K_2 + e_3) &= \frac{25}{4}T^2; \\ -(m_1 + g_1 + c) &= T^3; \\ l_3 &= \frac{25}{4}\Theta^3; \\ d_2 &= -B^3. \end{aligned} \quad (207)$$

Then, exploiting the right specification $\widehat{\Theta}_R(x_0^5 - x^5)$ of the step function, we get the following general form of the contravariant Killing 5-vector $\xi^A(x, x^5)$ for the 5-d phenomenological metric of the strong interaction:

$$\begin{aligned} \xi^0(x^1, x^2, x^3, x^5) &= \\ &= \widehat{\Theta}_R(x_0^5 - x^5) [-\zeta^1x^1 - \zeta^2x^2 + \zeta^5F(x^5)] - \zeta^3x^3 + T^0; \end{aligned} \quad (208)$$

$$\begin{aligned} \xi^1(x^0, x^2, x^3, x^5) &= \\ &= \frac{25}{2}\widehat{\Theta}_R(x_0^5 - x^5) [-\zeta^1x^0 - \theta^2x^3 - \Xi^1F(x^5)] + \frac{25}{2}\theta^3x^2 + \frac{25}{2}T^1; \end{aligned} \quad (209)$$

$$\begin{aligned} & \xi^2(x^0, x^1, x^3, x^5) = \\ & = \frac{25}{4} \widehat{\Theta}_R(x_0^5 - x^5) [-\zeta^2 x^0 + \theta^1 x^3 - \Xi^2 F(x^5)] - \frac{25}{4} \theta^3 x^1 + \frac{25}{4} T^2; \end{aligned} \quad (210)$$

$$\begin{aligned} & \xi^3(x^0, x^1, x^2, x^5) = \\ & = \widehat{\Theta}_R(x_0^5 - x^5) [\theta^2 x^1 - \theta^1 x^2 - \Xi^3 F(x^5)] - \zeta^3 x^0 + T^3; \end{aligned} \quad (211)$$

$$\begin{aligned} & \xi^5(x, x^5) = \\ & = \widehat{\Theta}_R(x_0^5 - x^5) \left\{ \mp (f(x^5))^{-\frac{1}{2}} [\zeta^5 x^0 + \Xi^1 x^1 + \Xi^2 x^2 + \Xi^3 x^3 - T^5] \right\}. \end{aligned} \quad (212)$$

By redefining

$$\begin{aligned} & \frac{25}{2} \begin{pmatrix} T^1 \\ \zeta^1 \\ \theta^2 \\ \Xi^1 \end{pmatrix} \equiv \begin{pmatrix} T^{1'} \\ \zeta^{1'} \\ \theta^{2'} \\ \Xi^{1'} \end{pmatrix} \\ & \frac{25}{4} \begin{pmatrix} T^2 \\ \zeta^2 \\ \theta^1 \\ \theta^3 \\ \Xi^2 \end{pmatrix} \equiv \begin{pmatrix} T^{2'} \\ \zeta^{2'} \\ \theta^{1'} \\ \theta^{3'} \\ \Xi^{2'} \end{pmatrix} \end{aligned} \quad (213)$$

in Eqs.(207)-(211) (and omitting the apices) one finds eventually:

$$\begin{aligned} & \xi^0(x^1, x^2, x^3, x^5) = \\ & = \widehat{\Theta}_R(x_0^5 - x^5) \left[-\frac{2}{25} \zeta^1 x^1 - \frac{4}{25} \zeta^2 x^2 + \zeta^5 F(x^5) \right] - \zeta^3 x^3 + T^0; \end{aligned} \quad (214)$$

$$\begin{aligned} & \xi^1(x^0, x^2, x^3, x^5) = \\ & = \widehat{\Theta}_R(x_0^5 - x^5) [-\zeta^1 x^0 - \theta^2 x^3 - \Xi^1 F(x^5)] + 2\theta^3 x^2 + T^1; \end{aligned} \quad (215)$$

$$\begin{aligned} & \xi^2(x^0, x^1, x^3, x^5) = \\ & = \widehat{\Theta}_R(x_0^5 - x^5) [-\zeta^2 x^0 + \theta^1 x^3 - \Xi^2 F(x^5)] - \theta^3 x^1 + T^2; \end{aligned} \quad (216)$$

$$\begin{aligned} & \xi^3(x^0, x^1, x^2, x^5) = \\ & = \widehat{\Theta}_R(x_0^5 - x^5) \left[\frac{2}{25} \theta^2 x^1 - \frac{4}{25} \theta^1 x^2 - \Xi^3 F(x^5) \right] - \zeta^3 x^0 + T^3; \end{aligned} \quad (217)$$

$$\begin{aligned} & \xi^5(x, x^5) = \\ & = \widehat{\Theta}_R(x_0^5 - x^5) \left\{ \mp (f(x^5))^{-\frac{1}{2}} \left[\zeta^5 x^0 + \frac{2}{25} \Xi^1 x^1 + \frac{4}{25} \Xi^2 x^2 + \Xi^3 x^3 - T^5 \right] \right\}. \end{aligned} \quad (218)$$

The 5-d. strong metric (185) in the energy range $0 < x^5 \leq x_0^5$ can be put in the form:

$$g_{AB,DR5}(x^5) = \text{diag} \left(g_{\mu\nu, \overline{M}_4}(x^5), \pm f(x^5) \right), \quad (219)$$

where \overline{M}_4 is a standard 4-d. Minkowskian manifold with the following coordinate rescaling (contraction):

$$\begin{aligned} x^1 & \longrightarrow \frac{\sqrt{2}}{5} x^1 \quad \Rightarrow \quad dx^1 \longrightarrow \frac{\sqrt{2}}{5} dx^1; \\ x^2 & \longrightarrow \frac{2}{5} x^2 \quad \Rightarrow \quad dx^2 \longrightarrow \frac{2}{5} dx^2. \end{aligned} \quad (220)$$

Considering slices at $dx^5 = 0$ of \mathfrak{R}_5 entails $\xi^5(x, x^5) = 0$ (see Eq.(170)). Then, the explicit form (213)-(217) of the Killing vector entails that — as expected — the Killing group is the standard Poincaré group $P(1, 3)$ (suitably rescaled):

$$[P(1, 3)_{STD.} = SO(1, 3)_{STD.} \otimes_s Tr.(1, 3)_{STD.}]|_{x^1 \rightarrow \frac{\sqrt{2}}{5} x^1, x^2 \rightarrow \frac{2}{5} x^2}. \quad (221)$$

For $x^5 > x_0^5$ the strong metric is given by Eq.(190). Therefore, it easily follows from Eqs.(213)-(217) that the 5-d Killing group of the constant-energy sections of \mathfrak{R}_5 is

$$\left(SO(2)_{STD., \Pi(x^1, x^2 \rightarrow \sqrt{2}x^2)} \otimes B_{x^3, STD.} \right) \otimes_s Tr.(1, 3)_{STD.}. \quad (222)$$

Here $SO(2)_{STD., \Pi(x^1, x^2 \rightarrow \sqrt{2}x^2)} = SO(2)_{STD., \Pi(x^1 \rightarrow \frac{\sqrt{2}}{5} x^1, x^2 \rightarrow \frac{2}{5} x^2)}$ is the 1-parameter group (generated by the usual, special-relativistic generator $S_{SR}^3|_{x^2 \rightarrow \sqrt{2}x^2}$) of the 2-d. rotations in the plane $\Pi(x^1, x^2)$ (characterized by the coordinate contractions (219)), and $B_{x^3, STD.}$ is the usual one-parameter group (generated by the special-relativistic generator K_{SR}^3) of the standard Lorentzian boosts along $\widehat{x^3}$. The direct and semidirect nature of the group products in (220) and (221) has the following explanation. In general (independently of contractions and/or dilations of coordinates) the standard mixed Lorentz algebra is given by the commutation relations (ESC on):

$$[S_{SR}^i, K_{SR}^j] = \epsilon_{ijl} K_{SR}^l, \forall i, j = 1, 2, 3, \quad (223)$$

where as usual ϵ_{ijl} is the Levi-Civita 3-tensor of rank 3 and S_{SR}^i and K_{SR}^i are the i -th generators of (true) rotations and Lorentz boosts, respectively. It follows that

$$[S_{SR}^3, K_{SR}^3]|_{x^2 \rightarrow \sqrt{2}x^2} \left(= [S_{SR}^3|_{x^2 \rightarrow \sqrt{2}x^2}, K_{SR}^3] \right) = 0 \quad (224)$$

what justifies the presence of the direct group product in Eq.(221). The semidirect product of $\left(SO(2)_{STD., \Pi(x^1, x^2 \rightarrow \sqrt{2}x^2)} \otimes B_{x^3, STD.} \right)$ by $Tr.(1, 3)_{STD.}$ is due instead to the

fact that the standard mixed Poincaré algebra (independently of contractions and/or dilations of coordinates) is defined by the following commutation relations (ESC on) ($\forall i, j, k = 1, 2, 3$):

$$\begin{aligned} [K_{SR}^i, \Upsilon_{SR}^0] &= -\Upsilon_{SR}^i; \\ [K_{SR}^i, \Upsilon_{SR}^j] &= -\delta^{ij} \Upsilon_{SR}^0; \\ [S_{SR}^i, \Upsilon_{SR}^0] &= 0; \\ [S_{SR}^i, \Upsilon_{SR}^k] &= \epsilon_{ikl} \Upsilon_{SR}^l, \end{aligned} \tag{225}$$

where $\Upsilon_{SR}^0, \Upsilon_{SR}^1, \Upsilon_{SR}^2$ and Υ_{SR}^3 are the generators of the standard space-time translations.

5.2.3 Solution of Strong Killing Equations above Threshold with Violated Υ -Hypothesis

In the energy range $x^5 > x_0^5$, if condition (197) is *not* satisfied, the hypothesis Υ of functional independence (110) does not hold for any value of μ . In this case the metric coefficient $f(x^5)$ obeys the equation

$$\frac{1}{2} \frac{f'(x^5)}{f(x^5)} - c \frac{f(x^5)}{x^5} + \frac{1}{x^5} = 0, c \in R. \tag{226}$$

Such ODE is separable $\forall c \in R$. By solving it, one gets the following form of the 5-d metric of the strong interaction (for $x^5 > x_0^5$ and when the Υ hypothesis (110) is violated):

$$\begin{aligned} g_{AB,DR5}(x^5) &= \\ &= \text{diag} \left(\left(\frac{x^5}{x_0^5} \right)^2, -\frac{2}{25}, -\frac{4}{25}, -\left(\frac{x^5}{x_0^5} \right)^2, \pm \frac{1}{\gamma \left(\frac{x^5}{x_0^5} \right)^2 + c} \right) \end{aligned} \tag{227}$$

with

$$c, \gamma \in R : \gamma \left(\frac{x^5}{x_0^5} \right)^2 + c > 0, \forall x^5 \in R_0^+ \Leftrightarrow c, \gamma \in R^+ (\text{not both zero}). \tag{228}$$

Solving the related Killing equations (after lengthy but elementary calculations) yields for the contravariant Killing 5-vector $\xi^A(x, x^5)$ the following compact form (valid for $c, \gamma \in R^+$ but not vanishing simultaneously):

$$\xi^0(x^3; c, \gamma) = (1 - \delta_{c,0}) \left[- (x_0^5)^2 ((1 - \delta_{\gamma,0}) d_3 x^3 + T_0) \right]; \tag{229}$$

$$\xi^1(x^2; \gamma) = - (1 - \delta_{\gamma,0}) \frac{25}{2} d_2 x^2 - \frac{25}{2} T_1; \tag{230}$$

$$\xi^2(x^1; \gamma) = (1 - \delta_{\gamma,0}) \frac{25}{4} d_2 x^1 - \frac{25}{4} T_2; \tag{231}$$

$$\xi^3(x^0; c, \gamma) = -(1 - \delta_{c,0})(1 - \delta_{\gamma,0})(x_0^5)^2(d_3x^0 + T_3); \quad (232)$$

$$\xi^5(x^5; c, \gamma) = \pm \delta_{c,0} \frac{\gamma\alpha}{(x_0^5)^2} x^5, \quad (233)$$

where we identified $-\varepsilon = T_0$ in Eq.(228), highlighted the parametric dependence of ξ^A on c and γ , and introduced the Kronecker δ .

The dimensions and ranges of the transformation parameters are

$$[\alpha] = l^2, \quad [d_3] = l^{-2}, \quad [T_0] = [T_3] = l^{-1}, \quad [T_1] = [T_2] = l, \quad [d_2] = l^0; \quad (234)$$

$$\alpha, d_2, d_3, T_0, T_1, T_2, T_3 \in R. \quad (235)$$

The 4-d.Killing group (*i.e.* the isometry group of the slices of \mathfrak{R}_5 at $dx^5 = 0$), too, can be written in the compact form

$$\left[Tr_{\widehat{x^1, x^2} STD.} \otimes (1 - \delta_{c,0}) Tr_{\widehat{x^0} STD.} \otimes (1 - \delta_{c,0})(1 - \delta_{\gamma,0}) Tr_{\widehat{x^3} STD.} \right] \otimes_s \\ \otimes_s \left[(1 - \delta_{\gamma,0}) SO(2)_{STD.(\Pi_2)} \otimes (1 - \delta_{c,0})(1 - \delta_{\gamma,0}) B_{STD.\widehat{x^3}} \right]. \quad (236)$$

Here, Π_2 is the 2-d. manifold (x^1, x^2) with metric rescaling $x^2 \longrightarrow \sqrt{2}x^2$ with respect to the Euclidean level, and $SO(2)_{STD.(\Pi_2)}$, $B_{STD.\widehat{x^3}}$ are the 1-parameter abelian groups generated by $S_{SR}^3|_{x^2 \longrightarrow \sqrt{2}x^2}$ and K_{SR}^3 , respectively. The semidirect nature of the group product is determined by the following commutation relations of the mixed, roto-translational, space-time Lorentz algebra (ESC on):

$$[S_{SR}^i, \Upsilon_{SR}^j] = \epsilon_{ijl} \Upsilon_{SR}^l. \quad (237)$$

The direct product of $SO(2)_{STD.(\Pi_2)}$ and $B_{STD.\widehat{x^3}}$ and of the translation groups $Tr_{\widehat{x^1, x^2} STD.}$, $Tr_{\widehat{x^0} STD.}$, $Tr_{\widehat{x^3} STD.}$ is instead a consequence of the commutativity of the generators:

$$[S_{SR}^3, K_{SR}^3]|_{x^2 \longrightarrow \sqrt{2}x^2} = 0; \quad (238)$$

$$[\Upsilon_{SR}^\mu, \Upsilon_{SR}^\nu] = 0. \quad (239)$$

5.3 Gravitational Interaction

5.3.1 Validity of the Υ -Hypothesis

Case a') (Minkowskian conditions). In the energy range $0 < x^5 \leq x_0^5$ the 5-d. metric for gravitational interaction (44) becomes:

$$g_{AB,DR5}(x^5) = diag(1, -b_1^2(x^5), -b_2^2(x^5), -1, \pm f(x^5)). \quad (240)$$

The "vectors" $A_\mu(x^5)$ and $B_\mu(x^5)$ (124) and (125) read therefore:

$$A_0(x^5) = A_3(x^5) = 0; \\ B_0(x^5) = B_3(x^5) = (f(x^5))^{\frac{1}{2}}; \quad (241)$$

$$\begin{aligned}
& A_i(x^5) = b_i(x^5)(f(x^5))^{-1/2} \\
& \cdot \left[- (b'_i(x^5))^2 + b_i(x^5)b''_i(x^5) - \frac{1}{2}b_i(x^5)b'_i(x^5)f'(x^5)(f(x^5))^{-1} \right], \\
& \qquad i = 1, 2;
\end{aligned} \tag{242}$$

$$B_i(x^5) = b_i(x^5)(f(x^5))^{1/2}, i = 1, 2; \tag{243}$$

$$\begin{aligned}
\frac{\pm A_i(x^5)}{B_i(x^5)} = \pm (f(x^5))^{-1} \left[- (b'_i(x^5))^2 + b_i(x^5)b''_i(x^5) - \frac{1}{2}b_i(x^5)b'_i(x^5)f'(x^5)(f(x^5))^{-1} \right], \\
i = 1, 2.
\end{aligned} \tag{244}$$

One has

$$\left. \begin{aligned}
& A_i(x^5) \neq 0 \Leftrightarrow \\
& \Leftrightarrow \left[- (b'_i(x^5))^2 + b_i(x^5)b''_i(x^5) - \frac{1}{2}b_i(x^5)b'_i(x^5)f'(x^5)(f(x^5))^{-1} \right] \neq 0; \\
& B_i(x^5) \neq 0 \quad \forall x^5 \in R_0^+ \text{ (no condition) }; \\
& \frac{\pm A_i(x^5)}{B_i(x^5)} \neq c, c \in R_0 \Leftrightarrow \\
& \Leftrightarrow - (b'_i(x^5))^2 + b_i(x^5)b''_i(x^5) - \frac{1}{2}b_i(x^5)b'_i(x^5)f'(x^5)(f(x^5))^{-1} \neq cf(x^5), \\
& c \in R_0
\end{aligned} \right\} \Leftrightarrow$$

$$\begin{aligned}
& \Leftrightarrow - (b'_i(x^5))^2 + b_i(x^5)b''_i(x^5) - \frac{1}{2}b_i(x^5)b'_i(x^5)f'(x^5)(f(x^5))^{-1} \neq \\
& \neq cf(x^5), c \in R, i = 1, 2.
\end{aligned} \tag{245}$$

Therefore the validity for $\mu = 1, 2$ of the Υ hypothesis (110) (not satisfied for $\mu = 0, 3$) depends on the nature and the functional form of the metric coefficients $b_1^2(x^5)$ and $b_2^2(x^5)$.

In general the 15 Killing equations corresponding to metric (239) are:

$$\left\{ \begin{array}{l}
 f(x^5)\xi_{0,0}(x^A) = 0; \\
 \xi_{0,1}(x^A) + \xi_{1,0}(x^A) = 0; \\
 \xi_{0,2}(x^A) + \xi_{2,0}(x^A) = 0; \\
 \xi_{0,3}(x^A) + \xi_{3,0}(x^A) = 0; \\
 \xi_{0,5}(x^A) + \xi_{5,0}(x^A) = 0; \\
 f(x^5)\xi_{1,1}(x^A) \mp b_1(x^5)b'_1(x^5)\xi_5(x^A) = 0; \\
 \xi_{1,2}(x^A) + \xi_{2,1}(x^A) = 0; \\
 \xi_{1,3}(x^A) + \xi_{3,1}(x^A) = 0; \\
 b_1(x^5)(\xi_{1,5}(x^A) + \xi_{5,1}(x^A)) - 2b'_1(x^5)\xi_1(x^A) = 0; \\
 f(x^5)\xi_{2,2}(x^A) \mp b_2(x^5)b'_2(x^5)\xi_5(x^A) = 0; \\
 \xi_{2,3}(x^A) + \xi_{3,2}(x^A) = 0; \\
 b_2(x^5)(\xi_{2,5}(x^A) + \xi_{5,2}(x^A)) - 2b'_2(x^5)\xi_2(x^A) = 0; \\
 f(x^5)\xi_{3,3}(x^A) = 0; \\
 \xi_{3,5}(x^A) + \xi_{5,3}(x^A) = 0; \\
 2f(x^5)\xi_{5,5}(x^A) - f'(x^5)\xi_5(x^A) = 0.
 \end{array} \right. \quad (246)$$

By making suitable assumptions on the functional form of the metric coefficients $b_i^2(x^5)$ ($i = 1, 2$), it is possible in 11 cases (which include all those of physical and mathematical interest) to solve the relevant Killing equations for the gravitational interaction and get the related isometries (see Appendix C).

Case b') (Non-Minkowskian conditions). In the energy range $x^5 > x_0^5$ the 5-d.

gravitational metric (44) reads:

$$g_{AB,DR5}(x^5) = \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_0^5} \right)^2, -b_1^2(x^5), -b_2^2(x^5), -\frac{1}{4} \left(1 + \frac{x^5}{x_0^5} \right)^2, \pm f(x^5) \right). \quad (247)$$

Eqs.(124) and (125) yield

$$A_0(x^5) = -A_3(x^5) = \frac{1}{8} \left(1 + \frac{x^5}{x_0^5} \right) \frac{x^5}{(x_0^5)^2} (f(x^5))^{-\frac{1}{2}} \left[\frac{1}{x^5} + \frac{1}{2} \left(1 + \frac{x_0^5}{x^5} \right) \frac{f'(x^5)}{f(x^5)} \right]; \quad (248)$$

$$B_0(x^5) = B_3(x^5) = \frac{1}{2} \left(1 + \frac{x_0^5}{x^5} \right) (f(x^5))^{\frac{1}{2}};$$

$$\frac{\pm A_0(x^5)}{B_0(x^5)} = \frac{\mp A_3(x^5)}{B_3(x^5)} = \pm \frac{1}{4} \frac{1}{f(x^5)} \frac{x^5}{(x_0^5)^2} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5}{x^5} \frac{f'(x^5)}{f(x^5)} \right], \quad (249)$$

whereas Eqs.(241)-(244) of the previous case still hold for $A_i(x^5)$ and $B_i(x^5)$ ($i = 1, 2$). Then, since $f(x^5)$ is strictly positive and

$$x^5, x_0^5 \in R_0^+ \underset{\substack{\Rightarrow \\ (\text{in gen.}) \\ \neq}}}{\Rightarrow} \left(1 + \frac{x_0^5}{x^5} \right) \in R_0^+, \quad (250)$$

the Υ -hypothesis of functional independence for the gravitational metric over threshold is satisfied at least for $\mu = 0, 3$ under the following conditions:

$$\left. \begin{aligned} & \frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5}{x^5} \frac{f'(x^5)}{f(x^5)} \neq 0; \\ & \frac{x^5}{f(x^5)} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5}{x^5} \frac{f'(x^5)}{f(x^5)} \right] \neq \mathfrak{C} \Leftrightarrow \\ & \frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5}{x^5} \frac{f'(x^5)}{f(x^5)} \neq \mathfrak{C} \frac{f(x^5)}{x^5}, \mathfrak{C} \in R_0, \end{aligned} \right\} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5}{x^5} \frac{f'(x^5)}{f(x^5)} \neq \mathfrak{C} \frac{f(x^5)}{x^5}, \mathfrak{C} \in R. \quad (251)$$

Therefore, in the energy range $x^5 > x_0^5$, if the Υ -hypothesis of functional independence for the gravitational metric is not satisfied for $\mu = 0, 3$, the metric coefficient $f(x^5)$ obeys the following equation

$$\frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5}{x^5} \frac{f'(x^5)}{f(x^5)} - \mathfrak{C} \frac{f(x^5)}{x^5} + \frac{1}{x^5} = 0, \mathfrak{C} \in R \quad (252)$$

which, since $x_0^5, x^5, f(x^5) \in R_0^+$, can be rewritten as

$$f'(x^5) + \frac{2}{x^5 + x_0^5} f(x^5) - \frac{2\mathfrak{C}}{x^5 + x_0^5} (f(x^5))^2 = 0, \mathfrak{C} \in R. \quad (253)$$

Such ordinary differential equation belongs to the separable subclass of the Bernoulli type $\forall \mathfrak{C} \in R$. Its only solution is

$$f(x^5) = \frac{1}{\gamma(x^5 + x_0^5)^2 + \mathfrak{C}}, \mathfrak{C}, \gamma \in R, \quad (254)$$

which can be expressed in dimensionless form as (by suitably rescaling the constants \mathfrak{C}, γ):

$$\left\{ \begin{array}{l} f(x^5) \equiv f\left(\frac{x^5}{x_0^5}\right) = \frac{1}{\gamma\left(1 + \frac{x^5}{x_0^5}\right)^2 + \mathfrak{C}}, \\ c, \gamma \in R : \gamma\left(1 + \frac{x^5}{x_0^5}\right)^2 + \mathfrak{C} > 0 \forall x^5 \in R_0^+ \Leftrightarrow \mathfrak{C}, \gamma \in R^+ \text{ (not both zero)}. \end{array} \right. \quad (255)$$

The corresponding 5-d gravitational metric is therefore

$$\begin{aligned} & g_{AB,DR5}(x^5) = \\ & = \text{diag} \left(\begin{array}{l} \frac{1}{4} \left(1 + \frac{x^5}{x_0^5}\right)^2, -b_1^2(x^5), -b_2^2(x^5), \\ -\frac{1}{4} \left(1 + \frac{x^5}{x_0^5}\right)^2, \pm \left(\gamma \left(1 + \frac{x^5}{x_0^5}\right)^2 + \mathfrak{C}\right)^{-1} \end{array} \right). \end{aligned} \quad (256)$$

Thus, in both energy ranges the Υ -hypothesis is violated for $\mu = 0, 3$. Below threshold this is automatically ensured by the form of the gravitational metric (239), whereas *above threshold such a requirement determines the expression of the fifth metric coefficient $f(x^5)$.*

5.3.2 The 5-d. Υ -Violating Metrics of Gravitation

We want now to discuss the 5-d. gravitational metrics which violate the Υ -hypothesis of functional independence $\forall \mu = 0, 1, 2, 3$ in either energy range $0 < x^5 \leq x_{0,grav}^5$ and $x^5 > x_{0,grav}^5$.

It was shown in Subsubsect.5.3.1 that, when the Υ -hypothesis is not satisfied by $\mu = 0, 3$, the gravitational metric is given by Eq.(239) and Eq.(255) below threshold and over threshold, respectively. If one imposes in addition that the Υ -hypothesis is violated also by $\mu = 1$ and/or 2, *such a requirement permits to get the expressions of the metric coefficients $b_1^2(x^5), b_2^2(x^5)$ in terms of $f(x^5)$.*

Indeed, in this case it follows from Eq.(244) that the metric coefficients $b_1^2(x^5)$, $b_2^2(x^5)$ and $f(x^5)$ satisfy the following ODE (ESC off)¹¹

$$\begin{aligned}
 - (b'_k(x^5))^2 + b_k(x^5)b''_k(x^5) - \frac{1}{2}b_k(x^5)b'_k(x^5)f'(x^5)(f(x^5))^{-1} - c_k f(x^5) = 0, \\
 c_k \in R, k = 1 \text{ and/or } 2,
 \end{aligned} \tag{257}$$

whose solution for $f(x^5)$ in terms of $b_k(x^5)$ is:

$$\begin{aligned}
 f(x^5) &= \frac{(b'_k(x^5))^2}{d_k b_k^2(x^5) - c_k} \Leftrightarrow \\
 &\Leftrightarrow d_k b_k^2(x^5) f(x^5) - (b'_k(x^5))^2 - c_k f(x^5) = 0, \\
 k &= 1 \text{ and/or } 2, d_k \in R^+, c_k \in R^- \text{ (not both zero)}.
 \end{aligned} \tag{258}$$

In dimensionless form for $f(x^5)$ and $b_k^2(x^5)$, we have

$$\begin{aligned}
 f(x^5) \equiv f\left(\frac{x^5}{x_0^5}\right) &= \frac{\left(b'_k\left(\frac{x^5}{x_0^5}\right)\right)^2}{d_k b_k^2\left(\frac{x^5}{x_0^5}\right) - c_k} \Leftrightarrow \\
 &\Leftrightarrow d_k b_k^2\left(\frac{x^5}{x_0^5}\right) f\left(\frac{x^5}{x_0^5}\right) - \left(b'_k\left(\frac{x^5}{x_0^5}\right)\right)^2 - c_k f\left(\frac{x^5}{x_0^5}\right) = 0, \\
 k &= 1 \text{ and/or } 2.
 \end{aligned} \tag{259}$$

By assuming $f(x^5)$ known, one gets the following implicit solution of (257) for $b_k(x^5)$ ($d_k > 0$):

$$\begin{aligned}
 \alpha_k \pm \int^{x^5} dx^{5'} \sqrt{-f(x^{5'})} + \frac{1}{\sqrt{d_k}} \arctan\left(\frac{\sqrt{d_k} b_k(x^5)}{\sqrt{c_k - d_k b_k^2(x^5)}}\right) = 0, \\
 k = 1 \text{ and/or } 2, \alpha_k \in R,
 \end{aligned} \tag{260}$$

under the constraint

$$c_k - d_k b_k^2\left(\frac{x^5}{x_0^5}\right) > 0, k = 1 \text{ and/or } 2 \forall x^5 \in R_0^+ \Leftrightarrow d_k \in R^+, c_k \in R^-. \tag{261}$$

Equation (259) can be solved for all possible pair of values (d_k, c_k) (even in the limit case of b_k constant). Precisely, one can distinguish the following three cases ($k = 1, 2$):

I) $(d_k, c_k) \in R_0^+ \times R_0^-$:

$$\begin{aligned}
 b_k^2(x^5) &= \frac{c_k}{d_k} \frac{\tanh^2[\sqrt{d_k}(\alpha_k \mp F(x^5))]}{\{\tanh^2[\sqrt{d_k}(\alpha_k \mp F(x^5))] - 1\}} = \\
 &= -\frac{c_k}{d_k} \left\{ \cosh\left[2\sqrt{d_k}(\alpha_k \mp F(x^5))\right] - 1 \right\}, \alpha_k \in R,
 \end{aligned} \tag{262}$$

¹¹ In the following, the lower index "k" in the constants means that, in general, these depend on the metric coefficient b_k considered.

where $F(x^5)$ is still given by Eq.(144).

II) $(d_k, c_k) \in \{0\} \times R_0^-$:

$$b_k^2(x^5) = (\pm\sqrt{-c_k}F(x^5) + \delta_k)^2, \delta_k \in R. \quad (263)$$

III) $(d_k, c_k) \in R_0^+ \times \{0\}$:

$$\begin{aligned} b_k^2(x^5) &= \kappa_k^2 \exp\left(2\sqrt{d_k}F(x^5)\right) = \\ &= \kappa_{k1}^2 \exp\left(2\sqrt{d_k}F(x^5)\right) + \kappa_{k2}^2 \exp\left(-2\sqrt{d_k}F(x^5)\right), \\ &\kappa_k, \kappa_{ki} \in R_0, i = 1, 2. \end{aligned} \quad (264)$$

Let us note that all the previous results hold true in general for any $\mu \in \{0, 1, 2, 3\}$. They have been discussed by considering $\mu = k = 1$ and/or 2 in order to apply the results to the case of the DR5 metric of the gravitational interaction, characterized by the indeterminacy of the metric coefficients $b_k(x^5)$, $k = 1, 2$, and requiring therefore a treatment of the Υ -violation for $\mu = k = 1$ and/or 2.

The above general formalism allows one to deal with the 5-d. metrics of DR5 for the gravitational interaction which violate $\Upsilon \forall \mu = 0, 1, 2, 3$ in the energy ranges $0 < x^5 \leq x_{0,grav}^5$ and $x^5 > x_{0,grav}^5$.

In the first case the functional form of $f(x^5)$ is undetermined, since in general it must only satisfy the condition $f > 0 \forall x^5 \in R_0^+$. As to the space coefficients $b_k(x^5)$, $k = 1, 2$, one has 9 possible cases, obtained by considering all the possible pairs $(\mathcal{I}_1, \mathcal{I}_2)$ ($\mathcal{I}_1, \mathcal{I}_2 = I, II, III$) of the functional typologies for $b_k(x^5)$ corresponding to the pairs of values (d_k, c_k) (see Eqs.(261)-(263)). Since the two space coefficients are expressed in terms of the fifth metric coefficient, one gets " $f(x^5)$ -dependent", *i.e.* in general "functionally parametrized" metrics.

In the energy range $x^5 > x_{0,grav}^5$, the fifth coefficient is determined by Eq.(252) with solution (254). The Υ -violating gravitational metric has the form (255), and the space coefficients $b_k(x^5)$, $k = 1, 2$, are still given by Eqs.(261)-(263). However, now the function $F(x^5)$ can be explicitly evaluated. One has:

$$\begin{aligned} F(x^5) &\equiv \int dx^5 (f(x^5))^{\frac{1}{2}} = + \int dx^5 \sqrt{f(x^5)} = \\ &= x_0^5 \int \frac{dx^5}{\sqrt{\gamma(x^5)^2 + 2\gamma x_0^5 x^5 + (\mathfrak{C} + \gamma)(x_0^5)^2}} = \\ &= \begin{cases} \frac{x^5}{\sqrt{\mathfrak{C}}} & \text{for } (\gamma, \mathfrak{C}) \in \{0\} \times R_0^+ \quad (\text{case A}); \\ \frac{x_0^5}{\sqrt{\gamma}} \ln\left(1 + \frac{x^5}{x_0^5}\right) & \text{for } (\gamma, \mathfrak{C}) \in R_0^+ \times \{0\} \quad (\text{case B}); \\ \frac{x_0^5}{\sqrt{\gamma}} \operatorname{arcsinh}\left[\sqrt{\frac{\gamma}{\mathfrak{C}}}\left(1 + \frac{x^5}{x_0^5}\right)\right] & \text{for } (\gamma, \mathfrak{C}) \in R_0^+ \times R_0^+ \quad (\text{case C}). \end{cases} \end{aligned} \quad (265)$$

Replacing the above expressions (264), corresponding to the 3 possible pairs (γ, \mathbb{C}) , in Eqs.(261)-(263), one gets all the possible forms of the coefficients $b_1^2(x^5)$ and $b_2^2(x^5)$ of the Υ -violating gravitational metric above threshold. The functional typologies of the spatial coefficients can be labelled by the pair $(\mathcal{L}, \mathcal{I})$ (with $\mathcal{L} = A, B, C$ labelling the 3 cases of Eq.(264) and $\mathcal{I} = I, II, III$ referring as before to the 3 expressions (261)-(263)). Then, one gets 27 possible forms for the 5-d gravitational metrics violating the hypothesis Υ in the whole energy range. They can be labelled by $(\mathcal{L}_1\mathcal{I}_1, \mathcal{L}_2\mathcal{I}_2)$ ($\mathcal{L}_1, \mathcal{L}_2 = A, B, C$, $\mathcal{I}_1, \mathcal{I}_2 = I, II, III$), in the notation exploited for the functional typology of the metric coefficients $b_k^2(x^5)$ for the indices 1 and 2, according to the above discussion. Their explicit form is easy to write down, and can be found in ref.[39].

In correspondence to the different gravitational metrics, one gets 27 systems of 15 Killing equations, which would require an explicit solution (or at least not exploiting the Υ -hypothesis), in order to find the corresponding isometries. However, solving these systems is far from being an easy task, even by using symbolic-algebraic manipulation programs.

A possible method of partial solution could be the Lie structural approach, based on the Lie symmetries obeyed by the system equations. Such a resolution could in principle be also applied to the general system (245), in which no assumption is made on the functional forms of the metric coefficients $b_\mu^2(x^5)$ ($\mu = 0, 1, 2, 3$) and $f(x^5)$.

6. Infinitesimal-Algebraic Structure of Killing Symmetries in \mathfrak{R}_5

6.1 Killing Algebra

From the knowledge of the Killing vectors for the 5-d. metrics of the four fundamental interactions, we can now discuss the algebraic-infinitesimal structure of the related Killing isometries.

As is well known, the M independent Killing vector fields of a differentiable, N -dimensional manifold S_N do span a linear space \mathcal{K} . The maximum number of independent Killing vectors (*i.e.* the maximum dimension of \mathcal{K}) is $N(N+1)/2 \geq M$. They can be related to the Lie groups of isometries of the manifold as follows. By virtue of the Poincaré-Birkhoff-Witt theorem and of the Lie theorems, any infinitesimal element δg of a Lie group G_L of order M can be always represented as:

$$\delta g \stackrel{ESC}{=} 1 + \alpha_A(g)T^A + O(\{\alpha_A^2(g)\}) \quad , \quad (266)$$

where $\{T^A\}_{A=1\dots M}$ is the generator basis of the Lie algebra of G_L and $\{\alpha_A = \alpha_A(g)\}_{A=1\dots M}$ is a set of M real parameters (of course depending on $g \in G_L$). Then, the infinitesimal transformation in S_N corresponding to δg can be written as

$$\begin{aligned} x'^A(x, \alpha) &= x^A + \delta x^A(x, \alpha) + O(\alpha^2) = \\ &= x^A + \xi^A(x, \alpha) + O(\alpha^2), \quad A = 1, 2, \dots, N, \end{aligned} \quad (267)$$

where the contravariant Killing N-vector $\xi^A(x, \alpha)$ of the manifold reads (ESC on):

$$\xi^A(x, \alpha) = \xi_{\mathcal{A}}^A(x) \alpha^{\mathcal{A}}. \quad (268)$$

Quantities $\xi_{\mathcal{A}}^A(x)$ are the components of the linearly-independent Killing vectors of \mathcal{K} , and are given by

$$\xi_{\mathcal{A}}^A(x) = \left. \frac{\partial x'^A(x, \alpha)}{\partial \alpha^{\mathcal{A}}} \right|_{\alpha^{\mathcal{B}}=0, \forall \mathcal{B}=1, \dots, 15}. \quad (269)$$

The general form of an infinitesimal metric automorphism of S_N is therefore

$$x'^A(x, \alpha) \stackrel{ESC \text{ on}}{=} x^A + \xi_{\mathcal{A}}^A(x) \alpha^{\mathcal{A}} + O(\alpha^2), \quad A = 1, 2, \dots, N, \quad (270)$$

By introducing the canonical vector basis $\{\partial_A \equiv \partial/\partial x^A\}$ in S_N , one has, for the Killing N-vector $\tilde{\xi}(x)$ (ESC on on A and \mathcal{A})¹²:

$$\tilde{\xi}(x) = \xi_{\mathcal{A}}^A(x) \alpha^{\mathcal{A}} \partial_A = \tilde{\xi}_{\mathcal{A}}(x) \alpha^{\mathcal{A}}, \quad (271)$$

where (ESC on)¹³

$$\tilde{\xi}_{\mathcal{A}}(x) = \xi_{\mathcal{A}}^A(x) \partial_A. \quad (272)$$

The M vectors $\tilde{\xi}_{\mathcal{A}}(x)$ are the infinitesimal generators of the algebra of the Killing group of symmetries G_L of S_N . The product of this algebra is, as usual, the commutator

$$\left[\tilde{\xi}_{\mathcal{A}}(x), \tilde{\xi}_{\mathcal{B}}(x) \right], \quad \mathcal{A}, \mathcal{B} = 1, 2, \dots, M. \quad (273)$$

The Killing algebra is then specified by the set of commutation relations

$$\left[\tilde{\xi}_{\mathcal{A}}(x), \tilde{\xi}_{\mathcal{B}}(x) \right] = \mathcal{C}_{\mathcal{AB}}^{\mathcal{C}} \tilde{\xi}_{\mathcal{C}}(x) \quad (274)$$

where $\mathcal{C}_{\mathcal{AB}}^{\mathcal{C}} = -\mathcal{C}_{\mathcal{BA}}^{\mathcal{C}}$ are the $M(M-1)/2$ structure constants of the algebra.

In the present case of the space \mathfrak{R}_5 , it is obviously $A = 0, 1, 2, 3, 5$. As to the dimension M (≤ 15) of the Killing manifold, it depends on the explicit solution $\xi^A(x, x^5)$ of the 15 Killing equations (83)-(94), and therefore on the metric $g_{DR5, int.}(x^5)$. In the following, we shall consider all possible cases of metrics of physical relevance.

6.2 Metric with Constant Space-Time Coefficients

Let us consider the 5.d-metric (136) $g_{AB, DR5}(x^5) = \text{diag}(a, -b, -c, -d, \pm f(x^5))$, special cases of which are the electromagnetic and weak metrics above threshold ($x^5 \geq x_{0e.m., weak}^5$), Eq.(135), and the strong metric below threshold ($0 < x^5 \leq x_{0, strong}^5$), Eq.(185).

¹² In this subsection, of course, the notation $\tilde{\mathbf{v}}$ means a N-vector.

¹³ Care must be exercised in distinguishing the two different vector spaces involved in Eqs.(270)-(271). On one hand, $\xi^A(x)$ are the (contravariant) components of the N -dimensional Killing vector $\xi(x)$, belonging to the (tangent space of) the manifold S_N . On the other side, $\tilde{\xi}_{\mathcal{A}}(x)$ are the components of the M -dimensional vector belonging to the M -d. Killing space. According to Eq.(271), each $\tilde{\xi}_{\mathcal{A}}(x)$ is in turn a vector in (the tangent space of) the manifold S_N .

The solution of the related Killing system for the contravariant Killing 5-vector $\xi^A(x, \alpha)$ is given by Eqs.(146)-(151). By making the following identifications

$$\begin{aligned} \alpha^1 &= \frac{1}{a}T^0; & \alpha^2 &= \frac{1}{a}T^1; & \alpha^3 &= \frac{1}{a}T^2; & \alpha^4 &= \frac{1}{a}T^3; \\ \alpha^5 &= -B^1; & \alpha^6 &= -B^2; & \alpha^7 &= -B^3; \\ \alpha^8 &= \Theta^3; & \alpha^9 &= \Theta^2; & \alpha^{10} &= \Theta^1; \\ \alpha_{(\pm)}^{11} &= \pm T^5; & \alpha^{12} &= \Xi^0; & \alpha^{13} &= \Xi^1; & \alpha^{14} &= \Xi^2; & \alpha^{15} &= \Xi^3, \end{aligned} \quad (275)$$

$\xi^A(x, \alpha)$ takes the form

$$\left\{ \begin{aligned} \xi^0(x^1, x^2, x^3, x^5) &= \alpha^1 + \frac{1}{a} [\alpha^5 x^1 + \alpha^6 x^2 + \alpha^7 x^3 + \alpha^{12} F(x^5)]; \\ \xi^1(x^0, x^2, x^3, x^5) &= \alpha^2 + \frac{1}{b} [\alpha^5 x^0 + \alpha^8 x^2 - \alpha^9 x^3 - \alpha^{13} F(x^5)]; \\ \xi^2(x^0, x^1, x^3, x^5) &= \alpha^3 + \frac{1}{c} [\alpha^6 x^0 - \alpha^8 x^1 + \alpha^{10} x^3 - \alpha^{14} F(x^5)]; \\ \xi^3(x^0, x^1, x^2, x^5) &= \alpha^4 + \frac{1}{d} [\alpha^7 x^0 + \alpha^9 x^1 - \alpha^{10} x^2 - \alpha^{15} F(x^5)]; \\ \xi^5(x, x^5) &= (f(x^5))^{-\frac{1}{2}} (\alpha^{11} \mp \alpha^{12} x^0 \mp \alpha^{13} x^1 \mp \alpha^{14} x^2 \mp \alpha^{15} x^3). \end{aligned} \right. \quad (276)$$

Then, it follows from Eq.(268):

$$\xi_{\mathcal{A}}^A(x) = \begin{pmatrix} \xi_{\mathcal{A},SR}^{\mu}(x) \\ 0 \end{pmatrix}, \quad \mathcal{A} = 1, 2, 3, 4; \quad (277)$$

$$\xi_5^A(x^0, x^1) = \begin{pmatrix} \frac{x^1}{a} \\ \frac{x^0}{b} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \delta_0^A \frac{x^1}{a} + \delta_1^A \frac{x^0}{b}; \quad (278)$$

$$\xi_6^A(x^0, x^2) = \begin{pmatrix} \frac{x^2}{a} \\ 0 \\ \frac{x^0}{c} \\ 0 \\ 0 \end{pmatrix} = \delta_0^A \frac{x^2}{a} + \delta_2^A \frac{x^0}{c}; \quad (279)$$

$$\xi_7^A(x^0, x^3) = \begin{pmatrix} \frac{x^3}{a} \\ 0 \\ 0 \\ \frac{x^0}{d} \\ 0 \end{pmatrix} = \delta_0^A \frac{x^3}{a} + \delta_3^A \frac{x^0}{d}; \quad (280)$$

$$\xi_8^A(x^1, x^2) = \begin{pmatrix} 0 \\ \frac{x^2}{b} \\ -\frac{x^1}{c} \\ 0 \\ 0 \end{pmatrix} = \delta_1^A \frac{x^2}{b} - \delta_2^A \frac{x^1}{c}; \quad (281)$$

$$\xi_9^A(x^1, x^3) = \begin{pmatrix} 0 \\ -\frac{x^3}{b} \\ 0 \\ \frac{x^1}{d} \\ 0 \end{pmatrix} = -\delta_1^A \frac{x^3}{b} + \delta_3^A \frac{x^1}{d}; \quad (282)$$

$$\xi_{10}^A(x^2, x^3) = \begin{pmatrix} 0 \\ \frac{x^3}{c} \\ \frac{x^2}{d} \\ 0 \end{pmatrix} = \delta_2^A \frac{x^3}{c} - \delta_3^A \frac{x^2}{d}; \quad (283)$$

$$\xi_{11}^A(x^5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mp (f(x^5))^{-\frac{1}{2}} \end{pmatrix} = \mp \delta_5^A (f(x^5))^{-\frac{1}{2}}; \quad (284)$$

$$\xi_{12}^A(x^0, x^5) = \begin{pmatrix} \frac{F(x^5)}{a} \\ 0 \\ 0 \\ 0 \\ \mp (f(x^5))^{-\frac{1}{2}} x^0 \end{pmatrix} = \delta_0^A \frac{F(x^5)}{a} \mp \delta_5^A (f(x^5))^{-\frac{1}{2}} x^0; \quad (285)$$

$$\xi_{13}^A(x^1, x^5) = \begin{pmatrix} 0 \\ \frac{F(x^5)}{b} \\ 0 \\ 0 \\ \mp (f(x^5))^{-\frac{1}{2}} x^1 \end{pmatrix} = \delta_1^A \frac{F(x^5)}{b} \mp \delta_5^A (f(x^5))^{-\frac{1}{2}} x^1; \quad (286)$$

$$\xi_{14}^A(x^2, x^5) = \begin{pmatrix} 0 \\ 0 \\ -\frac{F(x^5)}{c} \\ 0 \\ \mp (f(x^5))^{-\frac{1}{2}} x^2 \end{pmatrix} = -\delta_2^A \frac{F(x^5)}{c} \mp \delta_5^A (f(x^5))^{-\frac{1}{2}} x^2; \quad (287)$$

$$\xi_{15}^A(x^3, x^5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{F(x^5)}{d} \\ \mp (f(x^5))^{-\frac{1}{2}} x^3 \end{pmatrix} = -\delta_3^A \frac{F(x^5)}{d} \mp \delta_5^A (f(x^5))^{-\frac{1}{2}} x^3. \quad (288)$$

It is easy to see that the 4 Killing 5-vectors $\xi_{\mathcal{A}}^A(x)$, $\mathcal{A} = 1, 2, 3, 4$ (Eq.(276)) are the

generators Υ^μ ($\mu = 0, 1, 2, 3$) of the standard translation group $Tr.(1, 3)$ ¹⁴, whereas the 6 Killing 5-vectors $\xi_{\mathcal{A}}^A(x)$, $\mathcal{A} = 5, \dots, 10$ (Eqs.(277)-(282)) are the generators of the deformed Lorentz group $SO(1, 3)_{DEF}$, namely the 3-vectors \mathbf{S}_{DSR} and \mathbf{K}_{DSR} (associated to deformed space rotations and boosts, respectively: see ref.[40]) in the self-representation basis.

The other five Killing 5-vectors $\xi_{\mathcal{A}}^A(x)$, $\mathcal{A} = 11, \dots, 15$, can be identified with the *new* generators of the Killing algebra as follows¹⁵. By its expression (283), the Killing vector $\xi_{11}^A(x^5)$ is of course the translation generator along x^5 :

$$\xi_{11}^A(x^5) = \Upsilon^{5A}(x^5). \quad (289)$$

As to the other $\xi_{\mathcal{A}}^A(x)$ for $\mathcal{A} = 12, \dots, 15$, it is easily seen from Eqs.(284)-(287) that their interpretation as rotation or boost generators depends on the signature of the fifth coordinate x^5 , namely on its timelike or spacelike nature:

$$\xi_{12}^A(x^0, x^5) = \begin{cases} \text{'' + ''} : \Sigma^{1A}(x^0, x^5) \\ \text{'' - ''} : \Gamma^{1A}(x^0, x^5) \end{cases}; \quad (290)$$

$$\xi_{13}^A(x^1, x^5) = \begin{cases} \text{'' + ''} : \Gamma^{1A}(x^1, x^5) \\ \text{'' - ''} : \Sigma^{1A}(x^0, x^5) \end{cases}; \quad (291)$$

$$\xi_{14}^A(x^2, x^5) = \begin{cases} \text{'' + ''} : \Gamma^{2A}(x^2, x^5) \\ \text{'' - ''} : \Sigma^{2A}(x^2, x^5) \end{cases}; \quad (292)$$

$$\xi_{15}^A(x^3, x^5) = \begin{cases} \text{'' + ''} : \Gamma^{3A}(x^3, x^5) \\ \text{'' - ''} : \Sigma^{3A}(x^3, x^5) \end{cases}, \quad (293)$$

where Σ^{iA} and Γ^{iA} ($i = 1, 2, 3$) denote the new generators (with respect to the SR ones) corresponding, respectively, *to rotations and boosts involving the fifth coordinate*.

By direct evaluation of the commutators (274)¹⁶, it follows that the ensuing Killing al-

¹⁴This is due to the adopted choice of embodying the constants a, b, c, d in the definitions of the translational parameters α^μ (see Eqs.(274)). It is easily seen that, on the contrary, the identification $\alpha^\mu = T^\mu$ leads to the generators of the deformed translation group $Tr.(1, 3)_{DEF}$. [38] (with the consequent changes in the commutator algebra).

¹⁵Needless to say, the identifications of the Killing vectors $\xi_{\mathcal{A}}^A(x)$ with the generators of the Killing symmetry algebra hold except for a sign here and in the subsequent cases.

¹⁶Remember that the products in the commutator (22.132) has to be meant as row×column products of matrices, so that

$$\begin{aligned} [\xi_{\mathcal{A}}(x), \xi_{\mathcal{B}}(x)] &= [\xi_{\mathcal{A}}^B(x)\partial_B, \xi_{\mathcal{B}}^A(x)\partial_A] = \\ &= [\xi_{\mathcal{A}}^B(x)\xi_{\mathcal{B},B}^A(x) - \xi_{\mathcal{B}}^B(x)\xi_{\mathcal{A},B}^A(x)] \partial_A \rightarrow \\ \rightarrow [\xi_{\mathcal{A}}(x), \xi_{\mathcal{B}}(x)]^A &= \xi_{\mathcal{A}}^B(x)\xi_{\mathcal{B},B}^A(x) - \xi_{\mathcal{B}}^B(x)\xi_{\mathcal{A},B}^A(x). \end{aligned}$$

gebra is made of two pieces: the deformed Poincaré algebra $P(1, 3)_{DEF.} = \{su(2)_{DEF.} \otimes su(2)_{DEF.}\} \otimes_s tr.(1, 3)_{STD.}$, generated by $\mathbf{S}_{DSR.}$, \mathbf{K}_{DSR} and Υ^μ ($\mu = 0, 1, 2, 3$) (corresponding to $\mathcal{A} = 1, \dots, 10$), expressed by Eqs.(8.37)-(8.39), and the "mixed" algebra, involving also generators related to the energy dimension. This latter depends on the timelike or spacelike nature of x^5 , and is specified by the following commutation relations ($\forall \mu = 0, 1, 2, 3$ and $\forall i, j, k=1, 2, 3$)¹⁷:

1) Timelike x^5 :

$$\left. \begin{aligned}
 & \left[\tilde{\Upsilon}^5, \tilde{\Upsilon}^\mu \right] = 0 ; \\
 & \left[\tilde{\Upsilon}^0, \tilde{\Sigma}^1 \right] = -\tilde{\Upsilon}^5; \quad \left[\tilde{\Upsilon}^0, \tilde{\Gamma}^i \right] = 0; \\
 & \left[\tilde{\Upsilon}^i, \tilde{\Gamma}^j \right] = -\tilde{\Upsilon}^5 \delta^{ij}; \quad \left[\tilde{\Upsilon}^i, \tilde{\Sigma}^1 \right] = 0; \\
 & \left[\tilde{\mathbf{K}}^i, \tilde{\Gamma}^j \right] = \frac{1}{g_{ii,DR5}} \tilde{\Sigma}^1 \delta^{ij}; \quad \left[\tilde{\mathbf{K}}^i, \tilde{\Sigma}^1 \right] = \frac{1}{a} \tilde{\Gamma}^i = \frac{1}{g_{00,DR5}} \tilde{\Gamma}^i; \\
 & \left[\tilde{\mathbf{K}}^i, \tilde{\Upsilon}^5 \right] = 0; \quad \left[\tilde{\mathbf{S}}^i, \tilde{\Upsilon}^5 \right] = 0; \\
 & \left[\tilde{\mathbf{S}}^i, \tilde{\Sigma}^1 \right] = 0; \quad \left[\tilde{\mathbf{S}}^i, \tilde{\Gamma}^j \right] \stackrel{ESC \text{ on } k}{=} \epsilon_{ijk} \frac{1}{g_{jj,DR5}} \tilde{\Gamma}^k; \\
 & \left[\tilde{\Upsilon}^5, \tilde{\Sigma}^1 \right] = \frac{1}{a} \tilde{\Upsilon}^0 = \frac{1}{g_{00,DR5}} \tilde{\Upsilon}^0; \quad \left[\tilde{\Upsilon}^5, \tilde{\Gamma}^i \right] = -\tilde{\Upsilon}^i; \\
 & \left[\tilde{\Sigma}^1, \tilde{\Gamma}^i \right] = \tilde{\mathbf{K}}^i; \quad \left[\tilde{\Gamma}^i, \tilde{\Gamma}^j \right] \stackrel{ESC \text{ on } k}{=} -\epsilon_{ijk} \tilde{\mathbf{S}}^k.
 \end{aligned} \right\} \quad \text{" + " } \quad (294)$$

It contains the following subalgebras

$$\left. \begin{aligned}
 & \left[\tilde{\mathbf{S}}^i, \tilde{\mathbf{S}}^j \right] \stackrel{ESC \text{ on } k}{=} -\epsilon_{ijk} \frac{1}{g_{kk,DR5}} \tilde{\mathbf{S}}^k; \\
 & \left[\tilde{\mathbf{S}}^i, \tilde{\Gamma}^j \right] \stackrel{ESC \text{ on } k}{=} \epsilon_{ijk} \frac{1}{g_{jj,DR5}} \tilde{\Gamma}^k; \\
 & \left[\tilde{\Gamma}^i, \tilde{\Gamma}^j \right] \stackrel{ESC \text{ on } k}{=} -\epsilon_{ijk} \tilde{\mathbf{S}}^k,
 \end{aligned} \right\} su(2)_{DEF.} \otimes su(2)_{DEF.}; \quad (295)$$

¹⁷ In the following, for simplicity, we shall omit the DSR specification in the generator symbols.

$$\left. \begin{aligned} [\tilde{\Upsilon}^\nu, \tilde{\Upsilon}^\mu] &= 0 ; \\ [\tilde{\Upsilon}^5, \tilde{\Upsilon}^\mu] &= 0 , \end{aligned} \right\} tr.(2, 3)_{STD}. \quad (296)$$

2) Spacelike x^5 :

$$\left. \begin{aligned} & [\tilde{\Upsilon}^5, \tilde{\Upsilon}^\mu] = 0 ; \\ & [\tilde{\Upsilon}^0, \tilde{\Gamma}^1] = \tilde{\Upsilon}^5; \quad [\tilde{\Upsilon}^0, \tilde{\Sigma}^i] = 0; \\ & [\tilde{\Upsilon}^i, \tilde{\Sigma}^j] = \tilde{\Upsilon}^5 \delta^{ij}; \quad [\tilde{\Upsilon}^i, \tilde{\Gamma}^1] = 0; \\ & [\tilde{\mathbf{K}}^i, \tilde{\Sigma}^j] = -\frac{1}{g_{ii, DR5}} \tilde{\Gamma}^1 \delta^{ij}; \quad [\tilde{\mathbf{K}}^i, \tilde{\Gamma}^1] = \frac{1}{a} \tilde{\Sigma}^i = \frac{1}{g_{00, DR5}} \tilde{\Sigma}^i; \\ & [\tilde{\mathbf{K}}^i, \tilde{\Upsilon}^5] = 0; \quad [\tilde{\mathbf{S}}^i, \tilde{\Upsilon}^5] = 0; \\ & [\tilde{\mathbf{S}}^i, \tilde{\Gamma}^1] = 0; \quad [\tilde{\mathbf{S}}^i, \tilde{\Sigma}^j] \stackrel{ESC \text{ on } k}{=} -\epsilon_{ijk} \frac{1}{g_{jj, DR5}} \tilde{\Sigma}^k; \\ & [\tilde{\Upsilon}^5, \tilde{\Gamma}^1] = \frac{1}{g_{00, DR5}} \tilde{\Upsilon}^0; \quad [\tilde{\Upsilon}^5, \tilde{\Sigma}^i] = -\frac{1}{g_{ii, DR5}} \tilde{\Upsilon}^i; \\ & [\tilde{\Gamma}^1, \tilde{\Sigma}^i] = -\tilde{\mathbf{K}}^i; \quad [\tilde{\Sigma}^i, \tilde{\Sigma}^j] \stackrel{ESC \text{ on } k}{=} -\epsilon_{ijk} \tilde{\mathbf{S}}^k, \end{aligned} \right\} \quad (297)$$

with the subalgebras

$$\left. \begin{aligned} & [\tilde{\mathbf{S}}^i, \tilde{\mathbf{S}}^j] \stackrel{ESC \text{ on } k}{=} -\epsilon_{ijk} \frac{1}{g_{kk, DR5}} \tilde{\mathbf{S}}^k; \\ & [\tilde{\mathbf{S}}^i, \tilde{\Sigma}^j] \stackrel{ESC \text{ on } k}{=} -\epsilon_{ijk} \frac{1}{g_{jj, DR5}} \tilde{\Sigma}^k; \\ & [\tilde{\Sigma}^i, \tilde{\Sigma}^j] \stackrel{ESC \text{ on } k}{=} -\epsilon_{ijk} \tilde{\mathbf{S}}^k, \end{aligned} \right\} su(2)_{DEF.} \otimes su(2)_{DEF.}; \quad (298)$$

$$\left. \begin{aligned} [\tilde{\Upsilon}^\nu, \tilde{\Upsilon}^\mu] &= 0 ; \\ [\tilde{\Upsilon}^5, \tilde{\Upsilon}^\mu] &= 0 , \end{aligned} \right\} tr.(1, 4)_{STD}. \quad (299)$$

Since in this case there are 15 independent Killing vectors, the corresponding Riemann spaces, for the e.m. and weak interactions above threshold and for the strong one

below threshold, are maximally symmetric and have therefore constant curvature (zero for the e.m. and weak interactions, as it is straightforward to check directly by means of Eq.(135)).

6.3 Strong Metric for Violated Υ -Hypothesis

Let us consider the case of the strong metric above threshold ($x^5 \geq x_{0,strong}^5$) when the hypothesis Υ of functional independence is violated $\forall \mu = 0, 1, 2, 3$ (see Subsubsect.5.2.3). The metric is given by Eq.(226), and for $c = 0$ takes the form

$$g_{AB,DR5}(x^5) = \text{diag} \left(\left(\frac{x^5}{x_0^5} \right)^2, -\frac{2}{25}, -\frac{4}{25}, -\left(\frac{x^5}{x_0^5} \right)^2, \pm \frac{1}{\gamma \left(\frac{x^5}{x_0^5} \right)^2} \right). \quad (300)$$

The contravariant Killing vector $\xi^A(x, x^5)$ is given by Eqs.(228)-(232), which read now

$$\left\{ \begin{array}{l} \xi^0 = 0; \\ \xi^1(x^2) = -\frac{25}{2}d_2x^2 - \frac{25}{2}T_1 = \alpha^3x^2 + \alpha^1; \\ \xi^2(x^1) = \frac{25}{4}d_2x^1 - \frac{25}{4}T_2 = \frac{1}{2}\alpha^3x^1 + \alpha^2; \\ \xi^3 = 0; \\ \xi^5(x^5) = \pm \frac{\gamma\alpha}{(x_0^5)^2}x^5 = \pm\alpha^4x^5, \end{array} \right. \quad (301)$$

where the dependence on the transformation parameters α^A ($A=1,2,3,4$) has been made explicit.

The contravariant Killing vectors $\xi_{\mathcal{A}}^A(x)$ are therefore

$$\xi_{\mathcal{A}}^A(x) = \begin{pmatrix} \phantom{\xi_{\mathcal{A}}^A(x)} \\ \xi_{2,SR}^\mu(x) \\ \phantom{\xi_{\mathcal{A}}^A(x)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad (302)$$

$$\xi_2^A(x) = \begin{pmatrix} \xi_{3,SR}^\mu(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad (303)$$

$$\xi_3^A(x^1, x^2) = \begin{pmatrix} 0 \\ x^2 \\ -\frac{x^1}{2} \\ 0 \\ 0 \end{pmatrix}; \quad (304)$$

$$\xi_{4,\pm}^A(x^5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \pm x^5 \end{pmatrix}. \quad (305)$$

It is possible to do the following identifications:

$$\xi_1^A = \begin{pmatrix} \Upsilon_{SR}^{1\mu} \\ 0 \end{pmatrix} = \Upsilon^{1A}, \quad \alpha^1 = \mp T^{1\nu}; \quad (306)$$

$$\xi_2^A = \begin{pmatrix} \Upsilon_{SR}^{2\mu} \\ 0 \end{pmatrix} = \Upsilon^{2A}, \quad \alpha^2 = \mp T^{2\nu}; \quad (307)$$

$$\xi_3^A(x^1, x^2) = \begin{pmatrix} S^{3\mu}(x)|_{x^2 \rightarrow \sqrt{2}x^2} \\ 0 \end{pmatrix} = S^{3A}, \quad \alpha^3 = \mp \Theta^3; \quad (308)$$

$$\xi_{4,\pm}^A(x^5) = R(x^5). \quad (309)$$

In Eq.(307), the notation for S^3 has to be interpreted in the sense of Subsubsect.5.2.3, namely it is the generator of rotations in the 2-d. manifold $\Pi_2 = (x^1, x^2)$ with metric rescaling $x^2 \rightarrow \sqrt{2}x^2$. One therefore gets the following Killing algebra

$$\left\{ \begin{array}{l} [\tilde{\Upsilon}^1, \tilde{\Upsilon}^2] = 0; \\ [\tilde{\Upsilon}^1, \tilde{\mathbf{S}}^3] = -\frac{1}{2}\tilde{\Upsilon}^2; \\ [\tilde{\Upsilon}^2, \tilde{\mathbf{S}}^3] = \tilde{\Upsilon}^1; \\ [\tilde{\Upsilon}^i, \tilde{\mathbf{R}}] = 0, \quad i = 1, 2; \\ [\tilde{\mathbf{S}}^3, \tilde{\mathbf{R}}] = 0. \end{array} \right. \quad (310)$$

6.4 Power-Ansatz Metrics with Violated Υ -Hypothesis

We shall now discuss the infinitesimal algebraic structure of DR5 for the metrics in the Power Ansatz when the hypothesis Υ of functional independence is not satisfied by any $\mu = 0, 1, 2, 3$. According to Appendix B, this occurs in five cases (only three of which are independent). We will consider only the two which correspond to physical metrics.

6.4.1 Case 1

This corresponds to the VI class of solutions, characterized by the exponent set $\tilde{\mathbf{q}}_{VI} = (p, 0, 0, 0, p-2)$. The 5-d. metric is given by Eq.(B.75). The contravariant Killing 5-vector $\xi^A(x, x^5)$ depends on the timelike or spacelike nature of the fifth coordinate, and

writes (cfr. Eqs.(B.79)-(B.83))

$$\begin{aligned}
 & \xi^0(x^0, x^5; p) = \\
 & \left. \begin{aligned}
 & \quad \text{'' + '' :} \\
 & (x_0^5)^{p-1} \left[A \cos \left(\frac{p x^0}{2 x_0^5} \right) - B \sin \left(\frac{p x^0}{2 x_0^5} \right) \right] (x^5)^{-\frac{p}{2}} + \alpha (x_0^5)^p; \\
 & \quad \text{'' - '' :} \\
 & (x_0^5)^{p-1} \left[C \cosh \left(\frac{p x^0}{2 x_0^5} \right) + D \sinh \left(\frac{p x^0}{2 x_0^5} \right) \right] (x^5)^{-\frac{p}{2}} + \alpha (x_0^5)^p; \\
 & \xi^1(x^2, x^3) = -\Theta^3 x^2 - \Theta^2 x^3 + T^1; \\
 & \xi^2(x^1, x^3) = \Theta^3 x^1 - \Theta^1 x^3 + T^2; \\
 & \xi^3(x^1, x^2) = \Theta^2 x^1 + \Theta^1 x^2 + T^3; \\
 & \xi^5(x^0, x^5, p) = \\
 & \left. \begin{aligned}
 & \quad \text{'' + '' :} \\
 & (x_0^5)^{p-2} \left[A \sin \left(\frac{p x^0}{2 x_0^5} \right) + B \cos \left(\frac{p x^0}{2 x_0^5} \right) \right] (x^5)^{-\left(\frac{p}{2}-1\right)}; \\
 & \quad \text{'' - '' :} \\
 & - (x_0^5)^{p-2} \left[C \sinh \left(\frac{p x^0}{2 x_0^5} \right) + D \cosh \left(\frac{p x^0}{2 x_0^5} \right) \right] (x^5)^{-\left(\frac{p}{2}-1\right)}.
 \end{aligned} \right.
 \end{aligned}
 \tag{311}$$

By the identifications

$$\begin{aligned}
 & -\Theta^3 = \alpha^5; \quad \Theta^2 = \alpha^6; \quad \Theta^1 = -\alpha^7; \\
 & \alpha (x_0^5)^p = T^0(\alpha, x_0^5, p) = \alpha^1(\alpha, x_0^5, p); T^i = \alpha^{i+1}, i = 1, 2, 3; \\
 & \text{'' + ''} \begin{cases} A (x_0^5)^{\frac{p}{2}-2} = \alpha^8 \\ B (x_0^5)^{\frac{p}{2}-2} = \alpha^9 \end{cases}; \quad \text{'' - ''} \begin{cases} C (x_0^5)^{\frac{p}{2}-2} = \alpha^8 \\ D (x_0^5)^{\frac{p}{2}-2} = \alpha^9 \end{cases},
 \end{aligned}
 \tag{312}$$

the Killing vector can be written as¹⁸

$$\begin{aligned}
 & \xi^0(x^0, x^5; p) = \\
 = & \alpha^1 + \begin{cases} \text{'' + ''} : (x_0^5)^{\frac{p}{2}+1} \left[\alpha^8 \cos\left(\frac{p x^0}{2 x_0^5}\right) - \alpha^9 \sin\left(\frac{p x^0}{2 x_0^5}\right) \right] (x^5)^{-\frac{p}{2}}; \\ \text{'' - ''} : (x_0^5)^{\frac{p}{2}+1} \left[\alpha^8 \cosh\left(\frac{p x^0}{2 x_0^5}\right) + \alpha^9 \sinh\left(\frac{p x^0}{2 x_0^5}\right) \right] (x^5)^{-\frac{p}{2}}; \end{cases} \\
 & \xi^1(x^2, x^3) = \alpha^2 + \alpha^5 x^2 - \alpha^6 x^3; \\
 & \xi^2(x^1, x^3) = \alpha^3 - \alpha^5 x^1 + \alpha^7 x^3; \\
 & \xi^3(x^1, x^2) = \alpha^4 + \alpha^6 x^1 - \alpha^7 x^2; \\
 & \xi^5(x^0, x^5, p) = \\
 = & \begin{cases} \text{'' + ''} : (x_0^5)^{\frac{p}{2}} \left[\alpha^8 \sin\left(\frac{p x^0}{2 x_0^5}\right) + \alpha^9 \cos\left(\frac{p x^0}{2 x_0^5}\right) \right] (x^5)^{-\left(\frac{p}{2}-1\right)}; \\ \text{'' - ''} : -(x_0^5)^{\frac{p}{2}} \left[\alpha^8 \sinh\left(\frac{p x^0}{2 x_0^5}\right) + \alpha^9 \cosh\left(\frac{p x^0}{2 x_0^5}\right) \right] (x^5)^{-\left(\frac{p}{2}-1\right)}. \end{cases}
 \end{aligned} \tag{313}$$

One gets therefore, for the Killing vectors $\xi_{\mathcal{A}}^A(x)$ ($\mathcal{A} = 1, \dots, 9$):

$$\xi_{\mathcal{A}}^A(x) = \begin{pmatrix} \\ \xi_{\mathcal{A},SR}^\mu(x) \\ 0 \end{pmatrix}, \mathcal{A} = 1, 2, 3, 4; \tag{314}$$

$$\xi_5^A(x^1, x^2) = \begin{pmatrix} 0 \\ x^2 \\ -x^1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \\ \xi_{8,SR}^\mu(x) \\ 0 \end{pmatrix}; \tag{315}$$

¹⁸ The definitions of $\alpha^{\mathcal{A}}$ for $\mathcal{A} = 8, 9$ have been chosen so to make also these transformation parameters dimensionless.

$$\xi_6^A(x^1, x^3) = \begin{pmatrix} 0 \\ -x^3 \\ 0 \\ x^1 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi_{9,SR}^\mu(x) \\ 0 \end{pmatrix}; \quad (316)$$

$$\xi_7^A(x^2, x^3) = \begin{pmatrix} 0 \\ 0 \\ x^3 \\ -x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi_{10,SR}^\mu(x) \\ 0 \end{pmatrix}; \quad (317)$$

$$\xi_{8,\pm}^A(x^0, x^5, p) = \begin{pmatrix} (x_0^5)^{\frac{p}{2}+1} \cosh\left(\frac{p x^0}{2 x_0^5}\right) (x^5)^{-\frac{p}{2}} \\ 0 \\ 0 \\ 0 \\ \pm (x_0^5)^{\frac{p}{2}} \sinh\left(\frac{p x^0}{2 x_0^5}\right) (x^5)^{-\left(\frac{p}{2}-1\right)} \end{pmatrix}; \quad (318)$$

$$\xi_{9,\pm}^A(x^0, x^5, p) = \begin{pmatrix} \mp (x_0^5)^{\frac{p}{2}+1} \sinh\left(\frac{p x^0}{2 x_0^5}\right) (x^5)^{-\frac{p}{2}} \\ 0 \\ 0 \\ 0 \\ \pm (x_0^5)^{\frac{p}{2}} \cosh\left(\frac{p x^0}{2 x_0^5}\right) (x^5)^{-\left(\frac{p}{2}-1\right)} \end{pmatrix}. \quad (319)$$

The above Killing vectors can be identified with generators as follows:

$$\xi_1^A = \begin{pmatrix} \Upsilon_{SR}^{0\mu} \\ 0 \end{pmatrix} = \Upsilon^{1A}; \quad (320)$$

$$\xi_{i+1}^A = \begin{pmatrix} \Upsilon_{SR}^{i\mu} \\ 0 \end{pmatrix} = -\Upsilon^{iA}, i = 1, 2, 3; \quad (321)$$

$$\xi_{i+4}^A(x) = \begin{pmatrix} \xi_{i+7,SR}^\mu(x) \\ 0 \end{pmatrix} = \begin{pmatrix} S_{SR}^{3\mu}(x) \\ S_{SR}^{2\mu}(x) \\ S_{SR}^{1\mu}(x) \\ 0 \end{pmatrix} = S^{iA}, i = 1, 2, 3; \quad (322)$$

$$\xi_{8,\pm}^A(x^0, x^5; p) = Z_{\pm}^{1A}(x^0, x^5; p); \quad (323)$$

$$\xi_{9,\pm}^A(x^0, x^5; p) = Z_{\pm}^{2A}(x^0, x^5; p). \quad (324)$$

The Killing algebra is then specified by the following commutation relations:

$$\left\{ \begin{array}{l} [\tilde{\Upsilon}^\mu, \tilde{\Upsilon}^\nu] = 0 \quad \forall \mu, \nu = 0, 1, 2, 3; \\ [\tilde{\mathbf{S}}^i(x), \tilde{\mathbf{S}}^j(x)] \stackrel{ESC}{=} \epsilon_{ijk} \tilde{\mathbf{S}}^k \quad \forall i, j = 1, 2, 3; \\ [\tilde{\mathbf{S}}^i(x), \tilde{\Upsilon}^0] = 0 \quad \forall i, j = 1, 2, 3; \\ [\tilde{\mathbf{S}}^i(x), \tilde{\Upsilon}^j] \stackrel{ESC}{=} \epsilon_{ijk} \tilde{\Upsilon}^k \quad \forall i, j = 1, 2, 3; \\ [\tilde{\Upsilon}^0, \tilde{\mathbf{Z}}_{\pm}^1(x^0, x^5, p)] = \frac{p}{2x_0^5} \tilde{\mathbf{Z}}_{\pm}^2(x^0, x^5, p); \\ [\tilde{\Upsilon}^0, \tilde{\mathbf{Z}}_{\pm}^2(x^0, x^5, p)] = \mp \frac{p}{2x_0^5} \tilde{\mathbf{Z}}_{\pm}^1(x^0, x^5, p); \\ [\tilde{\Upsilon}^i, \tilde{\mathbf{Z}}_{\pm}^1(x^0, x^5, p)] = [\tilde{\Upsilon}^i, \tilde{\mathbf{Z}}_{\pm}^2(x^0, x^5, p)] = 0; \\ [\tilde{\mathbf{S}}^i(x), \tilde{\mathbf{Z}}_{\pm}^m(x^0, x^5, p)] = 0 \quad \forall i = 1, 2, 3, \quad \forall m = 1, 2. \end{array} \right. \quad (325)$$

It is easily seen that the Killing algebra for this case contains the subalgebra $su(2)_{STD} \otimes_s tr.(1, 3)_{STD}$.

6.4.2 Cases 2-4

The metrics belonging to classes II ($\tilde{\mathbf{q}}_{II} = (0, p, 0, 0, p - 2)$), IV ($\tilde{\mathbf{q}}_{IV} = (0, 0, 0, p, p - 2)$) and IX ($\tilde{\mathbf{q}}_{IX} = (0, 0, p, 0, p - 2)$) differ only for an exchange of spatial axes (see Eq.(A.86) for the first case). They are discussed in Subsects.B.2.2-B.2.4. The Killing algebra for all these three (physically equivalent) metrics can be dealt with by an unitary mathematical approach.

Let us label the three classes II, IV and IX by $i = 1, 2, 3$ respectively (according to the space axis involved). The metric coefficients are given, in compact form, by

$$g_{AB,DR5,i} \begin{cases} g_{00} = -g_{jj} = 1, & j \neq i, \quad j \in \{j_1, j_2\}, j_1 < j_2; \\ g_{ii}(x^5) = -\left(\frac{x^5}{x_0^5}\right)^p; & g_{55}(x^5) = \pm \left(\frac{x^5}{x_0^5}\right)^{p-2}. \end{cases} \quad (326)$$

As easily seen, the three indices i, j_1, j_2 take the values $\{123; 213; 312\}$. The contravariant Killing vector has the form (cfr. Eqs.(B.88)-(B.92))

$$\begin{aligned} \xi^0(x^{j_1}, x^{j_2}; \alpha) &= \alpha^1 + \alpha^5 x^{j_1} + \alpha^6 x^{j_2}; \\ \xi_{\pm}^i(x^i, x^5; \alpha) &= \alpha^{i+1}(\alpha, x_0^5, p) + \\ + \begin{cases} \text{" + " : } (x_0^5)^{\frac{p}{2}+1} \left[\alpha^8 \cosh\left(\frac{p x^i}{2 x_0^5}\right) + \alpha^9 \sinh\left(\frac{p x^i}{2 x_0^5}\right) \right] (x^5)^{-\frac{p}{2}}; \\ \text{" - " : } (x_0^5)^{\frac{p}{2}+1} \left[\alpha^8 \cos\left(\frac{p x^i}{2 x_0^5}\right) - \alpha^9 \sin\left(\frac{p x^i}{2 x_0^5}\right) \right] (x^5)^{-\frac{p}{2}}; \end{cases} \end{aligned} \quad (327)$$

$$\begin{aligned} \xi^{j_1}(x^0, x^{j_2}; \alpha) &= \alpha^{j_1+1} + \alpha^5 x^0 + \alpha^7 x^{j_2}; \\ \xi^{j_2}(x^0, x^{j_1}; \alpha) &= \alpha^{j_2+1} + \alpha^6 x^0 - \alpha^7 x^{j_1}; \\ \xi_{\pm}^5(x^i, x^5; \alpha) &= \\ = \begin{cases} \text{" + " : } (x_0^5)^{\frac{p}{2}} \left[\alpha^8 \sinh\left(\frac{p x^i}{2 x_0^5}\right) + \alpha^9 \cosh\left(\frac{p x^i}{2 x_0^5}\right) \right] (x^5)^{-\left(\frac{p}{2}-1\right)}; \\ \text{" - " : } - (x_0^5)^{\frac{p}{2}} \left[\alpha^8 \sin\left(\frac{p x^i}{2 x_0^5}\right) + \alpha^9 \cos\left(\frac{p x^i}{2 x_0^5}\right) \right] (x^5)^{-\left(\frac{p}{2}-1\right)}. \end{cases} \end{aligned} \quad (328)$$

$$(329)$$

where

$$\begin{aligned} \text{" + " } \begin{cases} A(x_0^5)^{\frac{p}{2}-2} = \alpha^8 \\ B(x_0^5)^{\frac{p}{2}-2} = \alpha^9 \end{cases}; \quad \text{" - " } \begin{cases} C(x_0^5)^{\frac{p}{2}-2} = \alpha^8 \\ D(x_0^5)^{\frac{p}{2}-2} = \alpha^9 \end{cases}; \\ \alpha^{i+1}(\alpha, x_0^5, p) = -\alpha(x_0^5)^p; \\ \alpha^7 = \begin{cases} \mp\Theta, & i = 1; \\ \pm\Theta, & i = 2; \\ \mp\Theta, & i = 3. \end{cases} \end{aligned} \quad (330)$$

The parameter α^7 corresponds to a true infinitesimal rotation in the 2-d. Euclidean plane $\Pi_{(x^{j_1}, x^{j_2})}$.

Therefore the explicit form of the Killing vectors $\xi_{\mathcal{A}}^A(x)$ ($\mathcal{A} = 1, \dots, 9$) is

$$\xi_{\mathcal{A}}^A(x) = \begin{pmatrix} \xi_{\mathcal{A},SR}^\mu(x) \\ 0 \end{pmatrix}, \mathcal{A} = 1, i+1, j_1+1, j_2+1; \quad (331)$$

$$\xi_5^A(x^0, x^{j_1}) = \begin{pmatrix} x^{j_1} \\ x^0 \\ \text{(j}_1\text{-th row)} \\ 0 \end{pmatrix} = \begin{cases} i=1: \begin{pmatrix} \xi_{6,SR}^\mu(x^0, x^2) \\ 0 \end{pmatrix} \\ i=2: \begin{pmatrix} \xi_{5,SR}^\mu(x^0, x^1) \\ 0 \end{pmatrix} \end{cases}; \quad (332)$$

$$\xi_6^A(x^0, x^{j_2}) = \begin{pmatrix} x^{j_2} \\ x^0 \\ \text{(j}_2\text{-th row)} \\ 0 \end{pmatrix} = \begin{cases} i=1: \begin{pmatrix} \xi_{7,SR}^\mu(x^0, x^3) \\ 0 \end{pmatrix} \\ i=2: \begin{pmatrix} \xi_{7,SR}^\mu(x^0, x^3) \\ 0 \end{pmatrix} \\ i=3: \begin{pmatrix} \xi_{6,SR}^\mu(x^0, x^2) \\ 0 \end{pmatrix} \end{cases}; \quad (333)$$

$$\xi_7^A(x^{j_1}, x^{j_2}) = \begin{pmatrix} 0 \\ x^{j_2} \\ \text{(j}_2\text{-th row)} \\ 0 \\ x^{j_1} \\ \text{(j}_1\text{-th row)} \\ 0 \end{pmatrix} = \begin{cases} i=1: \begin{pmatrix} \xi_{10,SR}^\mu(x^2, x^3) \\ 0 \end{pmatrix} \\ i=2: \begin{pmatrix} -\xi_{9,SR}^\mu(x^1, x^3) \\ 0 \end{pmatrix} \\ i=3: \begin{pmatrix} -\xi_{8,SR}^\mu(x^1, x^2) \\ 0 \end{pmatrix} \end{cases}; \quad (334)$$

$$\xi_{8,\pm}^A(x^i, x^5; p) = \begin{pmatrix} 0 \\ \left\{ \begin{array}{l} \text{'' + ''} : (x_0^5)^{\frac{p}{2}+1} \cosh\left(\frac{p x^i}{2 x_0^5}\right) (x^5)^{-\frac{p}{2}} \\ \text{'' - ''} : (x_0^5)^{\frac{p}{2}+1} \cos\left(\frac{p x^i}{2 x_0^5}\right) (x^5)^{-\frac{p}{2}} \end{array} \right. \\ \text{(i-th row)} \\ 0 \\ 0 \\ \left\{ \begin{array}{l} (x_0^5)^{\frac{p}{2}} \sinh\left(\frac{p x^i}{2 x_0^5}\right) (x^5)^{-\left(\frac{p}{2}-1\right)} \\ (x_0^5)^{\frac{p}{2}} \sin\left(\frac{p x^i}{2 x_0^5}\right) (x^5)^{-\left(\frac{p}{2}-1\right)} \end{array} \right. \end{pmatrix}; \quad (335)$$

$$\xi_{9,\pm}^A(x^i, x^5; p) = \begin{pmatrix} 0 \\ \left\{ \begin{array}{l} \text{'' + ''} : (x_0^5)^{\frac{p}{2}+1} \sinh\left(\frac{p x^i}{2 x_0^5}\right) (x^5)^{-\frac{p}{2}} \\ \text{'' - ''} : -(x_0^5)^{\frac{p}{2}+1} \sin\left(\frac{p x^i}{2 x_0^5}\right) (x^5)^{-\frac{p}{2}} \end{array} \right. \\ \text{(i-th row)} \\ 0 \\ 0 \\ \left\{ \begin{array}{l} -(x_0^5)^{\frac{p}{2}} \cosh\left(\frac{p x^i}{2 x_0^5}\right) (x^5)^{-\left(\frac{p}{2}-1\right)} \\ -(x_0^5)^{\frac{p}{2}} \cos\left(\frac{p x^i}{2 x_0^5}\right) (x^5)^{-\left(\frac{p}{2}-1\right)} \end{array} \right. \end{pmatrix}. \quad (336)$$

The 9 Killing vectors can be identified with the generators of the related algebra as follows:

$$\xi_1^A = \begin{pmatrix} \\ \Upsilon_{SR}^{0\mu} \\ 0 \end{pmatrix} = \Upsilon^{1A}, \quad \alpha^1 = T^0; \quad (337)$$

$$\xi_{\mathcal{A}}^A = \begin{pmatrix} \Upsilon_{SR}^{A\mu} \\ 0 \end{pmatrix} = \Upsilon^{AA}, \mathcal{A} = i, j_1, j_2, \begin{cases} \alpha^{i+1}(\alpha, x_0^5, p) = -T^i(\alpha, x_0^5, p) \\ \alpha^{j_1+1}(\alpha, x_0^5, p) = -T^{j_1}(\alpha, x_0^5, p) ; \\ \alpha^{j_2+1}(\alpha, x_0^5, p) = -T^{j_2}(\alpha, x_0^5, p) \end{cases} \quad (338)$$

$$\xi_5^A(x) = \begin{pmatrix} \begin{cases} i = 1 : K_{SR}^{2\mu}(x) \\ i = 2, 3 : K_{SR}^{1\mu}(x) \end{cases} \\ 0 \end{pmatrix}, \alpha^5 = \begin{cases} \rho^2, i = 1 \\ \rho^1, i = 2, 3 \end{cases} ; \quad (339)$$

$$\xi_6^A(x) = \begin{pmatrix} \begin{cases} i = 1, 2 : K_{SR}^{3\mu}(x) \\ i = 3 : K_{SR}^{2\mu}(x) \end{cases} \\ 0 \end{pmatrix}, \alpha^6 = \begin{cases} \rho^3, i = 1, 2 \\ \rho^2, i = 3 \end{cases} ; \quad (340)$$

$$\xi_7^A(x) = \begin{pmatrix} \begin{cases} i = 1 : S_{SR}^{1\mu}(x) \\ i = 2 : -S_{SR}^{2\mu}(x) \\ i = 3 : S_{SR}^{3\mu}(x) \end{cases} \\ 0 \end{pmatrix}, \alpha^7 = \begin{cases} -\Theta^1, i = 1 \\ \Theta^2, i = 2 \\ -\Theta^3, i = 3 \end{cases} ; \quad (341)$$

$$\xi_{8,\mp}^A(x^i, x^5; p) = Z_{\mp}^{1A}(x^i, x^5; p); \quad (342)$$

$$\xi_{9,\mp}^A(x^i, x^5; p) = Z_{\mp}^{2A}(x^i, x^5; p). \quad (343)$$

The commutation relations of the Killing algebra are therefore

$$[\tilde{\Upsilon}^\mu, \tilde{\Upsilon}^\nu] = 0 \quad \forall \mu, \nu = 0, 1, 2, 3;$$

$$\begin{cases} i = 1 : [\tilde{\mathbf{K}}^2(x), \tilde{\mathbf{K}}^3(x)] = -\tilde{\mathbf{S}}^1(x) \\ i = 2 : [\tilde{\mathbf{K}}^1(x), \tilde{\mathbf{K}}^3(x)] = \tilde{\mathbf{S}}^2(x) \quad ; \\ i = 3 : [\tilde{\mathbf{K}}^1(x), \tilde{\mathbf{K}}^2(x)] = -\tilde{\mathbf{S}}^3(x) \end{cases}$$

$$[\tilde{\mathbf{Z}}_{\mp}^1(x^i, x^5; p), \tilde{\mathbf{Z}}_{\mp}^2(x^0, x^5; p)] = 0;$$

$$\left[\tilde{\mathbf{Y}}^0, \begin{cases} i = 1 : \begin{cases} \tilde{\mathbf{K}}^2(x) \\ \tilde{\mathbf{K}}^3(x) \end{cases} \\ i = 2 : \begin{cases} \tilde{\mathbf{K}}^1(x) \\ \tilde{\mathbf{K}}^3(x) \end{cases} \\ i = 3 : \begin{cases} \tilde{\mathbf{K}}^1(x) \\ \tilde{\mathbf{K}}^2(x) \end{cases} \end{cases} \right] = \begin{cases} \tilde{\mathbf{Y}}^{j_1} \\ \tilde{\mathbf{Y}}^{j_2} \end{cases} ;$$

$$\left[\tilde{\mathbf{Y}}^0, \begin{cases} i = 1 : \tilde{\mathbf{S}}^1(x) \\ i = 2 : -\tilde{\mathbf{S}}^2(x) \\ i = 3 : \tilde{\mathbf{S}}^3(x) \end{cases} \right] = 0;$$

$$\left[\tilde{\mathbf{Y}}^i, \begin{cases} i = 1 : \begin{cases} \tilde{\mathbf{K}}^2(x) \\ \tilde{\mathbf{K}}^3(x) \\ \tilde{\mathbf{S}}^1(x) \end{cases} \\ i = 2 : \begin{cases} \tilde{\mathbf{K}}^1(x) \\ \tilde{\mathbf{K}}^3(x) \\ -\tilde{\mathbf{S}}^2(x) \end{cases} \\ i = 3 : \begin{cases} \tilde{\mathbf{K}}^1(x) \\ \tilde{\mathbf{K}}^2(x) \\ \tilde{\mathbf{S}}^3(x) \end{cases} \end{cases} \right] = 0;$$

$$\left[\tilde{\mathbf{Y}}^{j_1}, \begin{cases} i = 1 : \tilde{\mathbf{K}}^2(x) \\ i = 2, 3 : \tilde{\mathbf{K}}^1(x) \end{cases} \right] = \tilde{\mathbf{Y}}^0;$$

$$\begin{aligned}
& \left[\tilde{\Upsilon}^{j_1}, \begin{cases} i = 1, 2 : \tilde{\mathbf{K}}^3(x) \\ i = 3 : \tilde{\mathbf{K}}^2(x) \end{cases} \right] = 0; \\
& \left[\tilde{\Upsilon}^{j_1}, \begin{cases} i = 1 : \tilde{\mathbf{S}}^1(x) \\ i = 2 : -\tilde{\mathbf{S}}^2(x) \\ i = 3 : \tilde{\mathbf{S}}^3(x) \end{cases} \right] = -\tilde{\Upsilon}^{j_2}; \\
& \left[\tilde{\Upsilon}^{j_2}, \begin{cases} i = 1 : \tilde{\mathbf{K}}^2(x) \\ i = 2, 3 : \tilde{\mathbf{K}}^1(x) \end{cases} \right] = 0; \\
& \left[\tilde{\Upsilon}^{j_2}, \begin{cases} i = 1, 2 : \tilde{\mathbf{K}}^3(x) \\ i = 3 : \tilde{\mathbf{K}}^2(x) \end{cases} \right] = \tilde{\Upsilon}^0; \\
& \left[\tilde{\Upsilon}^{j_2}, \begin{cases} i = 1 : \tilde{\mathbf{S}}^1(x) \\ i = 2 : -\tilde{\mathbf{S}}^2(x) \\ i = 3 : \tilde{\mathbf{S}}^3(x) \end{cases} \right] = \tilde{\Upsilon}^{j_1}; \\
& \begin{cases} i = 1 : [\tilde{\mathbf{K}}^2(x), \tilde{\mathbf{S}}^1(x)] = -\tilde{\mathbf{K}}^3(x) \\ i = 2 : [\tilde{\mathbf{K}}^1(x), -\tilde{\mathbf{S}}^2(x)] = -\tilde{\mathbf{K}}^3(x) ; \\ i = 3 : [\tilde{\mathbf{K}}^1(x), \tilde{\mathbf{S}}^3(x)] = -\tilde{\mathbf{K}}^2(x) \end{cases} \\
& \begin{cases} i = 1 : [\tilde{\mathbf{K}}^3(x), \tilde{\mathbf{S}}^1(x)] = \tilde{\mathbf{K}}^2(x) \\ i = 2 : [\tilde{\mathbf{K}}^3(x), -\tilde{\mathbf{S}}^2(x)] = \tilde{\mathbf{K}}^1(x) ; \\ i = 3 : [\tilde{\mathbf{K}}^2(x), \tilde{\mathbf{S}}^3(x)] = \tilde{\mathbf{K}}^1(x) \end{cases} \\
& [\tilde{\Upsilon}^0, \tilde{\mathbf{Z}}_{\mp}^1(x; p)] = [\tilde{\Upsilon}^0, \tilde{\mathbf{Z}}_{\mp}^2(x; p)] = 0; \\
& [\tilde{\Upsilon}^i, \tilde{\mathbf{Z}}_{\mp}^1(x; p)] = \frac{p}{2x_0^5} \tilde{\mathbf{Z}}_{\mp}^2(x; p); \\
& [\tilde{\Upsilon}^i, \tilde{\mathbf{Z}}_{\mp}^2(x; p)] = \mp \frac{p}{2x_0^5} \tilde{\mathbf{Z}}_{\mp}^1(x; p); \\
& [\tilde{\Upsilon}^{j_1}, \tilde{\mathbf{Z}}_{\mp}^1(x; p)] = [\tilde{\Upsilon}^{j_1}, \tilde{\mathbf{Z}}_{\mp}^2(x; p)] = 0; \\
& [\tilde{\Upsilon}^{j_2}, \tilde{\mathbf{Z}}_{\mp}^1(x; p)] = [\tilde{\Upsilon}^{j_2}, \tilde{\mathbf{Z}}_{\mp}^2(x; p)] = 0;
\end{aligned}$$

$$\left[\tilde{\mathbf{Z}}_{\mp}^{1,2}(x; p), \left\{ \begin{array}{l} i = 1 : \left\{ \begin{array}{l} \tilde{\mathbf{K}}^2(x) \\ \tilde{\mathbf{K}}^3(x) \\ \tilde{\mathbf{S}}^1(x) \end{array} \right. \\ i = 2 : \left\{ \begin{array}{l} \tilde{\mathbf{K}}^1(x) \\ \tilde{\mathbf{K}}^3(x) \\ -\tilde{\mathbf{S}}^2(x) \end{array} \right. \\ i = 3 : \left\{ \begin{array}{l} \tilde{\mathbf{K}}^1(x) \\ \tilde{\mathbf{K}}^2(x) \\ \tilde{\mathbf{S}}^3(x) \end{array} \right. \end{array} \right\} \right] = 0. \quad (344)$$

It is easy to see that in all three cases $i = 1, 2, 3$ the Killing algebra contains the subalgebra $so(1, 2) \otimes_s tr.(1, 3)$, generated by

$$\underbrace{\tilde{\mathbf{Y}}^{\mu}}_{tr.(1,3)}; \left\{ \begin{array}{l} i = 1 : \tilde{\mathbf{K}}^2(x); \tilde{\mathbf{K}}^3(x); \tilde{\mathbf{S}}^1(x); \\ i = 2 : \tilde{\mathbf{K}}^1(x); \tilde{\mathbf{K}}^3(x); -\tilde{\mathbf{S}}^2(x); \\ i = 3 : \tilde{\mathbf{K}}^1(x); \tilde{\mathbf{K}}^2(x); \tilde{\mathbf{S}}^3(x). \end{array} \right. \quad (345)$$

$\underbrace{\hspace{15em}}_{so(1,2)}$

Some remarks are in order. For all metrics discussed in this section, it was possible, in general, neither to identify the global algebra obeyed by the Killing generators, nor even ascertain its Lie nature. Such an issue deserves further investigations. However, it is clearly seen from the explicit forms of the generators and of the commutation relations that, even in cases (like that corresponding to metric (135)) in which the space-time sector is Minkowskian, the presence of the fifth dimension implies transformations involving the energy coordinate (see definitions (289)-(292) of the generators $\tilde{\Sigma}^i, \tilde{\Gamma}^j$), and therefore entirely new physical symmetries. Moreover, it is easily seen from Eq.(266), expressing the general infinitesimal form of a metric automorphism in \mathfrak{R}_5 , and from the explicit form of the Killing vectors, that the isometric transformations are in general *nonlinear* (in particular in x^5). Then, the preliminary results obtained seemingly show that the isometries of \mathfrak{R}_5 related to the derived Killing algebras require *an invariance of physical laws under nonlinear coordinate transformations, in which energy is directly involved*.

7. Conclusions

The examples of Killing symmetries we just discussed show some peculiar features of \mathfrak{R}_5 isometries corresponding to the 5-d. phenomenological metrics of the four fundamental interactions. Indeed, due to the piecewise nature of such metrics, the respective symmetries are strongly related to the energy range considered. This is at variance with the DSR case, in which the (deformed) isometries of \widetilde{M} are *independent* of the energy parameter x^5 . This is why we never speak of a symmetry as related to a given interaction in the DSR context. On the contrary, in DR5 the metric nature of x^5 , and the consequent piecewise structure of the phenomenological metrics, is fundamental in determining the \mathfrak{R}_5 isometries. As a matter of fact, for a given interaction, in general *different* Killing symmetries are obtained in the two energy ranges below and above threshold. This is reflected in the discontinuous behavior of the \mathfrak{R}_5 Killing vectors at the energy threshold $x_{0,int.}^5$, namely one has in general

$$\lim_{x^5 \rightarrow x_{0,int.}^{5+}} \xi_{DR5,int.}^A(x^5) \neq \lim_{x^5 \rightarrow x_{0,int.}^{5-}} \xi_{DR5,int.}^A(x^5). \quad (346)$$

Conversely, at metric level, there is a continuity in the 5-d. metric tensor at the energy threshold (as clearly seen by their expressions in terms of the Heaviside function: see Subsubsect.3.5.1):

$$\begin{aligned} \lim_{x^5 \rightarrow x_{0,int.}^{5+}} g_{AB,DR5,int.}(x^5) &= \lim_{x^5 \rightarrow x_{0,int.}^{5-}} g_{AB,DR5,int.}(x^5) = \\ &= g_{AB,DR5,int.}(x_{0,int.}^5). \end{aligned} \quad (347)$$

This implies that symmetries present in an energy range in which the space-time sector is standard Minkowskian — or at least its metric coefficients are constant — may no longer hold when (in a different energy range) the space-time of \mathfrak{R}_5 becomes Minkowskian deformed, and *viceversa*.

As already stressed, this is essentially due to the change of nature (from parameter to coordinate) of the energy x^5 in the passage DSR \rightarrow DR5, *i.e.* in the geometrical embedding of \widetilde{M} in \mathfrak{R}_5 . In this process, at the metric level, the slicing properties (36)-(37) hold, namely the sections of \mathfrak{R}_5 at constant energy $x^5 = \overline{x^5}$ ($dx^5 = 0$) do possess the same metric structure of \widetilde{M} ($\overline{x^5}$). *This is no longer true at the level of the Killing symmetries.* We can write, symbolically:

$$\begin{aligned} &\text{Isometries of } \mathfrak{R}_5|_{dx^5=0 \Leftrightarrow x^5=\overline{x^5}} \neq \\ &\neq \text{Isometries of } \widetilde{M}(x^5 = \overline{x^5}) = \text{Deformed Poincaré group } P(1,3)_{DEF}^{10}. \end{aligned} \quad (348)$$

In fact, in increasing the dimension number by taking energy as the fifth coordinate, some of the 10 symmetry degrees of freedom of the maximal Killing group of DSR (*i.e.* the deformed Poincaré group) are lost.

The Killing isometries are therefore strictly related to the geometrical context considered. This is easily seen on account of the fact that the slicing process is carried out in

a genuine Riemannian geometric framework, in which the effect of the fifth coordinate is perceptible even at the four-dimensional level of space-time sections. From the point of view of the algebraic structure, this is reflected by the arising of new generators, associated to true or pseudo rotations involving both the space-time coordinates and the energy one. This situation exactly reminds that occurring in Special Relativity, where the presence of time as a genuine coordinate — no longer a parameter as in classical physics —, together with the ensuing geometrical structure of the Minkowski space, does affect the physics in the ordinary, Euclidean 3-space (the "shadow" of the pseudoeuclidean metric of M). This is a further evidence that *the embedding of \widetilde{M} in \mathfrak{R}_5 is not a mere formal artifact, but has deep physical motivations and implications.*

We shall see in a forthcoming paper [41] that similar considerations apply to dynamics, too.

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A Connection and Curvature in \mathfrak{R}_5

For reference, let us give the explicit expression of the main geometric quantities in the Riemannian space \mathfrak{R}_5 [20, 22].

- **Connection** $\Gamma_{BC}^A(x^5)$ (the prime denotes derivation with respect to x^5):

$$\left\{ \begin{array}{l} \Gamma_{05}^0 = \Gamma_{50}^0 = \frac{b'_0}{2b_0}; \quad \Gamma_{15}^1 = \Gamma_{51}^1 = \frac{b'_1}{2b_1}; \\ \Gamma_{25}^2 = \Gamma_{52}^2 = \frac{b'_2}{2b_2}; \quad \Gamma_{35}^3 = \Gamma_{53}^3 = \frac{b'_3}{2b_3}; \\ \Gamma_{00}^5 = -\frac{b_0 b'_0}{2f}; \quad \Gamma_{11}^5 = \frac{b_1 b'_1}{2f}; \quad \Gamma_{22}^5 = \frac{b_2 b'_2}{2f}; \\ \Gamma_{33}^5 = \frac{b_3 b'_3}{2f}; \quad \Gamma_{55}^5 = \frac{f'}{2f}. \end{array} \right. \quad (\text{A } 1)$$

The Riemann-Christoffel (curvature) tensor in \mathfrak{R}_5 is given by

$$R_{BCD}^A(x^5) = \partial_C \Gamma_{BD}^A - \partial_D \Gamma_{BC}^A + \Gamma_{KC}^A \Gamma_{BD}^K - \Gamma_{KD}^A \Gamma_{BC}^K. \quad (\text{A } 2)$$

- **Riemann-Christoffel tensor** $R_{ABCD}(x^5)$:

$$R_{0101} = \frac{b'_0 b'_1}{4f}; \quad R_{0202} = \frac{b'_0 b'_2}{4f}; \quad R_{0303} = \frac{b'_0 b'_3}{4f}; \quad (\text{A } 3.1)$$

$$R_{0505} = \frac{b'_0 f' b_0 + (b'_0)^2 - 2b''_0 b_0 f}{4b_0 f}; \quad (\text{A } 3.2)$$

$$R_{1212} = -\frac{b'_1 b'_2}{4f}; \quad R_{1313} = -\frac{b'_1 b'_3}{4f}; \quad R_{1515} = \frac{b'_1 f' b + (b'_1)^2 - 2b''_1 b_1 f}{4b_1 f}; \quad (\text{A } 3.3)$$

$$R_{2323} = -\frac{b'_2 b'_3}{4f}; \quad R_{2525} = \frac{b'_2 f' b_2 + (b'_2)^2 - 2b''_2 b_2 f}{4c f}; \quad (\text{A } 3.4)$$

$$R_{3535} = \frac{b'_3 f' b_3 + (b'_3)^2 - 2b''_3 b_3 f}{4b_3 f}. \quad (\text{A } 3.5)$$

- Ricci tensor $R_{AB}(x^5)$:

$$R_{00} = -\frac{1}{2} \frac{b''_0}{f} - \frac{b'_0}{4f} \left(-\frac{b'_0}{b_0} + \frac{b'_1}{b_1} + \frac{b'_2}{b_2} - \frac{f'}{f} \right); \quad (\text{A } 4)$$

$$R_{11} = \frac{1}{2} \frac{b''_1}{f} + \frac{b'_1}{4f} \left(\frac{b'_0}{b_0} - \frac{b'_1}{b_1} + \frac{b'_2}{b_2} + \frac{b'_3}{b_3} - \frac{f'}{f} \right); \quad (\text{A } 5)$$

$$R_{22} = \frac{1}{2} \frac{b''_2}{f} + \frac{b'_2}{4f} \left(\frac{b'_0}{b_0} + \frac{b'_1}{b_1} - \frac{b'_2}{b_2} + \frac{b'_3}{b_3} - \frac{f'}{f} \right); \quad (\text{A } 6)$$

$$R_{33} = \frac{1}{2} \frac{b''_3}{f} + \frac{b'_3}{4f} \left(\frac{b'_0}{b_0} + \frac{b'_1}{b_1} + \frac{b'_2}{b_2} - \frac{b'_3}{b_3} - \frac{f'}{f} \right); \quad (\text{A } 7)$$

$$R_{44} = -\frac{1}{2} \left(\frac{b'_0}{b_0} + \frac{b'_1}{b_1} + \frac{b'_2}{b_2} + \frac{b'_3}{b_3} \right)' + \frac{f'}{4f} \left(\frac{b'_0}{b_0} + \frac{b'_1}{b_1} + \frac{b'_2}{b_2} + \frac{b'_3}{b_3} \right) - \frac{1}{4} \left[\left(\frac{b'_0}{b_0} \right)^2 + \left(\frac{b'_1}{b_1} \right)^2 + \left(\frac{b'_2}{b_2} \right)^2 + \left(\frac{b'_3}{b_3} \right)^2 \right]. \quad (\text{A } 8)$$

- Scalar curvature $R(x^5)$:

$$\begin{aligned}
R(x^5) = & \frac{b_1 b_1' f (b_2' b_3 b_0 + b_3' b_2 b_0 + b_0' b_2 b_3) + b_2 b_3 [2b_1'' b_1 f - b_0 (b_1')^2 f - b_0 b_1' f' b_1]}{4b_1^2 f^2 b_2 b_3 b_0} + \\
& + \frac{b_2 b_2 f (b_1' b_3 b_0 + b_3' b_0 b_1 + b_0' b_3 b_1) + b_0 b_1 b_3 [2b_2'' b_2 f - (b_2')^2 f - b_2' f' b_2]}{4b_2^2 f^2 b_3 b_0 b_1} + \\
& + \frac{b_3 b_3 f (b_1' b_0 b_2 + b_2' b_0 b_1 + b_0' b_2 b_1) + b_0 b_1 b_2 [2b_3'' b_3 f - (b_3')^2 f - b_3' f' b_3]}{4b_3^2 f^2 b_2 b_0 b_1} + \\
& + \frac{b_0 b_0 f (b_1' b_2 b_3 + b_2' b_3 b_1 + b_3' b_2 b_1) + b_1 b_2 b_3 [2b_0'' b_0 f - (b_0')^2 f - b_0' f' b_0]}{4b_0^2 f^2 b_2 b_3 b_1} + \\
& + \frac{1}{4f^2 b_1^2 b_2^2} \{b_2^2 [2b_1'' b_1 f - (b_1')^2 f - b_1' f' b_1] + b_1^2 [2b_2'' b_2 f - (b_2')^2 f - b_2' f' b_2]\} + \\
& + \frac{1}{4f^2 b_0^2 b_3^2} \{b_0^2 [2b_3'' b_3 f - (b_3')^2 f - b_3' f' b_3] + b_3^2 [2b_0'' b_0 f - (b_0')^2 f - b_0' f' b_0]\}.
\end{aligned} \tag{A 9}$$

B Reductivity of the Υ -Hypothesis for the 12 Classes of the Vacuum Einstein's Equations in the Power Ansatz

We shall discuss here the possible reductivity of the Υ -hypothesis of functional independence, stated in Subsect.4.2, for the 12 classes of Power-Ansatz solutions of the Einstein equations in vacuum (derived in Sect.3.6), and solve explicitly the Killing equations in the 5 cases in which this hypothesis is violated. In the notation of Subsect.3.6, each class will be specified by an exponent set $\tilde{\mathbf{q}} \equiv (q_0, q_1, q_2, q_3, r)$.

B1 Analysis of Reductivity of the Υ -Hypothesis

B2 Class (I)

$$\tilde{\mathbf{q}}_I = \left(n, -n \left(\frac{2p+n}{2n+p} \right), n, p, \frac{p^2 - 2p + 2np - 4n + 3n^2}{2n+p} \right)$$

One gets

$$r + 2 = \frac{p^2 + 2np + 3n^2}{2n + p} \tag{B 1}$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = q_2 - r - 2 = \frac{3n^3 - 7n^2 - 4np + np^2 + 2n^2p - p^2}{2n + p}; \\ q_1 - r - 2 = -(2n + p); \\ q_3 - r - 2 = \frac{p^3 - 3p^2 - 6np + 2np^2 + 3n^2p - 3n^2}{2n + p}. \end{array} \right. \quad (\text{B } 2)$$

The condition of non-vanishing denominators yields

$$\text{Den.} \neq 0 \iff 2n + p \neq 0 \iff q_1 - r - 2 \neq 0. \quad (\text{B } 3)$$

Therefore, under the further assumptions

$$\left\{ \begin{array}{l} q_1 = -n \left(\frac{2p + n}{2n + p} \right) \neq 0 \iff \left\{ \begin{array}{l} n \neq 0 \\ 2p + n \neq 0 \end{array} \right. ; \\ r + 2 = \frac{p^2 + 2np + 3n^2}{2n + p} \neq 0 \iff p^2 + 2np + 3n^2 \neq 0, \end{array} \right. \quad (\text{B } 4)$$

one finds that the Υ -hypothesis is satisfied at least by $\mu = 1$.

Moreover, we have the following possible *degenerate cases* (*i.e.* those in which the Υ -hypothesis is violated for any value of μ):

I a) $n = 0 \Rightarrow p \neq 0$. Then

$$\left\{ \begin{array}{l} q_0 - r - 2 = q_2 - r - 2 = -p \neq 0; \\ q_1 - r - 2 = -(2n + p) \neq 0; \\ q_3 - r - 2 = p(p - 3), \end{array} \right. \quad (\text{B } 5)$$

whence for $p \neq 3$ the Υ -hypothesis is satisfied only by $\mu = 3$, and the case considered is not a degenerate one.

Therefore the true degenerate case of this class is characterized by

$$n = 0, p = 3 \quad (\text{B } 6)$$

and corresponds to the 5-d. metric

$$g_{AB,DR5power}(x^5) = \text{diag} \left(1, -1, -1, -\left(\frac{x^5}{x_0^5}\right)^3, \pm \frac{x^5}{x_0^5} \right), \quad (\text{B } 7)$$

special case for $p = 3$ of the metric

$$g_{AB,DR5power}(x^5) = \text{diag} \left(1, -1, -1, -\left(\frac{x^5}{x_0^5}\right)^p, \pm \left(\frac{x^5}{x_0^5}\right)^{p-2} \right), \quad (\text{B } 8)$$

that will be discussed in Subsect.B.2.4.

I b) $2n + p \iff n = -2p$. From Eq.(B 3) it follows

$$2n + p = -3p \neq 0 \quad (\text{B } 9)$$

and therefore

$$\begin{cases} q_0 - r - 2 = q_2 - r - 2 = p(6p + 7); \\ q_1 - r - 2 = 3p \neq 0; \\ q_3 - r - 2 = -3p \left(p - \frac{1}{3} \right). \end{cases} \quad (\text{B } 10)$$

Then, the Υ -hypothesis is satisfied for $p \neq -\frac{7}{6}$ by $\mu = 0, 2$, and $p \neq -\frac{1}{3}$ by $\mu = 3$.

For $p = -\frac{7}{6}$ one gets

$$\begin{cases} q_0 - r - 2 = q_2 - r - 2 = 0; \\ q_1 - r - 2 = -\frac{7}{2}; \\ q_3 - r - 2 = -\frac{21}{4}. \end{cases} \quad (\text{B } 11)$$

and the Υ -hypothesis is still satisfied by $\mu = 3$.

For $p = \frac{1}{3}$ it is

$$\begin{cases} q_0 - r - 2 = q_2 - r - 2 = 3; \\ q_1 - r - 2 = 1; \\ q_3 - r - 2 = 0. \end{cases} \quad (\text{B } 12)$$

and the Υ -hypothesis is still satisfied by $\mu = 0, 2$.

I c) $p^2 + 2np + 3n^2 = 0$. The only pair of real solutions of this equation is $(n, p) = (0, 0)$, that must be discarded because it entails the vanishing of the denominators.

B2.1 Class (II)

$$\tilde{\mathbf{q}}_{II} = (0, m, 0, 0, m - 2)$$

We have

$$r + 2 = m; \quad (\text{B } 13)$$

$$\begin{cases} q_0 - r - 2 = q_2 - r - 2 = q_3 - r - 2 = -m \\ q_1 - r - 2 = 0. \end{cases} \quad (\text{B } 14)$$

The Υ -hypothesis is violated $\forall m \in R$ and $\forall \mu \in \{0, 1, 2, 3\}$.

In general Class (II) corresponds to the 5-d. metric

$$g_{AB, DR5power}(x^5) = \text{diag} \left(1, -\left(\frac{x^5}{x_0^5}\right)^m, -1, -1, \pm \left(\frac{x^5}{x_0^5}\right)^{m-2} \right), \quad (\text{B } 15)$$

we shall consider in Sect.B.2.2.

In particular, for $m = 0$ one gets

$$g_{AB, DR5power}(x^5) = \text{diag} \left(1, -1, -1, -1, \pm \left(\frac{x^5}{x_0^5}\right)^{-2} \right), \quad (\text{B } 16)$$

whose space-time part is the same as the standard Minkowski space M . This metric is a special case of

$$g_{AB,DR5power}(x^5) = \text{diag}(a, -b, -c, -d, \pm f(x^5)), \quad (\text{B } 17)$$

whose Killing equations coincide with those of the metric with $a = b = c = d = 1$, solved in Subsubsect.5.1.1.

B2.2 Class (III)

$$\tilde{\mathbf{q}}_{III} = (n, -n, n, n, -2(1 - n))$$

It is

$$r + 2 = 2n; \quad (\text{B } 18)$$

$$\begin{cases} q_0 - r - 2 = q_2 - r - 2 = q_3 - r - 2 = -n, \\ q_1 - r - 2 = -3n. \end{cases} \quad (\text{B } 19)$$

The Υ -hypothesis is satisfied for $n \neq 0 \forall \mu \in \{0, 1, 2, 3\}$.

The degenerate case is characterized by $n = 0$ and corresponds to metric (B 17).

B2.3 Class (IV)

$$\tilde{\mathbf{q}}_{IV} = (0, 0, 0, p, p - 2)$$

One gets

$$r + 2 = m; \quad (\text{B } 20)$$

$$\begin{cases} q_0 - r - 2 = q_1 - r - 2 = q_3 - r - 2 = -p, \\ q_2 - r - 2 = 0. \end{cases} \quad (\text{B } 21)$$

The Υ -hypothesis is violated $\forall p \in R$ and $\forall \mu \in \{0, 1, 2, 3\}$.

In general this Class corresponds to the 5-d. metric (B 8), whose special case $p = 0$ is given by metric (B 16).

B2.4 Class (V)

$$\tilde{\mathbf{q}}_{V} = (-p, -p, -p, p, -(1 + p))$$

We have

$$r + 2 = 1 - p; \quad (\text{B } 22)$$

$$\begin{cases} q_0 - r - 2 = q_1 - r - 2 = q_2 - r - 2 = -1, \\ q_3 - r - 2 = 2p - 1. \end{cases} \quad (\text{B } 23)$$

Therefore the Υ -hypothesis is satisfied for $p \neq 0, p \neq 1$ by $\mu = 0, 1, 2$ (and for $p \neq 0, p \neq 1, p \neq \frac{1}{2}$ also by $\mu = 3$).

There are three possible degenerate cases:

V a) $p = 1$. One has

$$\begin{cases} q_0 = q_1 = q_2 = -1; \\ q_3 = 1; \\ r = -2. \end{cases} \quad (\text{B 24})$$

The Υ -hypothesis is not satisfied by any value of μ . The corresponding 5-d. metric is

$$\begin{aligned} & g_{AB,DR5power}(x^5) = \\ & = \text{diag} \left(\left(\frac{x^5}{x_0^5} \right)^{-1}, - \left(\frac{x^5}{x_0^5} \right)^{-1}, - \left(\frac{x^5}{x_0^5} \right)^{-1}, - \frac{x^5}{x_0^5}, \pm \left(\frac{x^5}{x_0^5} \right)^{-2} \right) \end{aligned} \quad (\text{B 25})$$

and is discussed in Sect.B.2.5.

V b) $p = 0$. It is

$$\begin{cases} q_0 = q_1 = q_2 = q_3 = 0; \\ r = -1. \end{cases} \quad (\text{B 26})$$

and the Υ -hypothesis is violated by any value of μ . The corresponding 5-d. metric is

$$g_{AB,DR5power}(x^5) = \text{diag} \left(1, -1, -1, -1, \pm \left(\frac{x^5}{x_0^5} \right)^{-1} \right), \quad (\text{B 27})$$

special case of the metric (B 17).

V c) $p = \frac{1}{2}$. One gets

$$r + 2 = \frac{1}{2}; \quad (\text{B 28})$$

$$\begin{cases} q_0 = q_1 = q_2 = -\frac{1}{2}, \\ q_3 = \frac{1}{2}. \end{cases} \quad (\text{B 29})$$

Therefore the Υ -hypothesis is satisfied by $\mu = 0, 1, 2$.

B2.5 Class (VI)

$$\tilde{\mathbf{q}}_{VI} = (q, 0, 0, 0, q - 2)$$

One has

$$r + 2 = q; \quad (\text{B 30})$$

$$\begin{cases} q_0 - r - 2 = 0, \\ q_1 - r - 2 = q_2 - r - 2 = q_3 - r - 2 = -q. \end{cases} \quad (\text{B 31})$$

The Υ -hypothesis is not satisfied $\forall q \in R$ and $\forall \mu \in \{0, 1, 2, 3\}$.

The corresponding 5-d. metric reads

$$g_{AB,DR5power}(x^5) = \text{diag} \left(\left(\frac{x^5}{x_0^5} \right)^q, -1, -1, -1, \pm \left(\frac{x^5}{x_0^5} \right)^{q-2} \right) \quad (\text{B } 32)$$

(with special case $q = 0$ given by (B 16)), and is discussed in Subsect.B.2.1.

B2.6 Class (VII)

$$\tilde{\mathbf{q}}_{VII} = (q, -q, -q, -q, -q, -2)$$

It is

$$r + 2 = -q; \quad (\text{B } 33)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = 2q, \\ q_1 - r - 2 = q_2 - r - 2 = q_3 - r - 2 = 0. \end{array} \right. \quad (\text{B } 34)$$

The Υ -hypothesis is satisfied for $q \neq 0$ by $\mu = 0$.

The degenerate case $q = 0$ corresponds to the 5-d. metric (B 16).

B2.7 Class (VIII)

$$\tilde{\mathbf{q}}_{VIII} = (0, 0, 0, 0, r \in R)$$

One has

$$q_0 - r - 2 = q_1 - r - 2 = q_2 - r - 2 = q_3 - r - 2 = -r - 2. \quad (\text{B } 35)$$

The Υ -hypothesis is violated $\forall r \in R$ and $\forall \mu \in \{0, 1, 2, 3\}$.

The Class (VIII) corresponds to the 5-d. metric

$$g_{AB,DR5power}(x^5) = \text{diag} \left(1, -1, -1, -1, \pm \left(\frac{x^5}{x_0^5} \right)^{-r} \right), \quad (\text{B } 36)$$

that generalizes metric (B 16) and is a special case of the metric (B 17).

B2.8 Class (IX)

$$\tilde{\mathbf{q}}_{IX} = (0, 0, n, 0, n - 2)$$

We have

$$r + 2 = n; \quad (\text{B } 37)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = q_1 - r - 2 = q_3 - r - 2 = -n, \\ q_2 - r - 2 = 0. \end{array} \right. \quad (\text{B } 38)$$

The Υ -hypothesis is not satisfied $\forall n \in R$ and $\forall \mu \in \{0, 1, 2, 3\}$.

The corresponding 5-d. metric is

$$g_{AB,DR5power}(x^5) = \text{diag} \left(1, -1, - \left(\frac{x^5}{x_0^5} \right)^n, -1, \pm \left(\frac{x^5}{x_0^5} \right)^{n-2} \right), \quad (\text{B } 39)$$

and is discussed in Subsect.B.2.3 (whereas $n = 0$ gives metric (B 16)).

B2.9 Class (X)

$$\tilde{\mathbf{q}}_X = \left(q, -\frac{pq+np+nq}{n+p+q}, n, p, \frac{(n+p+q)(n+p+q-2)-(pq+np+nq)}{n+p+q} \right)$$

One gets

$$r + 2 = \frac{(n + p + q)^2 - (pq + np + nq)}{n + p + q}; \quad (\text{B } 40)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = -\frac{p^2 + n^2 + np}{n + p + q}; \\ q_1 - r - 2 = -(n + p + q); \\ q_2 - r - 2 = -\frac{q^2 + p^2 + pq}{n + p + q}; \\ q_3 - r - 2 = -\frac{q^2 + n^2 + nq}{n + p + q}. \end{array} \right. \quad (\text{B } 41)$$

The condition of non-vanishing denominators yields

$$\text{Den.} \neq 0 \iff n + p + q \neq 0 \iff q_1 - r - 2 \neq 0. \quad (\text{B } 42)$$

Therefore, under the further assumptions

$$\left\{ \begin{array}{l} q_1 = -\frac{pq + np + nq}{n + p + q} \neq 0 \iff pq + np + nq \neq 0; \\ r + 2 = \frac{(n + p + q)^2 - (pq + np + nq)}{n + p + q} \neq 0 \iff \\ \iff p^2 + n^2 + q^2 + pq + np + nq \neq 0, \end{array} \right. \quad (\text{B } 43)$$

one finds that the Υ -hypothesis is satisfied at least by $\mu = 1$.

Then, we have the following possible degenerate cases:

X a) $pq + np + nq = 0$. Therefore

$$r + 2 = n + p + q \neq 0; \quad (\text{B } 44)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = -\frac{p^2 + n^2 + np}{n + p + q}; \\ q_1 - r - 2 = -(n + p + q) \neq 0; \\ q_2 - r - 2 = -\frac{q^2 + p^2 + pq}{n + p + q}; \\ q_3 - r - 2 = -\frac{q^2 + n^2 + nq}{n + p + q}. \end{array} \right. \quad (\text{B } 45)$$

One has to consider the following subcases:

X a.1) $q = 0$. Then

$$np = 0, n + p \neq 0 \iff \left\{ \begin{array}{l} X \text{ a.1.1) } n = 0, p \neq 0 \\ \text{or} \\ X \text{ a.1.2) } n \neq 0, p = 0. \end{array} \right. \quad (\text{B } 46)$$

X a.1.1) One gets

$$r = p - 2; \quad (\text{B 47})$$

$$\begin{cases} q_0 - r - 2 = -p \neq 0; \\ q_1 - r - 2 = p \neq 0; \\ q_2 - r - 2 = -p \neq 0; \\ q_3 - r - 2 = 0. \end{cases} \quad (\text{B 48})$$

Therefore the Υ -hypothesis is satisfied by no value of μ . The corresponding metric is given by Eq.(B 8) (Class (IV)).

X a.1.2) One finds

$$r = n - 2; \quad (\text{B 49})$$

$$\begin{cases} q_0 - r - 2 = q_1 - r - 2 = -n \neq 0; \\ q_2 - r - 2 = 0; \\ q_3 - r - 2 = -n \neq 0. \end{cases} \quad (\text{B 50})$$

Again, the Υ -hypothesis is not satisfied by any value of μ . The corresponding 5-d. metric is given by Eq.(B 39) (Class (IX)).

X a.2) $n = 0$. Then

$$pq = 0, p + q \neq 0 \iff \begin{cases} X \text{ a.2.1) } p = 0, q \neq 0 \\ \text{or} \\ X \text{ a.2.2) } p \neq 0, q = 0. \end{cases} \quad (\text{B 51})$$

X a.2.1) One gets

$$r = q - 2; \quad (\text{B 52})$$

$$\begin{cases} q_0 - r - 2 = 0; \\ q_1 - r - 2 = q_2 - r - 2 = q_3 - r - 2 = -q \neq 0. \end{cases} \quad (\text{B 53})$$

Therefore the Υ -hypothesis is satisfied by no value of μ . The corresponding metric coincides with that of Class (VI), Eq.(B 32).

The case *X a.2.2)* coincides with the case *X a.1.1)*.

X a.3) $p = 0$. It is

$$nq = 0, n + q \neq 0 \iff \begin{cases} X a.3.1) n = 0, q \neq 0 \\ \text{or} \\ X a.3.2) n \neq 0, q = 0, \end{cases} \quad (\text{B } 54)$$

and therefore the subcases *X a.3.1)* and *X a.3.2)* coincide with subcases *X a.2.1)* and *X a.1.2)*, respectively.

X a.4) $q_0 - r - 2 = 0 \iff p^2 + n^2 + np = 0$. The only possible pair of real solutions of this equation is $(p, n) = (0, 0)$. From the condition of non-vanishing denominators it then follows $q \neq 0$. Therefore such a case coincides with *X a.2.1)*.

X a.5) $q_1 - r - 2 = 0 \iff n + p + q = 0$. This condition expresses the vanishing of the denominators, and therefore this case is impossible.

X a.6) $q_2 - r - 2 = 0 \iff q^2 + p^2 + pq = 0$. The only real solution of this equation is $(q, p) = (0, 0)$. The condition of non-vanishing denominators entails $n \neq 0$, and then this case coincides with subcase *X a.1.2)*.

X a.7) $q_3 - r - 2 = 0 \iff q^2 + n^2 + nq = 0$. The only real solution is $(q, n) = (0, 0)$. The condition of non-vanishing denominators entails $p \neq 0$. Therefore this case coincides with subcase *X a.1.1)*.

X b) $p^2 + n^2 + q^2 + pq + np + nq = 0$. The only possible solution of such equation is $(p, n, q) = (0, 0, 0)$, which contradicts the non-vanishing condition of denominators. This case is impossible, too.

B2.10 Class (XI)

$$\tilde{\mathbf{q}}_{XI} = \left(q, -\frac{n(2q+n)}{2n+q}, n, n, \frac{3n^2-4n+2nq-2q+q^2}{2n+q} \right)$$

One gets

$$r + 2 = \frac{3n^2 + 2nq + q^2}{2n + q}; \quad (\text{B } 55)$$

$$\begin{cases} q_0 - r - 2 = -\frac{3n^2}{2n + q}; \\ q_1 - r - 2 = -(2n + q); \\ q_2 - r - 2 = q_3 - r - 2 = -\frac{q^2 + n^2 + nq}{2n + q}. \end{cases} \quad (\text{B } 56)$$

The condition of non-vanishing denominators yields

$$\text{Den.} \neq 0 \iff 2n + q \neq 0 \iff q_1 - r - 2 \neq 0. \quad (\text{B } 57)$$

Then, by assuming further

$$\left\{ \begin{array}{l} q_1 = -\frac{n(2q+n)}{2n+q} \neq 0 \iff \left\{ \begin{array}{l} n \neq 0; \\ 2q+n \neq 0, \end{array} \right. ; \\ r+2 = \frac{3n^2+2nq+q^2}{2n+q} \neq 0 \iff 3n^2+2nq+q^2 \neq 0 \end{array} \right. , \quad (\text{B } 58)$$

one gets that the Υ -hypothesis is satisfied at least by $\mu = 1$.

We have also the following possible degenerate cases:

XI a) $n = 0 \Rightarrow q \neq 0$. Then

$$r = q - 2; \quad (\text{B } 59)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = 0; \\ q_1 - r - 2 = q_2 - r - 2 = q_3 - r - 2 = -q \neq 0. \end{array} \right. \quad (\text{B } 60)$$

Therefore the Υ -hypothesis is satisfied by no value of μ . The corresponding metric coincides with that of Class (VI).

XI b) $2q + n = 0$ (which, together with $2n + q \neq 0$, entails $q \neq 0$). Then

$$r = -3q - 2; \quad (\text{B } 61)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = 4q \neq 0; \\ q_1 - r - 2 = 3q \neq 0; \\ q_2 - r - 2 = q_3 - r - 2 = q \neq 0. \end{array} \right. \quad (\text{B } 62)$$

Therefore the Υ -hypothesis is satisfied by $\mu = 0, 2, 3$.

XI c) $3n^2 + 2nq + q^2 = 0$. The only possible solution of such equation is $(n, q) = (0, 0)$, contradicting the non-vanishing condition of denominators, whence the impossibility of this case.

B2.11 Class (XII)

$$\tilde{\mathbf{q}}_{XII} = \left(q, n, n, -\frac{n(2q+n)}{2n+q}, \frac{p^2+pq-2p+np-2n+nq+n^2-2q+q^2}{n+p+q} \right)$$

We have

$$r + 2 = \frac{p^2 + n^2 + q^2 + pq + np + nq}{n + p + q}; \quad (\text{B } 63)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = -\frac{p^2 + n^2 + np}{n + p + q}; \\ q_1 - r - 2 = q_2 - r - 2 = -\frac{p^2 + q^2 + pq}{n + p + q}; \\ q_3 - r - 2 = \\ = -\frac{3n^3 + q^3 + 6n^2q + 5nq^2 + 2np^2 + 3pn^2 + q^2p + p^2q + 5npq}{(n + p + q)(2n + q)}. \end{array} \right. \quad (\text{B } 64)$$

The condition of non-zero denominators yields

$$\text{Den.} \neq 0 \iff \left\{ \begin{array}{l} 2n + q \neq 0; \\ n + p + q \neq 0. \end{array} \right. \quad (\text{B } 65)$$

Let us put

$$\left. \begin{array}{l} n + p + q = a \in R_0 \\ 2n + q = b \in R_0 \end{array} \right\} \Rightarrow p = a - \frac{q + b}{2}. \quad (\text{B } 66)$$

Considering *e.g.* $\mu = 0$, one has

$$\begin{aligned} q_0 - r - 2 &= -\frac{p^2 + n^2 + np}{n + p + q} = \\ &= \frac{1}{4a} [3(q - a)^2 + (a - b)^2] \underbrace{\geq}_{\substack{q \neq a \text{ or } a \neq b}} 0, \text{sgn}(a) = \begin{cases} 1 \\ -1 \end{cases}. \end{aligned} \quad (\text{B } 67)$$

By assuming $q \neq a$ and/or $a \neq b$, the vanishing of the denominators entails $q_0 - r - 2 \neq 0$. Then, under the further assumptions

$$\left\{ \begin{array}{l} q_0 = q \neq 0; \\ r + 2 = \frac{p^2 + n^2 + q^2 + pq + np + nq}{n + p + q} \neq 0 \iff \\ \iff p^2 + n^2 + q^2 + pq + np + nq \neq 0, \end{array} \right. \quad (\text{B } 68)$$

one gets that the Υ -hypothesis is satisfied at least by $\mu = 0$.

The possible degenerate cases are:

$$\text{XII a) } q = 0 \Rightarrow \left\{ \begin{array}{l} n \neq 0 \\ n + p \neq 0 \end{array} \right. . \text{ Then} \quad r + 2 = \frac{p^2 + n^2 + np}{n + p}; \quad (\text{B } 69)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = -\frac{p^2 + n^2 + np}{n + p}; \\ q_1 - r - 2 = q_2 - r - 2 = -\frac{p^2}{n + p}; \\ q_3 - r - 2 = -\frac{3n^2 + 2p^2 + 3pn}{2(n + p)}. \end{array} \right. \quad (\text{B } 70)$$

It is $r+2 \neq 0$ and $q_0 - r - 2 \neq 0$ because the only possible solution of $p^2 + n^2 + np = 0$ is the pair $(p, n) = (0, 0)$, incompatible with the non-vanishing of the denominators. Therefore this case is impossible.

We have the following subcases:

XII a.1) $p = 0$. Then

$$r = n - 2; \quad (\text{B } 71)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = -n \neq 0; \\ q_1 - r - 2 = q_2 - r - 2 = 0; \\ q_3 - r - 2 = -\frac{3}{2}n \neq 0. \end{array} \right. \quad (\text{B } 72)$$

Therefore the Υ -hypothesis is satisfied by $\mu = 3$.

XII a.2) $3n^2 + 2p^2 + 3pn = 0$. This equation admits as only real solution the pair $(p, n) = (0, 0)$, incompatible with the non-vanishing of the denominators. Therefore, this case is impossible.

XII b) $p^2 + n^2 + q^2 + pq + np + nq = 0$. This equation has the only real solution $(p, n, q) = (0, 0, 0)$, contradicting the non-vanishing of the denominators, and then this case has to be discarded.

XII c) $q = a = b \iff n = p = 0$. We have

$$r = a - 2; \quad (\text{B } 73)$$

$$\left\{ \begin{array}{l} q_0 - r - 2 = 0; \\ q_1 - r - 2 = q_2 - r - 2 = q_3 - r - 2 = -a \neq 0. \end{array} \right. \quad (\text{B } 74)$$

Consequently the Υ -hypothesis is satisfied by no value of μ . The metric obtained is the same of Class (VI).

B3 Solution of the 5-d. Killing Equations for Totally Violated Υ -Hypothesis

The analysis of the previous Subsection has shown that, in the framework of the Power Ansatz, there exist five cases (actually only three of them are independent) in which the hypothesis Υ of functional independence is violated $\forall \mu = 0, 1, 2, 3$. In the following, we shall explicitly solve the Killing equations in such cases.

B3.1 Case 1

In the framework of the Power Ansatz, the first case we shall consider in which the Υ -hypothesis is not satisfied by any value of μ corresponds to the 5-d. metric belonging to the VI class ($p \in R$)

$$g_{AB,DR5,1}(x^5) = \text{diag} \left(\left(\frac{x^5}{x_0^5} \right)^p, -1, -1, -1, \pm \left(\frac{x^5}{x_0^5} \right)^{p-2} \right). \quad (\text{B } 75)$$

For $p = 0$ one gets the metric

$$g_{AB,DR5}(x^5) = \text{diag} \left(1, -1, -1, -1, \pm \left(\frac{x^5}{x_0^5} \right)^{-2} \right) \quad (\text{B } 76)$$

that is a special case of the metric

$$g_{AB,DR5}(x^5) = \text{diag} (a, -b, -c, -d, \pm f(x^5)), \quad (\text{B } 77)$$

whose Killing equations (coincident with those relevant to the metric with $a = b = c = d = 1$) have been solved in Subsubsect.5.1.1. Therefore, one can assume $p \in R_0$.

The Killing system (83)-(94) in this case reads

$$\left\{ \begin{array}{l} \pm \xi_{0,0}(x^A) + \frac{p}{2} \frac{x^5}{(x_0^5)^2} \xi_5(x^A) = 0; \\ \xi_{0,1}(x^A) + \xi_{1,0}(x^A) = 0; \\ \xi_{0,2}(x^A) + \xi_{2,0}(x^A) = 0; \\ \xi_{0,3}(x^A) + \xi_{3,0}(x^A) = 0; \\ \xi_{0,5}(x^A)x^5 - p\xi_0(x^A) + \xi_{5,0}(x^A)x^5 = 0; \\ \xi_{1,1}(x^A) = 0; \\ \xi_{1,2}(x^A) + \xi_{2,1}(x^A) = 0; \\ \xi_{1,3}(x^A) + \xi_{3,1}(x^A) = 0; \\ \xi_{1,5}(x^A) + \xi_{5,1}(x^A) = 0; \\ \xi_{2,2}(x^A) = 0; \\ \xi_{2,3}(x^A) + \xi_{3,2}(x^A) = 0; \\ \xi_{2,5}(x^A) + \xi_{5,2}(x^A) = 0; \\ \xi_{3,3}(x^A) = 0; \\ \xi_{3,5}(x^A) + \xi_{5,3}(x^A) = 0; \\ -2\xi_{5,5}(x^A)x^5 + (p-2)\xi_5(x^A) = 0. \end{array} \right. \quad (\text{B } 78)$$

Its solution depends on the signature (timelike or spacelike) of x^5 . One gets, for the covariant Killing 5-vector:

$$\xi_0(x^0, x^5; p) = \begin{cases} \text{'' + '' :} \\ (x_0^5)^{-1} \left[A \cos \left(\frac{p x^0}{2 x_0^5} \right) - B \sin \left(\frac{p x^0}{2 x_0^5} \right) \right] (x^5)^{\frac{p}{2}} + \alpha(x^5)^p; \\ \text{'' - '' :} \\ (x_0^5)^{-1} \left[C \cosh \left(\frac{p x^0}{2 x_0^5} \right) - D \sinh \left(\frac{p x^0}{2 x_0^5} \right) \right] (x^5)^{\frac{p}{2}} + \alpha(x^5)^p; \end{cases} \quad (\text{B } 79)$$

$$\xi_1(x^2, x^3) = \Theta_3 x^2 + \Theta_2 x^3 - T_1; \quad (\text{B } 80)$$

$$\xi_2(x^1, x^3) = -\Theta_3 x^1 + \Theta_1 x^3 - T_2; \quad (\text{B } 81)$$

$$\xi_3(x^1, x^2) = -\Theta_2 x^1 + \Theta_1 x^2 - T_3; \quad (\text{B } 82)$$

$$\xi_5(x^0, x^5; p) = \begin{cases} \text{'' + '' :} \\ \left[A \cos \left(\frac{p x^0}{2 x_0^5} \right) + B \sin \left(\frac{p x^0}{2 x_0^5} \right) \right] (x^5)^{\frac{p}{2}-1}; \\ \text{'' - '' :} \\ \left[C \cosh \left(\frac{p x^0}{2 x_0^5} \right) + D \sinh \left(\frac{p x^0}{2 x_0^5} \right) \right] (x^5)^{\frac{p}{2}-1}, \end{cases} \quad (\text{B } 83)$$

with Θ_i, T_i ($i = 1, 2, 3$), $A, B, C, D, \alpha \in R$. Since $[x_0^5] = l$, the dimensions of the transformation parameters are

$$[A] = [B] = [C] = [D] = l^{-(\frac{p}{2}-1)}, \quad [\alpha] = l^{-p}, \quad [T_i] = l, \quad [\Theta_i] = l^0 \forall i. \quad (\text{B } 84)$$

The 3-d Killing group (of the Euclidean sections at $dx^5 = 0, dx^0 = 0$) is trivially the group of rototranslations of the Euclidean space E_3 with metric $\delta_{ij} = \text{diag}(-1, -1, -1)$ ($(i, j) \in \{1, 2, 3\}^2$):

$$SO(3)_{STD} \otimes_s Tr.(3)_{STD}. \quad (\text{B } 85)$$

B3.2 Case 2

The 5-d. metric for this case is

$$g_{AB, DR5,2}(x^5) = \text{diag} \left(1, - \left(\frac{x^5}{x_0^5} \right)^p, -1, -1, \pm \left(\frac{x^5}{x_0^5} \right)^{p-2} \right). \quad (\text{B } 86)$$

The corresponding Killing equations are

$$\left\{ \begin{array}{l} \xi_{0,0}(x^A) = 0; \\ \xi_{0,1}(x^A) + \xi_{1,0}(x^A) = 0; \\ \xi_{0,2}(x^A) + \xi_{2,0}(x^A) = 0; \\ \xi_{0,3}(x^A) + \xi_{3,0}(x^A) = 0; \\ \xi_{0,5}(x^A) + \xi_{5,0}(x^A) = 0; \\ \mp \xi_{1,1}(x^A) + \frac{p}{2} \frac{x^5}{(x_0^5)^2} \xi_5(x^A) = 0; \\ \xi_{1,2}(x^A) + \xi_{2,1}(x^A) = 0; \\ \xi_{1,3}(x^A) + \xi_{3,1}(x^A) = 0; \\ \xi_{1,5}(x^A)x^5 - p\xi_1(x^A) + \xi_{5,1}(x^A)x^5 = 0; \\ \xi_{2,2}(x^A) = 0; \\ \xi_{2,3}(x^A) + \xi_{3,2}(x^A) = 0; \\ \xi_{2,5}(x^A) + \xi_{5,2}(x^A) = 0; \\ \xi_{3,3}(x^A) = 0; \\ \xi_{3,5}(x^A) + \xi_{5,3}(x^A) = 0; \\ -2\xi_{5,5}(x^A)x^5 + (p-2)\xi_5(x^A) = 0. \end{array} \right. \quad (\text{B } 87)$$

having as solution the covariant Killing vector

$$\xi_0(x^2, x^3) = \zeta_2 x^2 + \zeta_3 x^3 + T_0; \quad (\text{B } 88)$$

$$\xi_1(x^1, x^5; p) = \begin{cases} \text{" + " :} \\ (x_0^5)^{-1} \left[A \cosh \left(\frac{p x^1}{2 x_0^5} \right) + B \sinh \left(\frac{p x^1}{2 x_0^5} \right) \right] (x^5)^{\frac{p}{2}} + \alpha (x^5)^p; \\ \text{" - " :} \\ (x_0^5)^{-1} \left[C \cos \left(\frac{p x^1}{2 x_0^5} \right) - D \sin \left(\frac{p x^1}{2 x_0^5} \right) \right] (x^5)^{\frac{p}{2}} + \alpha (x^5)^p; \end{cases} \quad (\text{B } 89)$$

$$\xi_2(x^0, x^3) = -\zeta_2 x^1 + \Theta_1 x^3 - T_2; \quad (\text{B } 90)$$

$$\xi_3(x^0, x^2) = -\zeta_3 x^1 - \Theta_1 x^2 - T_3; \quad (\text{B } 91)$$

$$\xi_5(x^1, x^5; p) = \begin{cases} \text{'' + '' :} \\ \left[A \sinh\left(\frac{p x^1}{2 x_0^5}\right) + B \cosh\left(\frac{p x^1}{2 x_0^5}\right) \right] (x^5)^{\frac{p}{2}-1}; \\ \text{'' - '' :} \\ \left[C \sin\left(\frac{p x^1}{2 x_0^5}\right) + D \cos\left(\frac{p x^1}{2 x_0^5}\right) \right] (x^5)^{\frac{p}{2}-1}, \end{cases} \quad (\text{B } 92)$$

with $\zeta_k (k = 2, 3)$, Θ_1, T_ν ($\nu = 0, 2, 3$), $A, B, C, D, \alpha \in R$. The dimensions of the transformation parameters are

$$[A] = [B] = [C] = [D] = l^{-(\frac{p}{2}-1)}, \quad [\alpha] = l^{-p}, \quad [T_\nu] = l, \quad [\zeta_i] = [\Theta_1] = l^0. \quad (\text{B } 93)$$

The 3-d Killing group (of the sections at $dx^5 = 0, dx^1 = 0$) is trivially the group of rototranslations of the pseudoeuclidean space E'_3 with metric $g_{\mu\nu} = \text{diag}(1, -1, -1)$ ($(\mu, \nu) \in \{0, 2, 3\}^2$):

$$SO(2, 1)_{STD} \otimes_s Tr.(2, 1)_{STD}. \quad (\text{B } 94)$$

B3.3 Case 3

In this case the 5-d. metric reads

$$g_{AB, DR5,3}(x^5) = \text{diag} \left(1, , -1, - \left(\frac{x^5}{x_0^5} \right)^p, -1, \pm \left(\frac{x^5}{x_0^5} \right)^{p-2} \right) \quad (\text{B } 95)$$

which is the same as Case 2, apart from an exchange of the space axes x and y . The Killing vector is therefore obtained from the previous solution (B 88)-(B 92) by the exchange $1 \leftrightarrow 2$.

B3.4 Case 4

The 5-d. metric of this case

$$g_{AB, DR5,4}(x^5) = \text{diag} \left(1, -1, -1, - \left(\frac{x^5}{x_0^5} \right)^p, \pm \left(\frac{x^5}{x_0^5} \right)^{p-2} \right) \quad (\text{B } 96)$$

amounts again to an exchange of space axes ($1 \leftrightarrow 3$) with respect to case 2. Accordingly, the solution for the Killing vector is obtained by such an exchange in the relevant equations.

B3.5 Case 5

The 5-d. metric of this case is given by

$$g_{AB, DR5,5}(x^5) = \text{diag} \left(\left(\frac{x^5}{x_0^5} \right)^{-1}, - \left(\frac{x^5}{x_0^5} \right)^{-1}, - \left(\frac{x^5}{x_0^5} \right)^{-1}, - \frac{x^5}{x_0^5}, \pm \left(\frac{x^5}{x_0^5} \right)^{-2} \right), \quad (\text{B } 97)$$

to which corresponds the Killing system

$$\left\{ \begin{array}{l} 2\xi_{0,0}(x^A)x_0^5 \mp \xi_5(x^A) = 0; \\ \xi_{0,1}(x^A) + \xi_{1,0}(x^A) = 0; \\ \xi_{0,2}(x^A) + \xi_{2,0}(x^A) = 0; \\ \xi_{0,3}(x^A) + \xi_{3,0}(x^A) = 0; \\ \xi_{0,5}(x^A)x^5 + \xi_0(x^A) + \xi_{5,0}(x^A)x^5 = 0; \\ 2\xi_{1,1}(x^A)x_0^5 \pm \xi_5(x^A) = 0; \\ \xi_{1,2}(x^A) + \xi_{2,1}(x^A) = 0; \\ \xi_{1,3}(x^A) + \xi_{3,1}(x^A) = 0; \\ \xi_{1,5}(x^A)x^5 + \xi_1(x^A) + \xi_{5,1}(x^A)x^5 = 0; \\ \xi_{2,2}(x^A)x_0^5 \pm \xi_5(x^A) = 0; \\ \xi_{2,3}(x^A) + \xi_{3,2}(x^A) = 0; \\ \xi_{2,5}(x^A)x^5 + \xi_2(x^A) + \xi_{5,2}(x^A)x^5 = 0; \\ \mp 2\xi_{3,3}(x^A)(x_0^5)^3 + (x^5)^2 \xi_5(x^A) = 0; \\ \xi_{3,5}(x^A)x^5 - \xi_3(x^A) + \xi_{5,3}(x^A)x^5 = 0; \\ \xi_{5,5}(x^A)x^5 + \xi_5(x^A) = 0. \end{array} \right. \quad (\text{B } 98)$$

Solving this system yields the covariant Killing vector

$$\xi_0(x^1, x^2, x^5) = \eta_1 \frac{x^1}{x^5} + \eta_2 \frac{x^2}{x^5} + \tau_0 \frac{1}{x^5}; \quad (\text{B } 99)$$

$$\xi_1(x^0, x^2, x^5) = -\eta_1 \frac{x^0}{x^5} + \Theta_3 \frac{x^2}{x^5} - \tau_1 \frac{1}{x^5}; \quad (\text{B } 100)$$

$$\xi_2(x^0, x^1, x^5) = -\eta_2 \frac{x^0}{x^5} - \Theta_3 \frac{x^1}{x^5} - \tau_2 \frac{1}{x^5}; \quad (\text{B } 101)$$

$$\xi_3 = 0; \quad (\text{B } 102)$$

$$\xi_5 = 0 \quad (\text{B } 103)$$

with $\eta_k (k = 1, 2)$, Θ_3 , $\tau_\nu (\nu = 0, 1, 2) \in R$. The dimensions of the transformation parameters are

$$[\tau_\nu] = l^2, \quad [\eta_k] = [\Theta_3] = l. \quad (\text{B } 104)$$

In this case the Killing group is the group of rototranslations of the pseudoeuclidean space M_3 with metric $g_{\mu\nu} = \text{diag} \left(\frac{x^5}{x_0^5} \right)^{-1} (1, -1, -1) ((\mu, \nu) \in \{0, 1, 2\}^2)$:

$$SO(2, 1)_{STD.M_3} \otimes_s Tr.(2, 1)_{STD.M_3}. \quad (\text{B } 105)$$

In all five cases discussed, the 5-d. contravariant Killing vectors $\xi^A(x, x^5)$ are obtained by means of the contravariant deformed metric tensor $g_{DR5}^{AB}(x^5)$ as

$$\xi^A(x, x^5) = g_{DR5}^{AB}(x^5)\xi_B(x, x^5). \quad (\text{B } 106)$$

For instance, in case 4, the contravariant metric tensor is

$$g_{DR5,4}^{AB}(x^5) = \text{diag} \left(1, -1, -1, -\left(\frac{x^5}{x_0^5}\right)^{-p}, \pm \left(\frac{x^5}{x_0^5}\right)^{-p+2} \right) \quad (\text{B } 107)$$

and therefore the contravariant components of the Killing vector read

$$\xi^0(x^1, x^2) = \zeta_1 x^1 + \zeta_2 x^2 + T_0; \quad (\text{B } 108)$$

$$\xi^1(x^0, x^2) = \zeta_1 x^0 - \Theta_3 x^2 + T_1; \quad (\text{B } 109)$$

$$\xi^2(x^0, x^1) = \zeta_2 x^0 + \Theta_3 x^1 + T_2; \quad (\text{B } 110)$$

$$\begin{aligned} & \xi^3(x^3, x^5; p) = \\ = & \begin{cases} \text{'' + '' :} \\ - (x_0^5)^{p-1} \left[A \cosh \left(\frac{p x^3}{2 x_0^5} \right) + B \sinh \left(\frac{p x^3}{2 x_0^5} \right) \right] (x^5)^{-\frac{p}{2}} + \alpha (x_0^5)^p; \\ \text{'' - '' :} \\ - (x_0^5)^{p-1} \left[C \cos \left(\frac{p x^3}{2 x_0^5} \right) - D \sin \left(\frac{p x^3}{2 x_0^5} \right) \right] (x^5)^{-\frac{p}{2}} + \alpha (x_0^5)^p; \end{cases} \quad (\text{B } 111) \end{aligned}$$

$$\begin{aligned} & \xi^5(x^3, x^5; p) = \\ = & \begin{cases} \text{'' + '' :} \\ (x_0^5)^{p-2} \left[A \sinh \left(\frac{p x^3}{2 x_0^5} \right) + B \cosh \left(\frac{p x^3}{2 x_0^5} \right) \right] (x^5)^{-\frac{p}{2}+1}; \\ \text{'' - '' :} \\ - (x_0^5)^{p-2} \left[C \sin \left(\frac{p x^3}{2 x_0^5} \right) + D \cos \left(\frac{p x^3}{2 x_0^5} \right) \right] (x^5)^{-\frac{p}{2}+1}. \end{cases} \quad (\text{B } 112) \end{aligned}$$

C Gravitational Killing Symmetries for Special Forms of $b_1^2(x^5)$ and $b_2^2(x^5)$

In this Appendix, we shall investigate the integrability of the Killing system for the gravitational interaction in different cases by assuming special forms for the spatial metric coefficients $b_1^2(x^5)$ and $b_2^2(x^5)$. For each case, we will consider the two energy ranges $0 < x^5 \leq x_0^5$ (subcase a)) and $x^5 > x_0^5$ (subcase b)).

C1 Form I

The phenomenological metric 5-d. is assumed to be

$$\begin{aligned}
 & g_{AB,DR5,grav.}(x^5) = \\
 & = \text{diag} \left(1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], -c_1, -c_2, \right. \\
 & \quad \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right), \\
 & \quad c_1, c_2 \in R_0^+, \text{ (in gen.: } c_1 \neq 1, c_2 \neq 1, c_1 \neq c_2 \text{)}.
 \end{aligned} \tag{C 1}$$

C1.1 I a)

Metric (C 1) becomes:

$$g_{AB,DR5}(x^5) = \text{diag} (1, -c_1, -c_2, -1, \pm f(x^5)) = \text{diag} (g_{\mu\nu, \overline{M}_4}(x^5), \pm f(x^5)), \tag{C 2}$$

where \overline{M}_4 is a standard 4-d. Minkowskian space with the following coordinate rescaling:

$$x^1 \longrightarrow \sqrt{c_1}x^1 \underset{\text{(in gen.)}}{\Rightarrow} dx^1 \longrightarrow \sqrt{c_1}dx^1; \tag{C 3.1}$$

$$x^2 \longrightarrow \sqrt{c_2}x^2 \underset{\text{(in gen.)}}{\Rightarrow} dx^2 \longrightarrow \sqrt{c_2}dx^2. \tag{C 3.2}$$

This case is therefore the same of the e.m. and weak interactions in the energy range $x^5 \geq x_0^5$ (Subsubject.5.1.1) and of the strong interaction in the range $0 < x^5 \leq x_0^5$ (Subsubject.5.2.1). Thus, the Υ -hypothesis of functional independence is violated for any $\mu \in \{0, 1, 2, 3\}$, and the contravariant Killing 5-vector $\xi^A(x, x^5)$ is given by Eqs.(139)-(143).

The Killing group of the sections at $dx^5 = 0$ of \mathfrak{R}_5 is therefore the standard Poincaré group, suitably rescaled:

$$[P(1, 3)_{STD.} = SO(1, 3)_{STD.} \otimes_s Tr.(1, 3)_{STD.}]|_{x^1 \rightarrow \sqrt{c_1}x^1, x^2 \rightarrow \sqrt{c_2}x^2}. \tag{C 4}$$

C1.2 I b)

The metric takes the form

$$\begin{aligned}
 & g_{AB,DR5}(x^5) = \\
 & = \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -c_1, -c_2, -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, \pm f(x^5) \right)
 \end{aligned} \tag{C 5}$$

and from (124)-(125) it follows

$$\begin{aligned} A_0(x^5) &= -A_3(x^5) = \\ &= \frac{1}{8} \left(1 + \frac{x^5}{x_0^5} \right) \frac{x^5}{(x_0^5)^2} (f(x^5))^{-\frac{1}{2}} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]; \quad (\text{C } 6) \\ A_1(x^5) &= A_2(x^5) = 0; \end{aligned}$$

$$B_0(x^5) = B_3(x^5) = \frac{1}{2} \left(1 + \frac{x_0^5}{x^5} \right) (f(x^5))^{\frac{1}{2}}; \quad (\text{C } 7.1)$$

$$\frac{1}{\sqrt{c_1}} B_1(x^5) = \frac{1}{\sqrt{c_2}} B_2(x^5) = (f(x^5))^{\frac{1}{2}}, \quad (\text{C } 7.2)$$

whence

$$\frac{\pm A_0(x^5)}{B_0(x^5)} = \frac{\mp A_3(x^5)}{B_3(x^5)} = \pm \frac{1}{4} \frac{1}{f(x^5)} \frac{x^5}{(x_0^5)^2} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]. \quad (\text{C } 8)$$

Therefore, the Υ -hypothesis is satisfied only for $\mu = 0, 3$ under condition

$$\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \neq c \frac{f(x^5)}{x^5}, c \in R. \quad (\text{C } 9)$$

Then, on the basis of the results of Subsect.4.3, the components of the contravariant Killing vector $\xi^A(x, x^5)$ in this case are given by Eqs. (118)-(122), in which (some of) the

real parameters are constrained by the following system (cfr. Eq. (123)):

$$\left\{ \begin{array}{l} (01) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 [d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2)] + \\ \quad + c_1 [h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + e_2)] = 0; \\ (02) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 (d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3) + \\ \quad + c_2 [l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4)] = 0; \\ (03) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 (d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2) + \\ \quad + \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 [m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + (m_5 + g_2)] = 0; \\ (12) \quad c_1 (h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3) + c_2 (l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3) = 0; \\ (13) \quad c_1 (h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6) + \\ \quad + \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 (m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3) = 0; \\ (23) \quad c_2 (l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8) + \\ \quad + \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 (m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2) = 0. \end{array} \right. \quad (C 10)$$

Solving system (C 9) one finally gets for $\xi^A(x, x^5)$:

$$\xi^0(x^3) = \widetilde{F}_0(x^3) = d_2 x^3 + (a_1 + d_1 + K_0); \quad (C 11)$$

$$\xi^1(x^2) = -\widetilde{F}_1(x^2) = \frac{c_2}{c_1} l_3 x^2 - (K_1 + h_5 + e_1); \quad (C 12)$$

$$\xi^2(x^1) = -\widetilde{F}_2(x^1) = -l_3 x^1 - (l_7 + K_2 + e_3); \quad (C 13)$$

$$\xi^3(x^0) = -\widetilde{F}_3(x^0) = d_2 x^0 - (m_1 + g_1 + c); \quad (C 14)$$

$$\xi^5 = 0. \quad (C 15)$$

It follows from the above equations that the 5-d. Killing group in the range considered is

$$\left(SO(2)_{STD., \Pi(x^1, x^2 \rightarrow \sqrt{\frac{c_2}{c_1}} x^2)} \otimes B_{x^3, STD.} \right) \otimes_s Tr.(1, 3)_{STD.} \quad (C 16)$$

where $SO(2)_{STD, \Pi(x^1, x^2 \rightarrow \sqrt{\frac{c_2}{c_1}} x^2)} = SO(2)_{STD, \Pi(x^1 \rightarrow \sqrt{c_1} x^1, x^2 \rightarrow \sqrt{c_2} x^2)}$ is the one-parameter group (generated by $S_{SR}^3|_{x^2 \rightarrow \sqrt{\frac{c_2}{c_1}} x^2}$) of the 2-d. rotations in the plane $\Pi(x^1, x^2)$ characterized by the scale transformation (C 3.1)-(C 3.2), $B_{x^3, STD}$ is the usual one-parameter group (generated by K_{SR}^3) of the standard Lorentzian boosts along \hat{x}^3 and $Tr.(1, 3)_{STD}$ is the usual space-time translation group.

Notice that, by introducing the right distribution $\hat{\Theta}_R(x_0^5 - x^5)$, putting

$$\begin{aligned} \frac{B^1}{c_1} &\equiv \zeta^1, \quad \frac{B^2}{c_2} \equiv \zeta^2, \quad B^3 \equiv \zeta^3, \\ \frac{\Theta^1}{c_2} &\equiv \theta^1, \quad \frac{\Theta^2}{c_1} \equiv \theta^2, \quad \frac{\Theta^3}{c_2} \equiv \theta^3, \\ \Xi^0 &\equiv \zeta^5, \quad \frac{\Xi^1}{c_1} \equiv \Xi^{1'}, \quad \frac{\Xi^2}{c_2} \equiv \Xi^{2'}, \\ \frac{T^1}{c_1} &\equiv T^{1'}, \quad \frac{T^2}{c_2} \equiv T^{2'}, \end{aligned} \quad (\text{C } 17)$$

and making the identifications

$$\begin{aligned} (a_1 + d_1 + K_0) &= T^0; \\ -(K_1 + h_5 + e_1) &= \frac{1}{c_1} T^1; \\ -(l_7 + K_2 + e_3) &= \frac{1}{c_2} T^2; \\ -(m_1 + g_1 + c) &= T^3; \\ l_3 &= \frac{1}{c_2} \Theta^3; \\ d_2 &= -B^3, \end{aligned} \quad (\text{C } 18)$$

it is possible to express the contravariant 5-vector $\xi^A(x, x^5)$ for the gravitational interaction in case I) in the following form, valid in the whole energy range ($x^5 \in R_0^+$):

$$\xi^0(x^1, x^2, x^3, x^5) = \hat{\Theta}_R(x_0^5 - x^5) [-c_1 \zeta^1 x^1 - c_2 \zeta^2 x^2 + \zeta^5 F(x^5)] - \zeta^3 x^3 + T^0; \quad (\text{C } 19)$$

$$\xi^1(x^0, x^2, x^3, x^5) = \hat{\Theta}_R(x_0^5 - x^5) [-\zeta^1 x^0 - \theta^2 x^3 - \Xi^1 F(x^5)] + \frac{c_2}{c_1} \theta^3 x^2 + T^1; \quad (\text{C } 20)$$

$$\xi^2(x^0, x^1, x^3, x^5) = \hat{\Theta}_R(x_0^5 - x^5) [-\zeta^2 x^0 + \theta^1 x^3 - \Xi^2 F(x^5)] - \theta^3 x^1 + T^2; \quad (\text{C } 21)$$

$$\xi^3(x^0, x^1, x^2, x^5) = \hat{\Theta}_R(x_0^5 - x^5) [c_1 \theta^2 x^1 - c_2 \theta^1 x^2 - \Xi^3 F(x^5)] - \zeta^3 x^0 + T^3; \quad (\text{C } 22)$$

$$\begin{aligned} \xi^5(x, x^5) &= \\ &= \hat{\Theta}_R(x_0^5 - x^5) \left\{ \mp (f(x^5))^{-\frac{1}{2}} [\zeta^5 x^0 + c_1 \Xi^1 x^1 + c_2 \Xi^2 x^2 + \Xi^3 x^3 - T^5] \right\}. \end{aligned} \quad (\text{C } 23)$$

C2 Form II

$$\begin{aligned}
& g_{AB,DR5,grav.}(x^5) = \\
& = \text{diag} \left(1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], \right. \\
& - \left. \{c_1 + \Theta(x^5 - x_{0,grav.}^5) [\beta_1^2(x^5) - c_1]\}, - \{c_2 + \Theta(x^5 - x_{0,grav.}^5) [\beta_2^2(x^5) - c_2]\}, \right. \\
& \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right); \quad (C 24)
\end{aligned}$$

$$c_1, c_2 \in R_0^+, \text{ (in gen.: } c_1 \neq 1, c_2 \neq 1, c_1 \neq c_2),$$

where the functions $\beta_1^2(x^5)$ and $\beta_2^2(x^5)$ have the properties:

$$\begin{aligned}
& \beta_1^2(x^5), \beta_2^2(x^5) \in R_0^+, \forall x^5 \in ([x_0^5, \infty)) \subset R_0^+; \\
& \beta_1^2(x^5) \neq \beta_2^2(x^5); \\
& \beta_1^2(x_0^5) = c_1, \beta_2^2(x_0^5) = c_2; \\
& \beta_1^2(x^5) \neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, \beta_2^2(x^5) \neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2.
\end{aligned} \quad (C 25)$$

C2.1 II a)

In the energy range $0 < x^5 \leq x_0^5$, the 5-d. metric has the same form (C 2) of case I a), and therefore the same results of that case hold true. In particular the Υ -hypothesis is not satisfied by any value of μ , and the contravariant vector $\xi^A(x, x^5)$ is still given by Eqs.(139)-(143).

C2.2 II b)

In this case the 5-d. gravitational metric reads

$$\begin{aligned}
& g_{AB,DR5}(x^5) = \\
& = \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -\beta_1^2(x^5), -\beta_2^2(x^5), -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, \pm f(x^5) \right) \quad (C 26)
\end{aligned}$$

The fake vectors (124), (125) become (ESC off):

$$A_0(x^5) = -A_3(x^5) = = \frac{1}{8} \left(1 + \frac{x^5}{x_0^5} \right) \frac{x^5}{(x_0^5)^2} (f(x^5))^{-\frac{1}{2}} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]; \quad (C 27)$$

$$\begin{aligned}
& A_i(x^5) \equiv \beta_i(x^5)(f(x^5))^{-1/2}. \\
& \cdot \left[-(\beta_i'(x^5))^2 + \beta_i(x^5)\beta_i''(x^5) - \frac{1}{2}\beta_i(x^5)\beta_i'(x^5)f'(x^5)(f(x^5))^{-1} \right], i = 1, 2; \quad (C 28)
\end{aligned}$$

$$B_0(x^5) = B_3(x^5) = \frac{1}{2} \left(1 + \frac{x_0^5}{x^5} \right) (f(x^5))^{\frac{1}{2}}; \quad (\text{C } 29)$$

$$B_i(x^5) \equiv \beta_i(x^5)(f(x^5))^{1/2}, i = 1, 2, \quad (\text{C } 30)$$

whence

$$\frac{\pm A_0(x^5)}{B_0(x^5)} = \frac{\mp A_3(x^5)}{B_3(x^5)} = \pm \frac{1}{4} \frac{1}{f(x^5)} \frac{x^5}{(x_0^5)^2} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right] \quad (\text{C } 31)$$

$$\begin{aligned} & \frac{\pm A_i(x^5)}{B_i(x^5)} = \\ & = \pm (f(x^5))^{-1} \left[-(\beta'_i(x^5))^2 + \beta_i(x^5)\beta''_i(x^5) - \frac{1}{2}\beta_i(x^5)\beta'_i(x^5)f'(x^5)(f(x^5))^{-1} \right], \\ & \quad i = 1, 2. \end{aligned} \quad (\text{C } 32)$$

Then, the Υ -hypothesis is satisfied for $\mu = 0, 3$ under constraint (C 9), and for $\mu = 1$ and/or 2 under the conditions

$$\left. \begin{aligned} & \left[-(\beta'_i(x^5))^2 + \beta_i(x^5)\beta''_i(x^5) - \frac{1}{2}\beta_i(x^5)\beta'_i(x^5)f'(x^5)(f(x^5))^{-1} \right] \neq 0; \\ & (f(x^5))^{-1} \left[-(\beta'_i(x^5))^2 + \beta_i(x^5)\beta''_i(x^5) - \frac{1}{2}\beta_i(x^5)\beta'_i(x^5)f'(x^5)(f(x^5))^{-1} \right] \neq c, \\ & c \in R_0, \end{aligned} \right\} \Leftrightarrow$$

$$\Leftrightarrow \left\{ \begin{aligned} & -(\beta'_i(x^5))^2 + \beta_i(x^5)\beta''_i(x^5) - \frac{1}{2}\beta_i(x^5)\beta'_i(x^5)f'(x^5)(f(x^5))^{-1} \neq cf(x^5), \\ & c \in R, \quad i = 1 \text{ and/or } 2. \end{aligned} \right. \quad (\text{C } 33)$$

By exploiting the results of Subsect.4.3, the components of the contravariant Killing vector $\xi^A(x, x^5)$ corresponding to form II) of the 5-d. gravitational metric over threshold are given by Eqs. (118)-(122), in which (some of) the real parameters are constrained by

the following system:

$$\left\{ \begin{array}{l}
 (01) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 [d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2)] + \\
 \quad + \beta_1^2(x^5) [h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + e_2)] = 0; \\
 (02) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 (d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3) + \\
 \quad + \beta_2^2(x^5) [l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4)] = 0; \\
 (03) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 [d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2 + \\
 \quad + m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + (m_5 + g_2)] = 0; \\
 (12) \quad \beta_1^2(x^5) (h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3) + \\
 \quad + \beta_2^2(x^5) (l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3) = 0; \\
 (13) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 (m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3) + \\
 \quad + \beta_1^2(x^5) (h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6) = 0; \\
 (23) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 (m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2) + \\
 \quad + \beta_2^2(x^5) (l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8) = 0.
 \end{array} \right. \tag{C 34}$$

Solving system (C 34) yields for $\xi^A(x, x^5)$ in this case

$$\xi^0(x^3) = \widetilde{F}_0(x^3) = d_2 x^3 + (a_1 + d_1 + K_0); \tag{C 35}$$

$$\xi^1 = -\widetilde{F}_1 = -(K_1 + h_5 + e_1); \tag{C 36}$$

$$\xi^2 = -\widetilde{F}_2 = -(l_7 + K_2 + e_3); \tag{C 37}$$

$$\xi^3(x^0) = -\widetilde{F}_3(x^0) = d_2 x^0 - (m_1 + g_1 + c); \tag{C 38}$$

$$\xi^5 = 0. \tag{C 39}$$

The Killing group in this range is

$$B_{x^3,STD} \otimes_s Tr.(1, 3)_{STD}. \tag{C 40}$$

By means of the distribution $\widehat{\Theta}_R(x_0^5 - x^5)$, by the ridenominations (C 17) of case I) and putting

$$\begin{aligned}
 (a_1 + d_1 + K_0) &= T^0; \\
 -(K_1 + h_5 + e_1) &= \frac{1}{c_1} T^1; \\
 -(l_7 + K_2 + e_3) &= \frac{1}{c_2} T^2; \\
 -(m_1 + g_1 + c) &= T^3; \\
 d_2 &= -B^3,
 \end{aligned} \tag{C 41}$$

one gets the following expression of the contravariant vector for the form II of the gravitational metric in the whole energy range:

$$\xi^0(x^1, x^2, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [-c_1 \zeta^1 x^1 - c_2 \zeta^2 x^2 + \zeta^5 F(x^5)] - \zeta^3 x^3 + T^0; \tag{C 42}$$

$$\xi^1(x^0, x^2, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) \left[-\zeta^1 x^0 + \frac{c_2}{c_1} \theta^3 x^2 - \theta^2 x^3 - \Xi^1 F(x^5) \right] + T^1; \tag{C 43}$$

$$\xi^2(x^0, x^1, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [-\zeta^2 x^0 - \theta^3 x^1 + \theta^1 x^3 - \Xi^2 F(x^5)] + T^2; \tag{C 44}$$

$$\xi^3(x^0, x^1, x^2, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [c_1 \theta^2 x^1 - c_2 \theta^1 x^2 - \Xi^3 F(x^5)] - \zeta^3 x^0 + T^3; \tag{C 45}$$

$$\begin{aligned}
 &\xi^5(x, x^5) = \\
 &= \widehat{\Theta}_R(x_0^5 - x^5) \left\{ \mp (f(x^5))^{-\frac{1}{2}} [\zeta^5 x^0 + c_1 \Xi^1 x^1 + c_2 \Xi^2 x^2 + \Xi^3 x^3 - T^5] \right\}.
 \end{aligned} \tag{C 46}$$

C3 Form III

$$\begin{aligned}
 &g_{AB, DR5, grav.}(x^5) = \\
 &= \text{diag} \left(1 + \Theta(x^5 - x_{0, grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0, grav.}^5} \right)^2 - 1 \right], \right. \\
 &\quad \left. -\beta_1^2(x^5), -\beta_2^2(x^5), \right. \\
 &\quad \left. - \left\{ 1 + \Theta(x^5 - x_{0, grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0, grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right),
 \end{aligned} \tag{C 47}$$

where the functions $\beta_1^2(x^5)$ and $\beta_2^2(x^5)$ have in general the properties:

$$\begin{aligned} \beta_1^2(x^5), \beta_2^2(x^5) &\in R_0^+, \forall x^5 \in R_0^+; \\ \beta_1^2(x^5) &\neq \beta_2^2(x^5); \\ \beta_1^2(x^5) &\neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5}\right)^2, \beta_2^2(x^5) \neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5}\right)^2. \end{aligned} \tag{C 48}$$

Therefore, the present case III) differs from the previous case II) for the nature strictly functional (and not composite, namely expressible in terms of one or more Heaviside functions) of $\beta_1^2(x^5)$ and $\beta_2^2(x^5)$.

C3.1 III a)

The 5-d metric reads

$$g_{AB,DR5}(x^5) = \text{diag} (1, -\beta_1^2(x^5), -\beta_2^2(x^5), -1, \pm f(x^5)). \tag{C 49}$$

Then, from definitions (124) and (125), there follow Eqs.(C 28),(C 30) and

$$A_0(x^5) = A_3(x^5) = 0; \tag{C 50.1}$$

$$B_0(x^5) = B_3(x^5) = (f(x^5))^{\frac{1}{2}}. \tag{C 50.2}$$

Therefore, the Υ -hypothesis is satisfied only for $\mu = 1$ and/or 2 under condition (244). From Subsect.4.3, the contravariant 5-vector $\xi^A(x, x^5)$ corresponding to form III) of the 5-d. gravitational metric below threshold is given by Eqs. (118)-(122), in which (some

of) the real parameters are constrained by the system:

$$\left\{ \begin{array}{l}
 (01) \quad [d_8x^2x^3 + d_7x^2 + d_6x^3 + (d_5 + a_2)] + \\
 \quad + \beta_1^2(x^5) [h_2x^2x^3 + h_1x^2 + h_8x^3 + (h_7 + e_2)] = 0; \\
 (02) \quad (d_8x^1x^3 + d_7x^1 + d_4x^3 + d_3) + \\
 \quad + \beta_2^2(x^5) [l_2x^1x^3 + l_1x^1 + l_6x^3 + (l_5 + e_4)] = 0; \\
 (03) \quad (d_8x^1x^2 + d_6x^1 + d_4x^2 + d_2) + \\
 \quad + [m_8x^1x^2 + m_7x^1 + m_6x^2 + (m_5 + g_2)] = 0; \\
 (12) \quad \beta_1^2(x^5) (h_2x^0x^3 + h_1x^0 + h_4x^3 + h_3) + \\
 \quad + \beta_2^2(x^5) (l_2x^0x^3 + l_1x^0 + l_4x^3 + l_3) = 0; \\
 (13) \quad \beta_1^2(x^5) (h_2x^0x^2 + h_8x^0 + h_4x^2 + h_6) + \\
 \quad + (m_8x^0x^2 + m_7x^0 + m_4x^2 + m_3) = 0; \\
 (23) \quad \beta_2^2(x^5) (l_2x^0x^1 + l_6x^0 + l_4x^1 + l_8) + \\
 \quad + (m_8x^0x^1 + m_6x^0 + m_4x^1 + m_2) = 0.
 \end{array} \right. \quad (C 51)$$

Then, from the solutions of the above system, one finds

$$\xi^0(x^3) = \widetilde{F}_0(x^3) = d_2x^3 + (a_1 + d_1 + K_0); \quad (C 52)$$

$$\xi^1 = -\widetilde{F}_1 = -(K_1 + h_5 + e_1); \quad (C 53)$$

$$\xi^2 = -\widetilde{F}_2 = -(l_7 + K_2 + e_3); \quad (C 54)$$

$$\xi^3(x^0) = -\widetilde{F}_3(x^0) = d_2x^0 - (m_1 + g_1 + c); \quad (C 55)$$

$$\xi^5 = 0. \quad (C 56)$$

Let us notice that the result obtained for $\xi^A(x, x^5)$ coincides with that of case II b), Eqs.(C 35)-(C 39).

C3.2 III b)

The form of the 5-d. metric is identical to that of case II b):

$$\begin{aligned}
 & g_{AB,DR5}(x^5) = \\
 & = \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, -\beta_1^2(x^5), -\beta_2^2(x^5), -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, \pm f(x^5) \right). \quad (C 57)
 \end{aligned}$$

Therefore, the same results of Subsect.C.2.2 hold. Moreover, we have just noted that case III a) yields the same results of case II b), and thus of case III b) too. Consequently, for the form III of the gravitational metric, the contravariant Killing vector of the gravitational metric is independent of the energetic range considered. In conclusion, putting

$$\begin{aligned}
(a_1 + d_1 + K_0) &\equiv T^0; \\
-(K_1 + h_5 + e_1) &\equiv T^1; \\
-(l_7 + K_2 + e_3) &\equiv T^2; \\
-(m_1 + g_1 + c) &= T^3; \\
d_2 &= -\zeta^3,
\end{aligned} \tag{C 58}$$

one gets the following general form for $\xi^A(x, x^5)$ for form III of the gravitational metric ($\forall x^5 \in R_0^+$):

$$\xi^0(x^3) = -\zeta^3 x^3 + T^0; \tag{C 59}$$

$$\xi^1 = +T^1; \tag{C 60}$$

$$\xi^2 = +T^2; \tag{C 61}$$

$$\xi^3(x^0) = -\zeta^3 x^0 + T^3; \tag{C 62}$$

$$\xi^5 = 0. \tag{C 63}$$

Moreover, $\forall x^5 \in R_0^+$ the 5-d. Killing group is

$$B_{x^3, STD.} \otimes_s Tr.(1, 3)_{STD.} \tag{C 64}$$

C4 Form IV

$$\begin{aligned}
&g_{AB, DR5, grav.}(x^5) = \\
&= diag \left(1 + \Theta(x^5 - x_{0, grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0, grav.}^5} \right)^2 - 1 \right], \right. \\
&- \left. \{ c + \Theta(x^5 - x_{0, grav.}^5) [\beta^2(x^5) - c] \}, - \{ c + \Theta(x^5 - x_{0, grav.}^5) [\beta^2(x^5) - c] \}, \right. \\
&- \left. \left\{ 1 + \Theta(x^5 - x_{0, grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0, grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right) \tag{C 65}
\end{aligned}$$

($c \in R_0^+$, $c \neq 1$), where the function $\beta^2(x^5)$ has the following properties

$$\begin{aligned}
\beta^2(x^5) &\in R_0^+, \forall x^5 \in ([x_0^5, \infty)) \subset R_0^+; \\
\beta^2(x_0^5) &= c; \\
\beta^2(x^5) &\neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0, grav.}^5} \right)^2.
\end{aligned} \tag{C 66}$$

Therefore this case is a special case of case II) with $\beta_1^2(x^5) = \beta_2^2(x^5)$.

C4.1 IV a)

The 5-d. metric in this case coincides with that of case I a) with $c_1 = c_2 = c$:

$$g_{AB,DR5}(x^5) = \text{diag} (1, -c, -c, -1, \pm f(x^5)). \quad (\text{C } 67)$$

and therefore all the results of Subsect.C.1.1 still hold with $c_1 = c_2 = c$.

C4.2 IV b)

The 5-d. metric is

$$\begin{aligned} g_{AB,DR5}(x^5) = \\ = \left(\text{diag} \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -\beta^2(x^5), -\beta^2(x^5), \right. \\ \left. -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, \pm f(x^5) \right). \end{aligned} \quad (\text{C } 68)$$

The results are the same of case II b) with $\beta_1^2(x^5) = \beta_2^2(x^5)$ and $c_1 = c_2 = c$.

C5 Form V

$$\begin{aligned} g_{AB,DR5,grav.}(x^5) = \\ = \text{diag} \left(1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], \right. \\ \left. -\beta^2(x^5), -\beta^2(x^5), \right. \\ \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right), \end{aligned} \quad (\text{C } 69)$$

where the function $\beta^2(x^5)$ has the following properties:

$$\begin{aligned} \beta^2(x^5) \in R_0^+, \forall x^5 \in R_0^+; \\ \beta^2(x^5) \neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2. \end{aligned} \quad (\text{C } 70)$$

Therefore this case is a special case of case III) with $\beta_1^2(x^5) = \beta_2^2(x^5)$.

C5.1 V a)

In the energy range considered the 5-d. metric (C 69) becomes

$$g_{AB,DR5}(x^5) = \text{diag} (1, -\beta^2(x^5), -\beta^2(x^5), -1, \pm f(x^5)). \quad (\text{C } 71)$$

Then, from definitions (124) and (125), one gets Eqs.(C 28),(C 30) and

$$A_0(x^5) = A_3(x^5) = 0; \quad (\text{C } 72.1)$$

$$B_0(x^5) = B_3(x^5) = (f(x^5))^{\frac{1}{2}}. \quad (\text{C } 72.2)$$

Therefore, the Υ -hypothesis is satisfied only for $\mu = 1,2$ under condition (244). From Subsect.4.3, the contravariant Killing 5-vector $\xi^A(x, x^5)$ corresponding to form V) of the 5-d. gravitational metric below threshold is given by Eqs. (118)-(122), in which (some of) the real parameters are constrained by the system:

$$\begin{aligned} (01) \quad & d_8x^2x^3 + d_7x^2 + d_6x^3 + (d_5 + a_2) + \\ & + \beta^2(x^5) [h_2x^2x^3 + h_1x^2 + h_8x^3 + (h_7 + e_2)] = 0; \\ (02) \quad & d_8x^1x^3 + d_7x^1 + d_4x^3 + d_3 + \\ & + \beta^2(x^5) [l_2x^1x^3 + l_1x^1 + l_6x^3 + (l_5 + e_4)] = 0; \\ (03) \quad & d_8x^1x^2 + d_6x^1 + d_4x^2 + d_2 + \\ & + m_8x^1x^2 + m_7x^1 + m_6x^2 + (m_5 + g_2) = 0; \\ (12) \quad & \beta^2(x^5) (h_2x^0x^3 + h_1x^0 + h_4x^3 + h_3) + \\ & + \beta^2(x^5) (l_2x^0x^3 + l_1x^0 + l_4x^3 + l_3) = 0; \\ (13) \quad & \beta^2(x^5) (h_2x^0x^2 + h_8x^0 + h_4x^2 + h_6) + \\ & + m_8x^0x^2 + m_7x^0 + m_4x^2 + m_3 = 0; \\ (23) \quad & \beta^2(x^5) (l_2x^0x^1 + l_6x^0 + l_4x^1 + l_8) + \\ & + m_8x^0x^1 + m_6x^0 + m_4x^1 + m_2 = 0. \end{aligned} \quad (\text{C } 73)$$

The solution of system (C 73) yields therefore, for $\xi^A(x, x^5)$:

$$\xi^0(x^3) = \widetilde{F}_0(x^3) = d_2x^3 + (a_1 + d_1 + K_0); \quad (\text{C } 74)$$

$$\xi^1(x^2) = -\widetilde{F}_1(x^2) = l_3x^2 - (K_1 + h_5 + e_1); \quad (\text{C } 75)$$

$$\xi^2(x^1) = -\widetilde{F}_2(x^1) = -l_3x^1 - (l_7 + K_2 + e_3); \quad (\text{C } 76)$$

$$\xi^3(x^0) = -\widetilde{F}_3(x^0) = d_2x^0 - (m_1 + g_1 + c); \quad (\text{C } 77)$$

$$\xi^5 = 0. \quad (\text{C } 78)$$

C5.2 V b)

The 5-d. metric has the same form of case IV b), Eq.(C 68). Consequently, all the results of Subject.C.4.2 hold. Moreover, the Killing vector has the same expression of case V a), and it is therefore independent of the energy range. By the following redenomination of the parameters

$$\begin{aligned}
 (a_1 + d_1 + K_0) &\equiv T^0; \\
 -(K_1 + h_5 + e_1) &\equiv T^1; \\
 -(l_7 + K_2 + e_3) &\equiv T^2; \\
 -(m_1 + g_1 + c) &= T^3; \\
 d_2 &= -\zeta^3; \\
 l_3 &= -\theta^3,
 \end{aligned} \tag{C 79}$$

the contravariant Killing vector $\xi^A(x, x^5)$ of the gravitational metric (C 68) can be written in the form (valid $\forall x^5 \in R_0^+$):

$$\xi^0(x^3) = -\zeta^3 x^3 + T^0; \tag{C 80}$$

$$\xi^1(x^2) = \theta^3 x^2 + T^1; \tag{C 81}$$

$$\xi^2(x^1) = -\theta^3 x^1 + T^2; \tag{C 82}$$

$$\xi^3(x^0) = -\zeta^3 x^0 + T^3; \tag{C 83}$$

$$\xi^5 = 0. \tag{C 84}$$

Then, $\forall x^5 \in R_0^+$, the 5-d. Killing group is

$$(SO(2)_{STD., \Pi(x^1, x^2)} \otimes B_{x^3, STD.}) \otimes_s Tr.(1, 3)_{STD.}. \tag{C 85}$$

C6 Form VI

$$\begin{aligned}
 &g_{AB, DR5, grav.}(x^5) = \\
 &= diag \left(1 + \Theta(x^5 - x_{0, grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0, grav.}^5} \right)^2 - 1 \right], \right. \\
 &\quad \left. - \left\{ 1 + \Theta(x^5 - x_{0, grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0, grav.}^5} \right)^2 - 1 \right] \right\}, \right. \\
 &\quad \left. - \{ c + \Theta(x^5 - x_{0, grav.}^5) [\beta^2(x^5) - c] \}, \right. \\
 &\quad \left. - \left\{ 1 + \Theta(x^5 - x_{0, grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0, grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right),
 \end{aligned} \tag{C 86}$$

($c \in R_0^+$, $c \neq 1$) where the function $\beta^2(x^5)$ has in general the properties

$$\begin{aligned} \beta^2(x^5) &\in R_0^+, \forall x^5 \in ([x_0^5, \infty)) \subset R_0^+; \\ \beta^2(x_0^5) &= c; \\ \beta^2(x^5) &\neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2. \end{aligned} \quad (C 87)$$

C6.1 VI a)

The form of the 5-d. metric is

$$g_{AB,DR5}(x^5) = \text{diag} (1, -1, -c, -1, \pm f(x^5)). \quad (C 88)$$

This is exactly case I a) with $c_1 = 1$, $c_2 = c$, and therefore all the results of Subsect.C.1.1 hold.

C6.2 VI b)

The 5-d. metric is

$$\begin{aligned} g_{AB,DR5}(x^5) &= \\ &= \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, \right. \\ &\quad \left. -\beta^2(x^5), -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, \pm f(x^5) \right) \end{aligned} \quad (C 89)$$

and from definitions (124), (125) it follows

$$\begin{aligned} A_0(x^5) &= -A_1(x^5) = -A_3(x^5) = \\ &= \frac{1}{8} \left(1 + \frac{x^5}{x_0^5} \right) \frac{x^5}{(x_0^5)^2} (f(x^5))^{-\frac{1}{2}} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]; \end{aligned} \quad (C 90)$$

$$B_0(x^5) = B_1(x^5) = B_3(x^5) = \frac{1}{2} \left(1 + \frac{x^5}{x_0^5} \right) (f(x^5))^{\frac{1}{2}};$$

$$\begin{aligned} \frac{\pm A_0(x^5)}{B_0(x^5)} &= \frac{\mp A_1(x^5)}{B_1(x^5)} = \frac{\mp A_3(x^5)}{B_3(x^5)} = \\ &= \pm \frac{1}{4} \frac{1}{f(x^5)} \frac{x^5}{(x_0^5)^2} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]; \end{aligned} \quad (C 91)$$

$$\begin{aligned} A_2(x^5) &\equiv \beta(x^5)(f(x^5))^{-1/2} \cdot \\ &\cdot \left[-(\beta'(x^5))^2 + \beta(x^5)\beta''(x^5) - \frac{1}{2}\beta(x^5)\beta'(x^5)f'(x^5)(f(x^5))^{-1} \right]; \end{aligned} \quad (C 92)$$

$$B_2(x^5) \equiv \beta(x^5)(f(x^5))^{1/2}; \quad (\text{C } 93)$$

$$\begin{aligned} & \frac{\pm A_2(x^5)}{B_2(x^5)} = \\ & = \pm (f(x^5))^{-1} \left[-(\beta'(x^5))^2 + \beta(x^5)\beta''(x^5) - \frac{1}{2}\beta(x^5)\beta'(x^5)f'(x^5)(f(x^5))^{-1} \right]. \end{aligned} \quad (\text{C } 94)$$

Then, the Υ -hypothesis is satisfied for $\mu = 0, 1, 3$ under condition (C 9), and for $\mu = 2$ under condition:

$$-(\beta'(x^5))^2 + \beta(x^5)\beta''(x^5) - \frac{1}{2}\beta(x^5)\beta'(x^5)f'(x^5)(f(x^5))^{-1} \neq \lambda f(x^5), \lambda \in R. \quad (\text{C } 95)$$

So, under at least one of the conditions (C 9), (C 94), the contravariant Killing 5-vector $\xi^A(x, x^5)$ of the gravitational metric in case VI b) is given by Eqs.(118)-(122), with (some of) the real parameters being constrained by the system

$$\begin{aligned} (01) \quad & d_8x^2x^3 + d_7x^2 + d_6x^3 + (d_5 + a_2) + \\ & + h_2x^2x^3 + h_1x^2 + h_8x^3 + (h_7 + e_2) = 0; \\ (02) \quad & \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 (d_8x^1x^3 + d_7x^1 + d_4x^3 + d_3) + \\ & + \beta^2(x^5) [l_2x^1x^3 + l_1x^1 + l_6x^3 + (l_5 + e_4)] = 0; \\ (03) \quad & d_8x^1x^2 + d_6x^1 + d_4x^2 + d_2 + \\ & + m_8x^1x^2 + m_7x^1 + m_6x^2 + (m_5 + g_2) = 0; \\ (12) \quad & \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 (h_2x^0x^3 + h_1x^0 + h_4x^3 + h_3) + \\ & + \beta^2(x^5) (l_2x^0x^3 + l_1x^0 + l_4x^3 + l_3) = 0; \\ (13) \quad & h_2x^0x^2 + h_8x^0 + h_4x^2 + h_6 + \\ & + m_8x^0x^2 + m_7x^0 + m_4x^2 + m_3 = 0; \\ (23) \quad & \beta^2(x^5) (l_2x^0x^1 + l_6x^0 + l_4x^1 + l_8) + \\ & + \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 (m_8x^0x^1 + m_6x^0 + m_4x^1 + m_2) = 0. \end{aligned} \quad (\text{C } 96)$$

Its solution yields the following explicit form of $\xi^A(x, x^5)$:

$$\xi^0(x^1, x^3) = \widetilde{F}_0(x^1, x^3) = -(h_7 + e_2)x^1 - (m_5 + g_2)x^3 + (a_1 + d_1 + K_0); \quad (\text{C } 97)$$

$$\xi^1(x^0, x^3) = -\widetilde{F}_1(x^0, x^3) = -(h_7 + e_2)x^0 - h_6x^3 - (K_1 + h_5 + e_1); \quad (\text{C } 98)$$

$$\xi^2 = -\widetilde{F}_2 = -(l_7 + K_2 + e_3); \quad (\text{C } 99)$$

$$\xi^3(x^0, x^1) = -\widetilde{F}_3(x^0, x^1) = -(m_5 + g_2)x^0 + h_6x^1 - (m_1 + g_1 + c); \quad (\text{C } 100)$$

$$\xi^5 = 0. \quad (\text{C } 101)$$

Then, it is easily seen that the 5-d. Killing group in this subcase is

$$SO(2, 1)_{STD.M_3} \otimes_s Tr.(1, 3)_{STD}. \quad (\text{C } 102)$$

Here, $SO(2, 1)_{STD.M_3}$ is the 3-parameter, homogeneous Lorentz group (generated by S_{SR}^2 , K_{SR}^1 , K_{SR}^3) of the 3-d. space M_3 endowed with the metric interval

$$ds_{M_3}^2 = \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2 \right) ((dx^0)^2 - (dx^1)^2 - (dx^3)^2) \quad (\text{C } 103)$$

and $Tr.(1, 3)_{STD}$ is the usual space-time translation group. Eq.(C 102) can be rewritten as

$$P(1, 2)_{STD.M_3} \otimes_s Tr.{}_{STD.,x^2} \quad (\text{C } 104)$$

where $P(1, 2)_{STD.M_3} = SO(1, 2)_{STD.M_3} \otimes_s Tr.(1)_{STD.M_3}$ is the Poincaré group of M_3 and $Tr.(1)_{STD.,x^2}$ is the 1-parameter group (generated by Υ_{SR}^2) of the translations along \widehat{x}^2 .

In case VI, too, it is possible to write the Killing vector $\xi^A(x, x^5)$ of the gravitational metric (C 86) in a form valid on the whole energy range by means of the (right) Heaviside distribution $\widehat{\Theta}_R(x_0^5 - x^5)$, by putting

$$\begin{aligned} B^1 &\equiv \zeta^1, \frac{B^2}{c} \equiv \zeta^2, B^3 \equiv \zeta^3; \\ \frac{\Theta^1}{c} &\equiv \theta^1, \Theta^2 \equiv \theta^2, \frac{\Theta^3}{c} \equiv \theta^3; \\ \Xi^0 &\equiv \zeta^5, \frac{\Xi^2}{c} \equiv \Xi^{2'}; \\ \frac{T^2}{c} &\equiv T^{2'} \end{aligned} \quad (\text{C } 105)$$

and identifying

$$\begin{aligned} (a_1 + d_1 + K_0) &= T^0; \\ -(K_1 + h_5 + e_1) &= T^1; \\ -(l_7 + K_2 + e_3) &= \frac{1}{c}T^2; \\ -(m_1 + g_1 + c) &= T^3; \\ m_5 + g_2 &= B^3; \\ h_6 &= \Theta^2; \\ h_7 + e_2 &= B^1. \end{aligned} \quad (\text{C } 106)$$

One gets

$$\xi^0(x^1, x^2, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [-c\zeta^2 x^2 + \zeta^5 F(x^5)] - \zeta^1 x^1 - \zeta^3 x^3 + T^0; \quad (\text{C } 107)$$

$$\xi^1(x^0, x^2, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [c\theta^3 x^2 - \Xi^1 F(x^5)] - \zeta^1 x^0 - \theta^2 x^3 + T^1; \quad (\text{C } 108)$$

$$\xi^2(x^0, x^1, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [-\zeta^2 x^0 + \theta^1 x^3 - \theta^3 x^1 - \Xi^2 F(x^5)] + T^2; \quad (\text{C } 109)$$

$$\xi^3(x^0, x^1, x^2, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [-c\theta^1 x^2 - \Xi^3 F(x^5)] - \zeta^3 x^0 + \theta^2 x^1 + T^3; \quad (\text{C } 110)$$

$$\begin{aligned} \xi^5(x, x^5) &= \\ &= \widehat{\Theta}_R(x_0^5 - x^5) \left\{ \mp (f(x^5))^{-\frac{1}{2}} [\zeta^5 x^0 + \Xi^1 x^1 + c\Xi^2 x^2 + \Xi^3 x^3 - T^5] \right\}. \end{aligned} \quad (\text{C } 111)$$

C7 Form VII

$$\begin{aligned} g_{AB,DR5,grav.}(x^5) &= \\ &= \text{diag} \left(1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], \right. \\ &\quad \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \beta^2(x^5), \right. \\ &\quad \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right), \end{aligned} \quad (\text{C } 112)$$

with the function $\beta^2(x^5)$ having in general the properties

$$\begin{aligned} \beta^2(x^5) &\in R_0^+, \forall x^5 \in R_0^+; \\ \beta^2(x^5) &\neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2. \end{aligned} \quad (\text{C } 113)$$

C7.1 VII a)

The 5-d. metric (C 112) reads

$$g_{AB,DR5}(x^5) = \text{diag} (1, -1, -\beta^2(x^5), -1, \pm f(x^5)). \quad (\text{C } 114)$$

Definitions (124) and (125) yield:

$$\begin{aligned} A_0(x^5) &= A_1(x^5) = A_3(x^5) = 0; \\ A_2(x^5) &\equiv \beta(x^5)(f(x^5))^{-1/2}. \\ &\cdot \left[-(\beta'(x^5))^2 + \beta(x^5)\beta''(x^5) - \frac{1}{2}\beta(x^5)\beta'(x^5)f'(x^5)(f(x^5))^{-1} \right]; \end{aligned} \quad (\text{C } 115)$$

$$\begin{aligned} B_0(x^5) &= B_1(x^5) = B_3(x^5) = (f(x^5))^{\frac{1}{2}}; \\ B_2(x^5) &\equiv \beta(x^5)(f(x^5))^{1/2}; \end{aligned} \quad (\text{C } 116)$$

$$\begin{aligned} &\frac{\pm A_2(x^5)}{B_2(x^5)} = \\ &= \pm (f(x^5))^{-1} \left[-(\beta'(x^5))^2 + \beta(x^5)\beta''(x^5) - \frac{1}{2}\beta(x^5)\beta'(x^5)f'(x^5)(f(x^5))^{-1} \right]. \end{aligned} \quad (\text{C } 117)$$

So, the Υ -hypothesis is satisfied only for $\mu = 2$ under condition (C 95). The contravariant Killing vector $\xi^A(x, x^5)$ for the gravitational metric in this subcase is still given by Eqs.(118)-(122), in which (some of) the real parameters satisfy the following constraint system:

$$\begin{aligned} (01) \quad & d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2) + \\ & + h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + e_2) = 0; \\ (02) \quad & d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3 + \\ & + \beta^2(x^5) [l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4)] = 0; \\ (03) \quad & d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2 + \\ & + m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + (m_5 + g_2) = 0; \\ (12) \quad & h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3 + \\ & + \beta^2(x^5) (l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3) = 0; \\ (13) \quad & h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6 + \\ & + m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3 = 0; \\ (23) \quad & \beta^2(x^5) (l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8) + \\ & + m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2 = 0, \end{aligned} \quad (\text{C } 118)$$

whose solution can be shown to coincide with that of subcase VI b). One gets therefore

$$\xi^0(x^1, x^3) = \widetilde{F}_0(x^1, x^3) = -(h_7 + e_2)x^1 - (m_5 + g_2)x^3 + (a_1 + d_1 + K_0); \quad (\text{C } 119)$$

$$\xi^1(x^0, x^3) = -\widetilde{F}_1(x^0, x^3) = -(h_7 + e_2)x^0 - h_6 x^3 - (K_1 + h_5 + e_1); \quad (\text{C } 120)$$

$$\xi^2 = -\widetilde{F}_2 = -(l_7 + K_2 + e_3); \quad (\text{C } 121)$$

$$\xi^3(x^0, x^1) = -\widetilde{F}_3(x^0, x^1) = -(m_5 + g_2)x^0 + h_6 x^1 - (m_1 + g_1 + c); \quad (\text{C } 122)$$

$$\xi^5 = 0. \quad (\text{C } 123)$$

C7.2 VII b)

In the energy range $x^5 > x_0^5$ the form of the metric coincides with that of the case VI in the same range, Eq.(C 89), and therefore all results of Subsect.C.6.2 still hold.

By putting

$$\begin{aligned}
 (a_1 + d_1 + K_0) &= T^0; \\
 -(K_1 + h_5 + e_1) &= T^1; \\
 -(l_7 + K_2 + e_3) &= T^2; \\
 -(m_1 + g_1 + c) &= T^3; \\
 m_5 + g_2 &= \zeta^3; \\
 h_7 + e_2 &= \zeta^1,
 \end{aligned} \tag{C 124}$$

one finds that the general form of $\xi^A(x, x^5)$ for the gravitational metric in form VII is independent of the energy range and coincides with that obtained in case VI in the range $x^5 > x_0^5$. It reads explicitly (cfr. Eqs.(C 107)-(C 111))

$$\xi^0(x^1, x^3) = -\zeta^1 x^1 - \zeta^3 x^3 + T^0; \tag{C 125}$$

$$\xi^1(x^0 x^3) = -\zeta^1 x^0 - \theta^2 x^3 + T^1; \tag{C 126}$$

$$\xi^2 = +T^2; \tag{C 127}$$

$$\xi^3(x^0, x^1) = -\zeta^3 x^0 + \theta^2 x^1 + T^3; \tag{C 128}$$

$$\xi^5(x, x^5) = 0. \tag{C 129}$$

Consequently, the 5-d. Killing group is, $\forall x^5 \in R_0^+$ (see case VI):

$$SO(1, 2)_{STD.M_3} \otimes_s Tr.(1, 3)_{STD.} = P(1, 2)_{STD.M_3} \otimes_s Tr.(1)_{STD.,x^2}. \tag{C 130}$$

C8 Form VIII

$$\begin{aligned}
 g_{AB,DR5,grav.}(x^5) &= \\
 &= diag \left(1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], \right. \\
 &\quad \left. - \{ c + \Theta(x^5 - x_{0,grav.}^5) [\beta^2(x^5) - c] \}, \right. \\
 &\quad \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \right. \\
 &\quad \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right)
 \end{aligned} \tag{C 131}$$

($c \in R_0^+$, $c \neq 1$), with

$$\begin{aligned} \beta^2(x^5) &\in R_0^+, \forall x^5 \in ([x_0^5, \infty)) \subset R_0^+; \\ \beta^2(x_0^5) &= c; \\ \beta^2(x^5) &\neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2. \end{aligned} \tag{C 132}$$

C9 VIII a)

In this subcase metric (C 131) becomes

$$g_{AB,DR5}(x^5) = \text{diag} (1, -c, -1, -1, \pm f(x^5)), \tag{C 133}$$

the same as case I a) with $c_1 = c$, $c_2 = 1$ (see Eq.(C 2)). Then, the results of Subsect.C.1.1 are valid.

C9.1 VIII b)

The metric (C 131) reads

$$\begin{aligned} g_{AB,DR5}(x^5) &= \\ &= \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, -\beta^2(x^5), -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, \right. \\ &\quad \left. -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, \pm f(x^5) \right). \end{aligned} \tag{C 134}$$

and coincides with metric (C 89) of case VI b) except for the exchange of the space axes $1 \rightleftharpoons 2$. By the usual procedure it is found that the Υ -hypothesis is satisfied for $\mu = 0, 2, 3$ under condition (C 9) and for $\mu = 1$ under condition (C 95). Therefore, under at least one of these two conditions, the contravariant Killing vector $\xi^A(x, x^5)$ of the gravitational metric in subcase VIII b) are given by Eqs.(118)-(122), in which (at least some of) the

real parameters are constrained by the following system:

$$\left\{ \begin{array}{l}
 (01) \quad \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 [d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2)] + \\
 \quad + \beta^2(x^5) [h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + e_2)] = 0; \\
 (02) \quad d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3 + \\
 \quad + l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4) = 0; \\
 (03) \quad d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2 + \\
 \quad + m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + (m_5 + g_2) = 0; \\
 (12) \quad \beta^2(x^5) (h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3) + \\
 \quad + \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 (l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3) = 0; \\
 (13) \quad \beta^2(x^5) (h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6) + \\
 \quad + \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 (m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3) = 0; \\
 (23) \quad l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8 + \\
 \quad + m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2 = 0.
 \end{array} \right. \quad (C 135)$$

Then, $\xi^A(x, x^5)$ is given by

$$\xi^0(x^2, x^3) = \widetilde{F}_0(x^2, x^3) = -(l_5 + e_4) x^2 - (m_5 + g_2) x^3 + (a_1 + d_1 + K_0); \quad (C 136)$$

$$\xi^1 = -\widetilde{F}_1 = -(K_1 + h_5 + e_1); \quad (C 137)$$

$$\xi^2(x^0, x^3) = -\widetilde{F}_2(x^0, x^3) = -(l_5 + e_4) x^0 - l_8 x^3 - (l_7 + K_2 + e_3); \quad (C 138)$$

$$\xi^3(x^0, x^2) = -\widetilde{F}_3(x^0, x^2) = -(m_5 + g_2) x^0 + l_8 x^2 - (m_1 + g_1 + c); \quad (C 139)$$

$$\xi^5 = 0. \quad (C 140)$$

It follows that the 5-d. Killing group is

$$SO(1, 2)_{STD.\overline{M}_3} \otimes_s Tr.(1, 3)_{STD.}. \quad (C 141)$$

Here $SO(1, 2)_{STD.\overline{M}_3}$ is the 3-parameter Lorentz group (generated by S_{SR}^1 , K_{SR}^2 , K_{SR}^3) of the 3-d. manifold \overline{M}_3 with metric interval

$$ds_{\overline{M}_3}^2 = \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 \right) ((dx^0)^2 - (dx^2)^2 - (dx^3)^2) \quad (C 142)$$

and $Tr.(1, 3)_{STD}$ is the usual space-time translation group. Eq.(C 141) can be rewritten in the form

$$P(1, 2)_{STD.\overline{M}_3} \otimes_s Tr.(1)_{STD.,x^1}, \quad (C 143)$$

where we introduced the 6-parameter Poincaré group of \overline{M}_3 , $P(1, 2)_{STD.\overline{M}_3} = SO(1, 2)_{STD.\overline{M}_3} \otimes_s Tr.(1)_{STD.\overline{M}_3}$, and $Tr.(1)_{STD.,x^1}$ is the one-parameter translation group along \hat{x}^1 (generated by Υ_{SR}^1).

Moreover, it is easy to see that the compact form of the contravariant Killing vector for the gravitational metric in case VIII, valid $\forall x^5 \in R_0^+$, is still given by Eqs.(B.107)-(B.111), with the exchange $1 \rightleftharpoons 2$.

C10 Form IX

$$\begin{aligned} g_{AB,DR5,grav.}(x^5) = & \\ = \text{diag} \left(1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], \right. & \\ -\beta^2(x^5), - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, & \\ \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right) & \end{aligned} \quad (C 144)$$

where

$$\begin{aligned} \beta^2(x^5) \in R_0^+, \forall x^5 \in R_0^+; & \\ \beta^2(x^5) \neq \frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2. & \end{aligned} \quad (C 145)$$

C10.1 IX a)

In the range $0 < x^5 \leq x_0^5$ metric (C 144) becomes

$$g_{AB,DR5}(x^5) = \text{diag} (1, -\beta^2(x^5), -1, -1, \pm f(x^5)), \quad (C 146)$$

namely the same of case VII a) with the exchange $2 \rightarrow 1$. Therefore, performing such an exchange in Eqs. (C 114)-(C 117), one finds that the hypothesis Υ of functional independence is satisfied only for $\mu = 1$ under condition (C 95). Under such a condition, the contravariant Killing 5-vector $\xi^A(x, x^5)$ of the gravitational metric in the subcase IX a) is given by Eqs.(118)-(122), with (some of) the real parameters being constrained by

the system

$$\begin{aligned}
(01) \quad & [d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2)] + \\
& + \beta^2(x^5) [h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + e_2)] = 0; \\
(02) \quad & d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3 + \\
& + l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4) = 0; \\
(03) \quad & d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2 + \\
& + m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + m_5 + g_2 = 0; \\
(12) \quad & \beta^2(x^5) (h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3) + \\
& + l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3 = 0; \\
(13) \quad & \beta^2(x^5) (h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6) + \\
& + m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3 = 0; \\
(23) \quad & l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8 + \\
& + m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2 = 0.
\end{aligned} \tag{C 147}$$

The solution of the above system is the same obtained in subcase VIII b). So, the explicit form of $\xi^A(x, x^5)$ for the gravitational metric in form IX under threshold coincides with that of case VIII over threshold, Eqs.(C 136)-(C 140).

C10.2 IX b)

In the energy range $x^5 > x_0^5$ the form of the metric coincides with that of case VIII in the same range:

$$\begin{aligned}
& g_{AB,DR5}(x^5) = \\
& = \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, -\beta^2(x^5), \right. \\
& \left. -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav}^5} \right)^2, \pm f(x^5) \right), \tag{C 148}
\end{aligned}$$

so all the results of Subsect.C.8.2 hold. The contravariant Killing vector $\xi^A(x, x^5)$ is therefore independent of the energy range, and (on account of the discussion of Subsect.C.8.2) its expression is obtained from Eqs.(C 125)-(C 130) by the exchange $1 \Leftrightarrow 2$.

As to the 5-d. Killing group, it is therefore, $\forall x^5 \in R_0^+$ (cfr. case VIII b)):

$$SO(1,2)_{STD.\overline{M}_3} \otimes_s Tr.(1,3)_{STD.} = P(1,2)_{STD.\overline{M}_3} \otimes_s Tr.(1)_{STD.,x^2}. \quad (C 149)$$

C11 Form X

$$\begin{aligned} g_{AB,DR5,grav.}(x^5) = & \\ = & \text{diag} \left(1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], \right. \\ & - \left. \left\{ c_1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{c_1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - c_1 \right] \right\} \right. \\ & - \left. \left\{ c_2 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{c_2}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - c_2 \right] \right\}, \right. \\ & \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right) \end{aligned} \quad (C 150)$$

($c_1, c_2 \in R_0^+$, in gen.: $c_1 \neq 1, c_2 \neq 1, c_1 \neq c_2$).

C11.1 X a)

In the energy range $0 < x^5 \leq x_0^5$ the form of the 5-d. metric is identical to that of cases I) and II) in the same range, Eq.(C 2). All the results of Subsects.C.1.1 and C.2.1 are valid. The Υ -hypothesis is violated $\forall \mu \in \{0, 1, 2, 3\}$. The Killing equations, and therefore the Killing vector, coincide with those corresponding to the e.m. and weak metrics above threshold (Subsubsect.5.1.1) and to the strong metric below threshold (Subsubsect.5.2.1). Therefore the contravariant Killing 5-vector $\xi^A(x, x^5)$ is given by Eqs.(139)-(143), and the Killing group of the sections at $dx^5 = 0$ of \mathfrak{R}_5 is of course the rescaled Poincaré group (C 4).

C11.2 X b)

The 5-d. metric becomes

$$\begin{aligned} g_{AB,DR5}(x^5) = & \\ = & \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -\frac{c_1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, \right. \\ & \left. -\frac{c_2}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, \pm f(x^5) \right). \end{aligned} \quad (C 151)$$

Therefore

$$\begin{aligned} A_0(x^5) &= -A_3(x^5) = \\ &= \frac{1}{8} \left(1 + \frac{x^5}{x_0^5}\right) \frac{x^5}{(x_0^5)^2} (f(x^5))^{-\frac{1}{2}} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]; \end{aligned} \quad (\text{C } 152)$$

$$B_0(x^5) = B_3(x^5) = \frac{1}{2} \left(1 + \frac{x^5}{x_0^5}\right) (f(x^5))^{\frac{1}{2}};$$

$$\frac{\pm A_0(x^5)}{B_0(x^5)} = \mp \frac{A_3(x^5)}{B_3(x^5)} = \pm \frac{1}{4} \frac{1}{f(x^5)} \frac{x^5}{(x_0^5)^2} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]; \quad (\text{C } 153)$$

$$\begin{aligned} A_i(x^5) &= \\ &= -\frac{(c_i)^{\frac{3}{2}}}{8} \left(1 + \frac{x^5}{x_0^5}\right) \frac{x^5}{(x_0^5)^2} (f(x^5))^{-\frac{1}{2}} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]; \\ B_i(x^5) &= \frac{(c_i)^{\frac{1}{2}}}{2} \left(1 + \frac{x^5}{x_0^5}\right) (f(x^5))^{\frac{1}{2}}, \end{aligned} \quad (\text{C } 154)$$

$$i = 1, 2;$$

$$\frac{\pm A_i(x^5)}{B_i(x^5)} = \mp \frac{c_i}{4} \frac{1}{f(x^5)} \frac{x^5}{(x_0^5)^2} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right], i = 1, 2. \quad (\text{C } 155)$$

It follows that the Υ -hypothesis is satisfied by any $\mu = 0, 1, 2, 3$ under condition (C 9). Then, under such condition, the contravariant Killing vector $\xi^A(x, x^5)$ for the gravitational metric in this subcase is given by Eqs.(118)-(122), in which (some of) the real

parameters satisfy the constraint system

$$\begin{aligned}
 (01) \quad & d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2) + \\
 & + c_1 [h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + e_2)] = 0; \\
 (02) \quad & d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3 + \\
 & + c_2 [l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4)] = 0; \\
 (03) \quad & d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2 + \\
 & + m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + (m_5 + g_2) = 0;
 \end{aligned} \tag{C 156}$$

$$\begin{aligned}
 (12) \quad & c_1 (h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3) + \\
 & + c_2 (l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3) = 0;
 \end{aligned}$$

$$\begin{aligned}
 (13) \quad & c_1 (h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6) + \\
 & + m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3 = 0;
 \end{aligned}$$

$$\begin{aligned}
 (23) \quad & c_2 (l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8) + \\
 & + m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2 = 0.
 \end{aligned}$$

Then, $\xi^A(x, x^5)$ explicitly reads, in this subcase:

$$\begin{aligned}
 \xi^0(x^1, x^2, x^3) &= \widetilde{F}_0(x^1, x^2, x^3) = \\
 &= -c_1(h_7 + e_2)x^1 - c_2(l_5 + e_4)x^2 - (m_5 + g_2)x^3 + (a_1 + d_1 + K_0);
 \end{aligned} \tag{C 157}$$

$$\begin{aligned}
 \xi^1(x^0, x^2, x^3) &= -\widetilde{F}_1(x^0, x^2, x^3) = \\
 &= -(h_7 + e_2)x^0 - h_3x^2 - h_6x^3 - (K_1 + h_5 + e_1);
 \end{aligned} \tag{C 158}$$

$$\begin{aligned}
 \xi^2(x^0, x^1, x^3) &= -\widetilde{F}_2(x^0, x^1, x^3) = \\
 &= -(l_5 + e_4)x^0 + \frac{c_1}{c_2}h_3x^1 - l_8x^3 - (l_7 + K_2 + e_3);
 \end{aligned} \tag{C 159}$$

$$\begin{aligned}
 \xi^3(x^0, x^1, x^2) &= -\widetilde{F}_3(x^0, x^1, x^2) = \\
 &= -(m_5 + g_2)x^0 + c_1h_6x^1 + c_2l_8x^2 - (m_1 + g_1 + c);
 \end{aligned} \tag{C 160}$$

$$\xi^5 = 0. \tag{C 161}$$

By introducing as usual the distribution $\widehat{\Theta}_R(x_0^5 - x^5)$, putting

$$\begin{aligned} \frac{B^1}{c_1} &\equiv \zeta^1, \frac{B^2}{c_2} \equiv \zeta^2, B^3 \equiv \zeta^3; \\ \frac{\Theta^1}{c_2} &\equiv \theta^1, \frac{\Theta^2}{c_1} \equiv \theta^2, \frac{\Theta^3}{c_2} \equiv \theta^3; \\ \Xi^0 &\equiv \zeta^5, \frac{\Xi^1}{c_1} \equiv \Xi^{1'}, \frac{\Xi^2}{c_2} \equiv \Xi^{2'}; \\ \frac{T^1}{c_1} &\equiv T^{1'}, \frac{T^2}{c_2} \equiv T^{2'}, \end{aligned} \quad (\text{C } 162)$$

and identifying

$$\begin{aligned} (a_1 + d_1 + K_0) &= T^0; \\ -(K_1 + h_5 + e_1) &= \frac{1}{c_1} T^1; \\ -(l_7 + K_2 + e_3) &= \frac{1}{c_2} T^2; \\ -(m_1 + g_1 + c) &= T^3; \\ h_7 + e_2 &= \frac{B_1}{c_1}; l_5 + e_4 = \frac{B^2}{c_2}; m_5 + g_2 = B^3; \\ l_8 &= -\frac{\Theta^1}{c_2}; h_6 = \frac{\Theta^2}{c_1}; h_3 = -\frac{\Theta^3}{c_1}, \end{aligned} \quad (\text{C } 163)$$

one gets the following general form of the Killing vector in case X, valid $\forall x^5 \in R_0^+$:

$$\xi^0(x^1, x^2, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [\zeta^5 F(x^5)] - c_1 \zeta^1 x^1 - c_2 \zeta^2 x^2 - \zeta^3 x^3 + T^0; \quad (\text{C } 164)$$

$$\xi^1(x^0, x^2, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [-\Xi^1 F(x^5)] - \zeta^1 x^0 + \frac{c_2}{c_1} \theta^3 x^2 - \theta^2 x^3 + T^1; \quad (\text{C } 165)$$

$$\xi^2(x^0, x^1, x^3, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [-\Xi^2 F(x^5)] - \zeta^2 x^0 - \theta^3 x^1 + \theta^1 x^3 + T^2; \quad (\text{C } 166)$$

$$\xi^3(x^0, x^1, x^2, x^5) = \widehat{\Theta}_R(x_0^5 - x^5) [-\Xi^3 F(x^5)] - \zeta^3 x^0 + c_1 \theta^2 x^1 - c_2 \theta^1 x^2 + T^3; \quad (\text{C } 167)$$

$$\begin{aligned} \xi^5(x, x^5) &= \\ &= \widehat{\Theta}_R(x_0^5 - x^5) \left\{ \mp (f(x^5))^{-\frac{1}{2}} [\zeta^5 x^0 + c_1 \Xi^1 x^1 + c_2 \Xi^2 x^2 + \Xi^3 x^3 - T^5] \right\}. \end{aligned} \quad (\text{C } 168)$$

The 5-d. Killing group in this range is the Poincaré one, suitably rescaled

$$[P(1, 3)_{STD.} = SO(1, 3)_{STD.} \otimes_s Tr.(1, 3)_{STD.}]|_{x^1 \rightarrow \sqrt{c_1} x^1, x^2 \rightarrow \sqrt{c_2} x^2} \quad (\text{C } 169)$$

as in the energy range $0 < x^5 \leq x_0^5$.

C12 Form XI

$$\begin{aligned}
& g_{AB,DR5,grav.}(x^5) = \\
& = \text{diag} \left(1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right], \right. \\
& \quad \left. -\frac{c_1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -\frac{c_2}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, \right. \\
& \quad \left. - \left\{ 1 + \Theta(x^5 - x_{0,grav.}^5) \left[\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right) \quad (C 170)
\end{aligned}$$

($c_1, c_2 \in R_0^+$. In gen.: $c_1 \neq 1, c_2 \neq 1, c_1 \neq c_2$).

C12.1 XI a)

Metric (C 170) reads

$$\begin{aligned}
& g_{AB,DR5}(x^5) = \\
& = \text{diag} \left(1, -\frac{c_1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -\frac{c_2}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2, -1, \pm f(x^5) \right). \quad (C 171)
\end{aligned}$$

Therefore

$$A_0(x^5) = A_3(x^5) = 0; \quad (C 172.1)$$

$$B_0(x^5) = B_3(x^5) = (f(x^5))^{\frac{1}{2}}; \quad (C 172.2)$$

$$\begin{aligned}
& A_i(x^5) = \\
& = -\frac{(c_i)^{\frac{3}{2}}}{8} \left(1 + \frac{x^5}{x_0^5} \right) \frac{x^5}{(x_0^5)^2} (f(x^5))^{-\frac{1}{2}} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right]; \quad (C 173) \\
& B_i(x^5) = \frac{(c_i)^{\frac{1}{2}}}{2} \left(1 + \frac{x_0^5}{x^5} \right) (f(x^5))^{\frac{1}{2}}, i = 1, 2;
\end{aligned}$$

$$\frac{\pm A_i(x^5)}{B_i(x^5)} = \mp \frac{c_i}{4} \frac{1}{f(x^5)} \frac{x^5}{(x_0^5)^2} \left[\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} + \frac{1}{2} \frac{x_0^5 f'(x^5)}{x^5 f(x^5)} \right], i = 1, 2. \quad (C 174)$$

So the Υ -hypothesis is satisfied for $\mu = 1, 2$ under condition (C 9), which ensures that the contravariant Killing vector $\xi^A(x, x^5)$ for the gravitational metric in this subcase is given

by Eqs.(118)-(122), where (some of) the real parameters are constrained by the system

$$\begin{aligned}
 (01) \quad & d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2) + \\
 & + \frac{c_1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 [h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + e_2)] = 0; \\
 (02) \quad & (d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3) + \\
 & + \frac{c_2}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 [l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4)] = 0; \\
 (03) \quad & (d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2) + \\
 & + [m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + (m_5 + g_2)] = 0;
 \end{aligned} \tag{C 175}$$

$$\begin{aligned}
 (12) \quad & c_1 (h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3) + \\
 & + c_2 (l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3) = 0;
 \end{aligned}$$

$$\begin{aligned}
 (13) \quad & \frac{c_1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 (h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6) + \\
 & + (m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3) = 0;
 \end{aligned}$$

$$\begin{aligned}
 (23) \quad & \frac{c_2}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 (l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8) + \\
 & + (m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2) = 0.
 \end{aligned}$$

From the solution of this system one gets the explicit form of the Killing 5-vector:

$$\xi^0(x^3) = \widetilde{F}_0(x^3) = -(m_5 + g_2) x^3 + (a_1 + d_1 + K_0); \tag{C 176}$$

$$\xi^1(x^2) = -\widetilde{F}_1(x^2) = -h_3 x^2 - (K_1 + h_5 + e_1); \tag{C 177}$$

$$\xi^2(x^1) = -\widetilde{F}_2(x^1) = \frac{c_1}{c_2} h_3 x^1 - (l_7 + K_2 + e_3); \tag{C 178}$$

$$\xi^3(x^0) = -\widetilde{F}_3(x^0) = -(m_5 + g_2) x^0 - (m_1 + g_1 + c); \tag{C 179}$$

$$\xi^5 = 0. \tag{C 180}$$

The metric (C 171) can be rewritten as

$$g_{AB,DR5,grav.}(x^5) = \text{diag} \left(\frac{1}{4} \left(1 + \frac{x^5}{x_{0,grav.}^5} \right)^2 g_{\mu\nu, \overline{M}_4}(x^5), \pm f(x^5) \right) \tag{C 181}$$

where \overline{M}_4 is the standard 4-d. Minkowski space with the coordinate scale transformation (C 3.1)-(C 3.2) introduced in Subsect.C.1.1. Therefore, in the energy range $0 < x^5 \leq x_0^5$ the 5-d. Killing group is the rescaled Poincaré group

$$[P(1,3)_{STD.} = SO(1,3)_{STD.} \otimes_s Tr.(1,3)_{STD.}]|_{x^1 \rightarrow \sqrt{c_1}x^1, x^2 \rightarrow \sqrt{c_2}x^2}. \quad (C 182)$$

C12.2 XI b)

In the energy range $x^5 > x_0^5$ the metric (C 170) coincides with that of case X) in the same range, Eq.(C 151). Therefore, the results for this case coincide with those obtained in Subsect.C.10.2. In particular, solution (C 157)-(C 161) holds for the contravariant Killing vector $\xi^A(x, x^5)$

The 5-d. Killing group in this range is

$$(SO(2)_{STD.(x^1 \rightarrow \sqrt{c_1}x^1, x^2 \rightarrow \sqrt{c_2}x^2)} \otimes B_{x^3, STD.}) \otimes_s Tr.(1,3)_{STD.}, \quad (C 183)$$

where $SO(2)_{STD.(x^1 \rightarrow \sqrt{c_1}x^1, x^2 \rightarrow \sqrt{c_2}x^2)} = SO(2)_{STD., \Pi(x^1 \rightarrow \sqrt{\frac{c_1}{c_2}}x^1)}$ is the 1-parameter group (generated by S_{SR}^3) of the 2-d. rotations in the plane $\Pi(x^1, x^2)$ characterized by the scale transformations (C 3.1)-(C 3.2), $B_{x^3, STD.}$ is the usual 1-parameter group (generated by K_{SR}^3) of the Lorentz boosts along \hat{x}^3 and $Tr.(1,3)_{STD.}$ is the group of standard space-time translations.

Since

$$\begin{aligned} & (SO(2)_{STD.(x^1 \rightarrow \sqrt{c_1}x^1, x^2 \rightarrow \sqrt{c_2}x^2)} \otimes B_{x^3, STD.}) \otimes_s Tr.(1,3)_{STD.} \subsetneq \\ & \subsetneq [P(1,3)_{STD.} = SO(1,3)_{STD.} \otimes_s Tr.(1,3)_{STD.}]|_{x^1 \rightarrow \sqrt{c_1}x^1, x^2 \rightarrow \sqrt{c_2}x^2} \end{aligned} \quad (C 184)$$

we can state that the present case XI is the only one — in the framework of the Ansatz of solution of Killing equations for the 5-d phenomenological metric of gravitational interaction — in which the 5-d. Killing group in the range $0 < x^5 \leq x_0^5$ is a proper (non-abelian) subgroup of the Killing group in the range $x^5 > x_0^5$.

By introducing the left distribution $\hat{\Theta}_L(x^5 - x_0^5)$ (see Eq.(63))¹⁹ and putting

$$\begin{aligned} (a_1 + d_1 + K_0) & \equiv T^0; \\ -(K_1 + h_5 + e_1) & \equiv T^1; \\ -(l_7 + K_2 + e_3) & \equiv T^2; \\ -(m_1 + g_1 + c) & \equiv T^3; \\ m_5 + g_2 & \equiv \zeta^3; l_5 + e_4 \equiv \zeta^2; h_7 + e_2 \equiv \zeta^1; \\ l_8 & \equiv -\theta^1; h_6 \equiv \theta^2; h_3 \equiv -\frac{c_2}{c_1}\theta^3, \end{aligned} \quad (C 185)$$

¹⁹ The use of the left distribution $\hat{\Theta}_L(x^5 - x_0^5)$ (instead of the right one $\hat{\Theta}_R(x_0^5 - x^5)$ used in all the other cases) is due to the fact already stressed that in the present case the 5-d. Killing group in the range $0 < x^5 \leq x_0^5$ is a proper (non-abelian) subgroup of the Killing group in the range $x^5 > x_0^5$. Indeed, let us recall the complementary nature of the two distributions, expressed by Eq.(65).

the contravariant Killing vector $\xi^A(x, x^5)$ of the gravitational metric in case XI can be written in the following form (valid $\forall x^5 \in R_0^+$):

$$\xi^0(x^1, x^2, x^3, x^5) = \widehat{\Theta}_L(x^5 - x_0^5) [-c_1 \zeta^1 x^1 - c_2 \zeta^2 x^2] - \zeta^3 x^3 + T^0; \quad (\text{C } 186)$$

$$\xi^1(x^0, x^2, x^3, x^5) = \widehat{\Theta}_L(x^5 - x_0^5) [-\zeta^1 x^0 - \theta^2 x^3] + \frac{c_2}{c_1} \theta^3 x^2 + T^1; \quad (\text{C } 187)$$

$$\xi^2(x^0, x^1, x^3, x^5) = \widehat{\Theta}_L(x^5 - x_0^5) [-\zeta^2 x^0 + \theta^1 x^3] - \theta^3 x^1 + T^2; \quad (\text{C } 188)$$

$$\xi^3(x^0, x^1, x^2, x^5) = \widehat{\Theta}_L(x^5 - x_0^5) [c_1 \theta^2 x^1 - c_2 \theta^1 x^2] - \zeta^3 x^0 + T^3; \quad (\text{C } 189)$$

$$\xi^5 = 0. \quad (\text{C } 190)$$

Notice that in this case the dependence of (C 186)-(C 190) on x^5 is fictitious, because actually ξ^μ depends on x^5 through the distribution $\widehat{\Theta}_L(x^5 - x_0^5)$ only.

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Noncommutative Lemaitre-Tolman-Bondi like Metric and Cosmology

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Abstract: Using noncommutative deformed canonical commutation relations, a model describing gravitation is constructed. A noncommutative Lemaitre-Tolman-Bondi like metric is proposed and non static solutions are discussed. It turns out that in spite of its smallness, the noncommutativity of the geometry plays an important role in unifying the dark matter and energy without any ad hoc assumption, giving a plausible explanation of matter-antimatter asymmetry and controlling the evolution of the universe.

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1. Introduction

Despite its success, the cosmological standard model " The Hot Big Bang " contains certain unsolved problems (horizon problems, relic particles abundances, etc..)[1] – [8]. This has led to some model extensions, such as the inflationary model [9] – [17]. Recently, another cyclic model giving an alternative possibility to explain the origin and dynamics of the universe evolution was proposed by Turok [18] – [19]. It differs from the inflationary and classical Friedman cosmological models in the sense that our universe undergoes periodic and infinite sequences of contractions " The Big Crunch " and expansions " The Big Bang ". During these cycles, the temperature and density of the universe remain finite and two hyper surfaces (branes) separated by a finite space extra dimension have enough energy and momenta to interact through gravity. Thus, the branes are brought together by these interactions and collide then bounce at regular intervals [18] – [19]. Furthermore, in the last two decades, non commutative geometry becomes the focus of interest of

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many research activities, especially those of model building [20] – [27]. There are several motivations to speculate that the space-time becomes non commutative at very short distances when quantum gravity becomes relevant and even better, if we believe that the extra dimension idea can push the non commutativity scale lower. Moreover, in string theories, the non commutative gauge theory appears as a certain limit in the presence of a background field [28] – [29]. One approach to the non commutative geometry (NCG) is the one based on the deformation of the space-time [20] – [27]. In this context, a gauge field theory with star products and Seiberg-Witten maps is used [22]. It provides a systematic way to compute various observables which may contain a signature to the hypothetical non commutative space-time structure. Another approach is to deform just the canonical relations between the space-time coordinates and their canonical conjugates while the remaining commutation relations vanish. The purpose of this paper is to build and discuss a model describing gravity within the noncommutative deformed canonical commutation relations approach. In section 2, we present the formalism and the non-commutative Lemaitre-Tolman-Bondi like metric. In section 3, we discuss the various resulted noncommutative models and scenarios. Finally in section 4, we present some of the results and draw our conclusions

2. Formalism

In what follows, we take $\hbar = c = 1$ and consider a noncommutative geometry, characterized by the space-time coordinates \hat{x}^μ and momenta \hat{p}^μ which are non commuting operators satisfying the following matrices valued commutation relations:

$$[\hat{x}_\mu, \hat{x}_\nu] = 0 \quad (1)$$

$$[\hat{x}_\mu, \hat{p}_\nu] = i(\delta_{\mu\nu}I + \theta_{\mu\nu}) \quad (2)$$

and

$$[\hat{p}_\mu, \hat{p}_\nu] = 0 \quad (3)$$

(I is 4×4 identity matrix) where $\theta_{\mu\nu}$ are matrices valued parameters and taken to be proportional to the Dirac matrices $\gamma_{\mu\nu}$ in the curved space-time such that:

$$\theta_{\mu\nu} = \frac{1}{4}\xi(x)\gamma_{\mu\nu} = \frac{1}{4}\xi(x)[\gamma_\mu, \gamma_\nu] \quad (4)$$

(here $\xi(x)$ is a scalar function of the space-time variable x_μ). Notice that although the above commutation relations do not fit into the case where the noncommutativity parameters are c-numbers, there is nothing fundamentally wrong with this choice.

2.1 Non commutative Gravity Model

The operators \hat{x}_ν and \hat{p}_ν have as representations:

$$\hat{x}_\nu = x_\nu, \quad \hat{p}_\nu = -i\hat{\partial}_\nu \quad (5)$$

where the noncommutative matrix derivative $\hat{\partial}_\nu$ has as expression

$$\hat{\partial}_\nu = \partial_\nu + i\theta_{\nu\alpha}\partial^\alpha \quad (6)$$

and

$$\partial^\alpha = \hat{g}^{\mu\alpha}\partial_\mu \quad (7)$$

(x_ν and ∂_ν are the ordinary coordinates and derivative respectively). $\hat{g}^{\mu\alpha}$ is the inverse of the noncommutative symmetric metric $\hat{g}_{\mu\alpha}$ (which is not a matrix) such that

$$\hat{g}_{\nu\mu}\hat{g}^{\mu\alpha} = \delta_\nu^\alpha \quad (8)$$

The modified affine connection (which is not Riemannian) denoted by $\hat{\Gamma}^\nu_{\mu\lambda}$ takes the form:

$$\hat{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2}\hat{g}^{\mu\nu} \left(\hat{\partial}_\beta\hat{g}_{\nu\alpha} + \hat{\partial}_\alpha\hat{g}_{\nu\beta} - \hat{\partial}_\nu\hat{g}_{\alpha\beta} \right) \quad (9)$$

which can be rewritten as:

$$\hat{\Gamma}^\mu_{\alpha\beta} = \bar{\Gamma}^\mu_{\alpha\beta} + \tilde{\Gamma}^\mu_{\alpha\beta} \quad (10)$$

where

$$\bar{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2}\hat{g}^{\mu\nu} (\partial_\beta\hat{g}_{\nu\alpha} + \partial_\alpha\hat{g}_{\nu\beta} - \partial_\nu\hat{g}_{\alpha\beta}) \quad (11)$$

and

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{i}{2}\hat{g}^{\mu\nu} (\theta_{\beta\sigma}\partial^\sigma\hat{g}_{\nu\alpha} + \theta_{\alpha\sigma}\partial^\sigma\hat{g}_{\nu\beta} - \theta_{\nu\sigma}\partial^\sigma\hat{g}_{\alpha\beta}) \quad (12)$$

Here $\tilde{\Gamma}^\mu_{\alpha\beta}$ represents a non metricity like a tensor. We remind the reader that in differential geometry, the affine connection on a differentiable manifold with a metric can be always decomposed into the sum of a Levi-Civita (metric) connection, a non metricity tensor and a torsion. This is the case of theories with more complicated geometrical structure like the Riemann-Cartan space with general metric-affine spaces (curvature, torsion and non-metricity) and the Weyl-Cartan space which is a connected differentiable manifold with a Lorentz metric obeying the Weyl non-metricity condition. In the Riemannian space of general relativity the metric and the connection (which are considered respectively as a potential and strength of the gravitational field) are linked through the requirement of metric homogeneity (metricity condition). The latter assures that the length of a vector transported parallel in any direction remains invariant. Regarding the noncommutative matrices curvature and Ricci tensors $\hat{R}^\sigma_{\lambda\mu\nu}$ and $\hat{R}_{\mu\nu}$, they are given by:

$$\widehat{R}^{\mu}_{\alpha\beta\lambda} = \widehat{\partial}_{\beta}\widehat{\Gamma}^{\mu}_{\alpha\lambda} - \widehat{\partial}_{\lambda}\widehat{\Gamma}^{\mu}_{\alpha\beta} + \widehat{\Gamma}^{\mu}_{\sigma\beta}\widehat{\Gamma}^{\sigma}_{\alpha\lambda} - \widehat{\Gamma}^{\mu}_{\sigma\lambda}\widehat{\Gamma}^{\sigma}_{\alpha\beta} \quad (13)$$

and

$$\widehat{R}_{\mu\nu} = \widehat{R}^{\lambda}_{\mu\lambda\nu} = \widehat{\partial}_{\nu}\widehat{\Gamma}^{\lambda}_{\mu\lambda} - \widehat{\partial}_{\lambda}\widehat{\Gamma}^{\lambda}_{\mu\nu} + \widehat{\Gamma}^{\lambda}_{\mu\sigma}\widehat{\Gamma}^{\sigma}_{\lambda\nu} - \widehat{\Gamma}^{\sigma}_{\mu\nu}\widehat{\Gamma}^{\lambda}_{\sigma\lambda} \quad (14)$$

We can also define a noncommutative matrix Einstein tensor $\widehat{G}_{\mu\nu}$ as:

$$\widehat{G}_{\mu\nu} = \widehat{R}_{\mu\nu} - \frac{1}{2}\widehat{g}_{\mu\nu}\widehat{R} \quad (15)$$

where the non commutative matrix scalar curvature \widehat{R} is :

$$\widehat{R} = \widehat{g}^{\mu\nu}\widehat{R}_{\mu\nu} \quad (16)$$

Now, we define the noncommutative Hilbert-Einstein I_{NCG} by

$$I_{NCG} = \frac{1}{64\pi\kappa} \int d^4x \sqrt{\widehat{g}}\widehat{g}^{\mu\nu} Tr\widehat{R}_{\mu\nu} \quad (17)$$

Where κ is the gravitational constant, \widehat{g} stands for $|\det(\widehat{g}_{\mu\nu})|$ and Tr is the trace over the gamma Dirac matrices. Since $Tr\theta^{\beta\sigma} = 0$, thus the terms linear in $\theta^{\beta\sigma}$ do not contribute in the expression of $Tr\widehat{R}^{\mu}_{\alpha\beta\lambda}$ and therefore:

$$\mathcal{R}_{\alpha\lambda} \equiv Tr\widehat{R}_{\alpha\lambda} = \overline{R}_{\alpha\lambda} + Tr\widetilde{R}_{\alpha\lambda} \quad (18)$$

where $\widetilde{R}_{\mu\nu}$ and $\overline{R}_{\mu\nu}$ are given by:

$$\widetilde{R}_{\mu\nu} = \left(i\theta_{\beta\sigma}\partial^{\sigma}\widetilde{\Gamma}^{\beta}_{\mu\nu} - i\theta_{\nu\sigma}\partial^{\sigma}\widetilde{\Gamma}^{\beta}_{\mu\beta} + \widetilde{\Gamma}^{\beta}_{\sigma\beta}\widetilde{\Gamma}^{\sigma}_{\mu\nu} - \widetilde{\Gamma}^{\beta}_{\sigma\nu}\widetilde{\Gamma}^{\sigma}_{\mu\beta} \right) \quad (19)$$

and

$$\overline{R}_{\mu\nu} = 4(\partial_{\nu}\overline{\Gamma}^{\lambda}_{\mu\lambda} - \partial_{\lambda}\overline{\Gamma}^{\lambda}_{\mu\nu} + \overline{\Gamma}^{\lambda}_{\mu\sigma}\overline{\Gamma}^{\sigma}_{\lambda\nu} - \overline{\Gamma}^{\sigma}_{\mu\nu}\overline{\Gamma}^{\lambda}_{\sigma\lambda}) \quad (20)$$

It is worth to mention that it is easy to show that, the principle of a least action leads to the following noncommutative Einstein field equations in the vacuum:

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}^{\alpha}_{\alpha} = 0 \quad (21)$$

which is equivalent to:

$$G_{\mu\nu} = \kappa\mathcal{T}_{\mu\nu} \quad (22)$$

where $G_{\mu\nu}$ has the form:

$$G_{\mu\nu} = \overline{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overline{R}^{\alpha}_{\alpha} \quad (23)$$

and $\mathcal{T}_{\mu\nu}$ is an effective matter energy-momentum tensor induced by the non commutativity of the space and has as expression:

$$\mathcal{T}_{\mu\nu} = \frac{-1}{\kappa} (\text{Tr} \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \text{Tr} \tilde{R}^{\alpha}_{\alpha}) \quad (24)$$

This means that the non commutativity deforms the space and generates a more complex structure with a non metricity contributing to the field equations and induce an effective macroscopic matter energy-momentum tensor as an additional source of gravity. This is not surprising about the role of the deformed canonical commutation relations in quantum mechanics where it was shown in ref.[27], that there exists an intimate connection to the curved space. Moreover, a suitable choice of the position-momentum commutator can elegantly describe many features of gravity, including the IR/UV correspondence and dimensional reduction (holography)[28]. In what follows, we set:

$$\xi^2(x) = 4\eta^2\phi(x) \quad (25)$$

($\eta \ll 1$ is a constant parameter which characterizes the space noncommutativity). Since $\hat{g}_{\nu\alpha}$ is a solution of the field equations, it is assumed to have the form:

$$\hat{g}_{\nu\alpha} = g_{\nu\alpha} + \eta^2 g_{\nu\alpha}^{(1)} \quad (26)$$

where $g_{\nu\alpha}$ is the classical Einstein Riemannian metric and $g_{\nu\alpha}^{(1)}$ is a non commutative correction to be determined later. Furthermore, to simplify our calculation, we assume

that the only non vanishing matrix valued parameters are θ_{01} (of course our qualitative results remain valid in the general case). Then, it is easy to show that at the $O(\eta^2)$, one has:

$$\frac{1}{4} \text{Tr} \theta_{01} \theta_{01} \approx \eta^2 \phi(x) (g_{01} g_{01} - g_{00} g_{11}) \quad (27)$$

It is important to mention that the noncommutative Hilbert-Einstein action given in eq.(17) is invariant under general coordinate transformations.

2.2 Non Commutative Lemaitre-Tolman-Bondi Like Metric:

Standard cosmology is based on Friedman-Lemaitre-Robertson-Walter (FLRW) models which are characterized by spatially homogeneous and isotropic geometry and a matter content represented by a perfect fluid although it was proven by Tupper [30] the possibility of associating to the same metric a more general non perfect fluid with a heat conductor vector and anisotropic pressure tensor. FLRW universes were very successful in explaining the major features of the observed universe (observed galactic redshifts, remnant black body radiation and element abundance predictions and observations). However, these models do not describe the real universe well in an essential way, in that the highly idealized degree of symmetry does not correspond exactly to the real universe. Thus, they can serve as basic models giving the largest-scale smoothed out features of the observable physical universe, but one needs to perturb them to get realistic ('almost-FLRW') universe models that can be used to examine the inhomogeneities and anisotropies arising

during structure formation, and that can be compared in detail with observations. It is worth to mention that small anisotropies in the microwave background imply that the universe is almost FLRW. We remind also that the FLRW universe is characterized by two functions, the Hubble rate H and the density parameter ρ which depend only on time but are independent of the spatial location. However, one should keep in mind that their values cannot be extracted directly from the observations but must be deduced from the properties of light coming from the past light cone. In the context of the FLRW model this is almost trivial, since the redshift and scale factor are everywhere related. This theoretical simplicity should however not cloud the fact that all cosmological parameter determination requires an element of interpretation of the data. Of course, the FLRW interpretation of the properties of the past light cone has served cosmology well, giving a good fit to observations and, until the late 90's, implying a matter dominated universe with a density $\Omega \approx \Omega_M$. (Ω_M is a matter density). The situation changed dramatically

with the WMAP [31] and distant supernova data,[32], [33]. In fact, considering the recent data from supernovae [34], [35], galaxy distributions [36] and anisotropies of the cosmic microwave background [37], the simplest FLRW model would lead to a highly contradictory picture of the universe. With the actual best fit values for the average matter density, one has $\Omega_M \sim 1$ from the Cosmic microwave background, $\Omega_M \sim 0.3$ from the Galaxy surveys and $\Omega_M \sim 0$ from type Ia supernovae. The discrepancies between the

different data sets have conventionally been remedied by introducing the cosmological constant or vacuum energy to the Einstein equations. This gives rise to an accelerated expansion of the universe. As a consequence, the apparent dimming of the luminosity of distant supernovae finds, in the context of perfectly homogeneous universe, a natural explanation. However, although the cosmological concordance where the CDM-model [38] fits the observations well, there is no theoretical understanding of the origin of the cosmological constant or its magnitude. Despite a large number of different dark energy models[38], [39] which attempt to provide a dynamical explanation for the cosmological constant, but none of them are compelling from particle physics point of view and very often they require fine-tuning. Facing such difficulties, one might be tempted to consider a relinquishing of the FLRW assumption concerning the perfect homogeneity of the universe. After all, inhomogeneities are abundant in the universe: there are not only clusters of galaxies but also large voids. Since general relativity is a non-linear theory, any relatively small local inhomogeneities with a sufficiently large density contrast could in principle give rise to cosmological evolution that is not accessed by the usual cosmological perturbation theory in an FLRW background. In fact, the potentially interesting consequences of the inhomogeneities were recognized already at the time when the homogeneous and isotropic models of the universe were first studied, but their impact on the global dynamics of the universe is still largely unknown [40]. Now, can the acceleration of the universe be just a trick of light, a misinterpretation that arises due to the oversimplification of the real, inhomogeneous universe inherent in the FLRW model?. Light, while traveling through inhomogeneities, does not see the average Hubble expansion but

rather feels its variations, which could sum up to an important correction. This effect is particularly important for the case of large scale inhomogeneities. This will be a good motivation for our work which consists essentially to consider a spherically symmetric universe with radial inhomogeneities due to the noncommutativity of the space geometry manifested through the deformation of the canonical commutation relations of eqs.(1)-(3). Choosing spatial coordinates to comove ($dx_i/dt = 0$) with the matter, the spatial origin ($x_i = 0$) as the symmetry center, and the time coordinate ($x_0 \equiv t$) to measure the proper time of the comoving fluid, the line element takes the general form [41]– [43].

$$ds^2 = -dt^2 + \widehat{U}(r, t) dr^2 + \widehat{V}(r, t) (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (28)$$

where the functions $\widehat{U}(r, t)$ and $\widehat{V}(r, t)$ have both temporal and spatial dependence and parametrized as:

$$\widehat{U}(r, t) = e^{\beta(r,t)}, \quad \widehat{V}(r, t) = e^{\alpha(r,t)} \quad (29)$$

(r, θ and φ stand for the spherical coordinates and $\alpha(r, t)$ and $\beta(r, t)$ are functions to be determined later). The metric in eq.(28) is called the Lemaitre-Tolman-Bondi (LTB) metric. The noncommutative metric tensor components read:

$$\widehat{g}_{00} = \widehat{g}_{tt} = -1, \quad \widehat{g}_{11} = \widehat{g}_{rr} = e^{\beta}, \quad \widehat{g}_{22} = \widehat{g}_{\theta\theta} = e^{\alpha}, \quad \widehat{g}_{33} = \widehat{g}_{\varphi\varphi} = \widehat{g}_{22} \sin^2 \theta \quad (30)$$

By setting

$$\phi(x) = \sigma(x)e^{-\beta} \quad (31)$$

one can show easily that the trace of the noncommutative Ricci tensor $Tr \widetilde{R}_{\mu\nu}$ components have the following expressions:

$$Tr \widetilde{R}_{01} = -\frac{1}{2}\eta^2 \sigma e^{-\beta} \left\{ 2\dot{\alpha}' - 2\alpha'\dot{\beta} + (\alpha' - \beta')\dot{\alpha} \right\} - \frac{1}{2}\eta^2 \dot{\sigma} \alpha' e^{-\beta} \quad (32)$$

$$Tr \widetilde{R}_{00} = -\frac{1}{2}\eta^2 \sigma e^{-2\beta} \left(2\alpha'' + \beta'' + \frac{1}{2}(q-2)\beta'^2 + q\alpha'^2 - 2\alpha'\beta' \right) - \frac{1}{4}\eta^2 e^{-\beta} (\beta' + 2\alpha') \dot{\sigma} \quad (33)$$

$$Tr \widetilde{R}_{11} = \frac{1}{2}\eta^2 \sigma e^{-\beta} \left\{ \beta'' - \frac{1}{2}\beta'^2 + \alpha'\beta' \right\} - \frac{1}{2}\eta^2 \sigma \left(2\ddot{\sigma} + \dot{\alpha}^2 - \dot{\alpha}\dot{\beta} \right) - \frac{1}{2}\eta^2 \dot{\alpha}\dot{\sigma} - \frac{1}{4}\eta^2 e^{-\beta} \dot{\sigma} \quad (34)$$

$$Tr \widetilde{R}_{22} = \frac{1}{2}\eta^2 \sigma e^{-2\beta} e^{\alpha} \left\{ \alpha'' + \alpha'^2 - \frac{1}{2}\alpha'\beta' \right\} - \frac{1}{4}\eta^2 e^{\alpha-\beta} \dot{\alpha} + \frac{1}{4}\eta^2 e^{-2\beta} e^{\alpha} \alpha' \sigma' \quad (35)$$

$$-\frac{1}{2}\eta^2 \sigma e^{\alpha-\beta} \left(\ddot{\alpha} + \frac{3}{2}\dot{\alpha}^2 - \frac{1}{2}\dot{\alpha}\dot{\beta} \right) \dot{\sigma} \quad (36)$$

$$Tr \widetilde{R}_{33} = Tr \widetilde{R}_{22} \sin^2 \theta \quad (37)$$

and the non vanishing components of $\overline{R}_{\alpha\lambda}$ are given by :

$$\bar{R}_{01} = \dot{\alpha}' + \frac{1}{2}\dot{\alpha}\alpha' - \frac{1}{2}\alpha'\dot{\beta} \quad (38)$$

$$\bar{R}_{00} = \frac{1}{2} \left\{ \ddot{\beta} + 2\ddot{\alpha} + \frac{1}{2}\dot{\beta}^2 + \dot{\alpha}^2 \right\} \quad (39)$$

$$\bar{R}_{11} = \alpha'' + \frac{1}{2}\alpha'^2 - \frac{1}{2}\alpha'\beta' - \frac{1}{2}e^{\beta} \left\{ \ddot{\beta} + \frac{1}{2}\dot{\beta} + 2\dot{\alpha}\dot{\beta} \right\} \quad (40)$$

$$\bar{R}_{22} = -1 + \frac{1}{2}e^{\alpha-\beta} \left\{ \alpha'' + \alpha'^2 - \frac{1}{2}\alpha'\beta' \right\} - \frac{1}{2}e^{\alpha} \left\{ \ddot{\alpha} + \dot{\alpha}^2 + \frac{1}{2}\dot{\alpha}\dot{\beta} \right\} \quad (41)$$

and

$$\bar{R}_{33} = \bar{R}_{22} \sin^2 \theta \quad (42)$$

Here the notations $''$, $'\prime$ and $'''$ are for the first space, time and second space derivatives respectively. In what follows, we take $\sigma = \sigma(r, t)$ and assume the following separable form of the functions α and β :

$$\alpha = \alpha(r, t) = \alpha(r) + a(t) \quad (43)$$

$$\beta = \beta(r, t) = \beta(r) + b(t) \quad (44)$$

Then, we look for perturbative solutions around the classical ones at the $O(\eta^2)$ with respect to the noncommutativity parameter η^2 that is:

$$\alpha(r) \approx \alpha_0(r) + \eta^2 \alpha_1(r) \equiv \alpha_0 + \eta^2 \alpha_1 \quad (45)$$

$$\beta(r) \approx \beta_0(r) + \eta^2 \beta_1(r) \equiv \beta_0 + \eta^2 \beta_1 \quad (46)$$

$$a(t) \approx a_0(t) + \eta^2 a_1(t) \quad (47)$$

and

$$b(t) \approx b_0(t) + \eta^2 b_1(t) \quad (48)$$

(here $a_0(t) = b_0(t)$). The functions $\alpha_0(r)$, $\beta_0(r)$ and $a_0(t)$ are the classical solutions of the Friedman equations[44] – [48]. As in the commutative case, the noncommutative Einstein equations in the presence of matter and a cosmological constant are given by:

$$Tr \hat{G}_{\mu\nu} = 8\pi\kappa \tilde{T}_{\mu\nu} \quad (49)$$

where in our model, the shifted noncommutative momenta-energy stress tensor $\tilde{T}_{\mu\nu}$ is assumed to have the form:

$$\tilde{T}_{\mu\nu} = \tilde{P} \hat{g}_{\mu\nu} + (\tilde{\rho} + \tilde{P}) u_\mu u_\nu \quad (50)$$

with

$$\tilde{P} = P - \frac{\Lambda}{8\pi\kappa} \quad (51)$$

and

$$\tilde{\rho} = \rho + \frac{\Lambda}{8\pi\kappa} \quad (52)$$

(κ is the Newton constant and $P = P(t)$, $\rho = \rho(t)$ and u^μ denote the non commutative pressure, energy density and the four-vector velocity of the particle). It is to be noted that eq.(50) does not mean that our universe is filled with a perfect fluid since $P(r, t)$ and $\rho(r, t)$ are assumed to depend also on the noncommutativity parameter η and can be expressed in a perturbative expansion around the classical values $P_0(t)$ and $\rho_0(t)$ (functions only of time) respectively as:

$$\rho(r, t) = \rho_0(t) + \eta^2 \rho_1(t) \quad (53)$$

and

$$P(r, t) = P_0(t) + \eta^2 P_1(t) \quad (54)$$

After straightforward simplifications and up to the $O(\eta^2)$, the noncommutative Einstein equations lead to the following independent differential equations:

$$\left(\dot{a}_1 - \dot{b}_1\right) e^{a_0} + \dot{a}_0 \sigma - \dot{\sigma} = 0 \quad (55)$$

$$2\ddot{a}_1 + \ddot{b}_1 + 2\dot{a}_0 \dot{a}_1 + \dot{a}_0 \dot{b}_1 - \sigma e^{-2a_0} \{2\alpha_0'' + \alpha_0'^2\} - e^{-a_0} \alpha_0' \sigma' = -8\pi\kappa \left(\tilde{\rho}_1 + 3\tilde{P}_1\right) \quad (56)$$

$$-\frac{2}{r}\beta_1' - \left(\ddot{b}_1 + 2\dot{a}_0 \dot{b}_1 + \dot{a}_0 \dot{a}_1\right) e^{a_0} - 2\sigma \ddot{a}_0 - \dot{a}_0 \dot{\sigma} = -8\pi\kappa \left(\tilde{\rho}_1 - \tilde{P}_1\right) e^{a_0} \quad (57)$$

and

$$\begin{aligned} -\frac{1}{r}\beta_1' - \frac{2}{r^2}\beta_1 + \frac{2}{r^2}(a_1 - b_1 + \sigma e^{-a_0}) + \frac{1}{r}e^{-a_0}\sigma' - \sigma(\ddot{a}_1 + \dot{a}_0^2) - \frac{1}{2}\dot{a}_0 \dot{\sigma} \\ -(\ddot{a}_1 + \frac{5}{2}\dot{a}_0 \dot{a}_1 + \frac{1}{2}\dot{a}_0 \dot{b}_1)e^{a_0} = -8\pi\kappa \left(\tilde{\rho}_1 - \tilde{P}_1\right) e^{a_0} \end{aligned} \quad (58)$$

where a_0 satisfies the commutative classical equations[33] – [34] :

$$2\ddot{a}_0 + \dot{a}_0^2 = -\frac{16\pi\kappa}{3} \left(\tilde{\rho}_0 + 3\tilde{P}_0\right) \quad (59)$$

$$\dot{a}_0^2 = \frac{32\pi\kappa}{3} \tilde{\rho}_0 - 4ke^{-a_0} \quad (60)$$

$\tilde{\rho}_0$ and \tilde{P}_0 are the shifted classical matter density and pressure respectively. Notice that the non linear coupled differential equations (55)-(58) are very complicated to be solved.

Thus, in order to make some simplifications and keep our calculations clear and transparent (since we are interested just in a qualitative study of the noncommutativity effect) we consider only the case where the integration constant $k = 0$ (present in the spatial part of the classical FLRW metric) and take $\beta_0 = \alpha_1 = \sigma' = 0$. Therefore, the noncommutative Einstein equations of eqs.(55)-(58) take the form:

$$\left(\dot{a}_1 - \dot{b}_1\right) e^{a_0} + \dot{a}_0 \sigma - \dot{\sigma} = 0 \quad (61)$$

$$2\ddot{a}_1 + \ddot{b}_1 + 2\dot{a}_0 \dot{a}_1 + \dot{a}_0 \dot{b}_1 = -8\pi\kappa \left(\tilde{\rho}_1 + 3\tilde{P}_1\right) \quad (62)$$

$$-\frac{2}{r}\beta_1' - \left(\ddot{b}_1 + 2\dot{a}_0 \dot{b}_1 + \dot{a}_0 \dot{a}_1\right) e^{a_0} - \left(2\sigma\ddot{a}_0 + \dot{a}_0 \dot{\sigma}\right) = -8\pi\kappa \left(\tilde{\rho}_1 - \tilde{P}_1\right) e^{a_0} \quad (63)$$

and

$$\begin{aligned} -\frac{1}{r}\beta_1' - \frac{2}{r^2}\beta_1 + \frac{2}{r^2} \{a_1 - b_1 + \sigma e^{-a_0}\} - \left\{\ddot{a}_1 + \frac{5}{2}\dot{a}_0 \dot{a}_1 + \frac{1}{2}\dot{a}_0 \dot{b}_1\right\} e^{a_0} \\ - \frac{1}{2} \left\{(2\ddot{a}_0 + 2\dot{a}_0^2) \sigma + \dot{a}_0 \dot{\sigma}\right\} = -8\pi\kappa \left(\tilde{\rho}_1 - \tilde{P}_1\right) e^{a_0} \end{aligned} \quad (64)$$

After direct simplifications we obtain:

$$\sigma(t) = A \exp a_0 \quad (65)$$

and therefore, eq.(61) leads to:

$$b_1(t) = a_1(t) \quad (66)$$

The β_1 function satisfies the following differential equations:

$$-\frac{2}{r}\beta_1' = k_1 \quad (67)$$

and

$$-\frac{1}{r}\beta_1' - \frac{2}{r^2}\beta_1 + \frac{2c}{r^2} = k_2 \quad (68)$$

Eqs. (67) and (68) lead to:

$$\beta_1 = -\frac{1}{4}k_1 r^2 + D \quad (69)$$

and

$$k_1 = k_2, \quad c = 1 \quad (70)$$

where k_1, k_2, c and D are integration constants. Thus eqs.(62)-(64) can be rewritten as:

$$3\ddot{a}_1 + 3\dot{a}_0\dot{a}_1 = -8\pi\kappa \left(\tilde{\rho}_1 + 3\tilde{P}_1 \right) \quad (71)$$

$$k_1 + \left(\ddot{a}_1 + 3\dot{a}_0\dot{a}_1 \right) e^{a_0} + A \left(2\ddot{a}_0 + \dot{a}_0^2 \right) e^{a_0} = 8\pi\kappa \left(\tilde{\rho}_1 - \tilde{P}_1 \right) e^{a_0} \quad (72)$$

and

$$k_2 + \left(\ddot{a}_1 + 3\dot{a}_0\dot{a}_1 \right) e^{a_0} + A \left(\ddot{a}_0 + \frac{3}{2}\dot{a}_0^2 \right) e^{a_0} = 8\pi\kappa \left(\tilde{\rho}_1 - \tilde{P}_1 \right) e^{a_0} \quad (73)$$

Notice that from the last two eqs.(72) and (73), we deduce that

$$\ddot{a}_0 - \frac{1}{2}\dot{a}_0^2 = 0 \quad (74)$$

Since the classical Hubble parameter $H_0(t)$ is defined as:

$$H_0(t) = \frac{1}{2}\dot{a}_0 \quad (75)$$

it is easy to show that:

$$\dot{H}_0 - H_0^2 = 0 \quad (76)$$

and consequently

$$H_0^2 = B e^{a_0} \quad (77)$$

(B is a positive integration constant). Moreover the solution of eq.(77) gives:

$$H_0(t) = -\frac{1}{t+c} \quad (78)$$

and therefore,

$$R_0(t) = -BH_0 = +\frac{B}{t+c} \sim \frac{1}{t} \quad (79)$$

Now, from eqs.(61) and(62) one can show easily that:

$$\dot{a}_0^2 = -\frac{8\pi\kappa}{3} \left(\tilde{\rho}_0 + 3\tilde{P}_0 \right) = \frac{32\pi\kappa}{3} \tilde{\rho}_0 \quad (80)$$

implying that :

$$5\tilde{\rho}_0 + 3\tilde{P}_0 = 0 \quad (81)$$

or equivalently:

$$5\rho_0 + 3P_0 = -\frac{2\Lambda_0}{8\pi\kappa} \quad (82)$$

Notice that the result in eq.(82) does not seem that it is a consequence of the space

deformation. The left or right hand sides do not disappear when the noncommutativity parameter vanishes. In fact, the situation is similar to the one loop calculation of the beta function in a noncommutative QED where the contribution coming from the non planar Feynman diagrams give a result which does not depend on the noncommutativity parameter [49]. Thus, one has to be carefull in the sense that when the noncommutativity parameter vanishes we do not have at all eq. (82). Now, it is easy to show that the function $\tilde{\rho}_0(t)$ gets the form:

$$\tilde{\rho}_0(t) = \frac{3}{8\pi\kappa} \left(\frac{1}{t+c} \right)^2 \quad (83)$$

The result in eq. (82) is very interesting. It shows a kind of unification between dark energy and dark matter imposed by the noncommutativity of the geometry and manifested by a relation between the cosmological constant Λ_0 pression P_0 and matter density ρ_0 respectively. Notice also that if $\Lambda_0 \geq 0$ (Friedman or De Sitter like space), we are dealing with a fluid which satisfies the constraint:

$$P_0 \leq -\frac{5}{3}\rho_0 \quad (84)$$

It is worth to mention that, the pressure P_0 is not subject to the same constraint since it can be related to the density ρ_0 by an adiabatic index γ through the state equation:

$$P_0 = \gamma\rho_0$$

or another type of dark energy, the so-called Chaplygin gas which obeys an equation of state like[50] – [51]

$$P_0 = A\rho_0 - B/\rho_0, (A, B > 0) \quad (85)$$

In our case $\gamma \leq -\frac{5}{3}$ and since the matter energy density is defined semi-positive, we will get a negative pression and violate the so called strong energy condition. This, will allow for an expanding universe with an accelerating rate. However, if $\Lambda_0 < 0$ (anti De Sitter like space), γ has to be greater than $-\frac{5}{3}$, if the pressure is negative. It is worth to mention

that recent observations of the luminosity of type Ia Supernovae indicate[52]– [53] an accelerated expansion of the Universe and lead to the search for a new type of matter which violates the strong energy condition i.e., $P_0 < 0$. The matter content responsible for such a condition to be satisfied at a certain stage of evolution of the universe is referred to as a dark energy. Now, the NCG field eqs.(71), (72) and (73) can be rewritten as:

$$6\dot{H}_1 + 12H_0H_1 = -8\pi\kappa \left(\tilde{\rho}_1 + 3\tilde{P}_1 \right) \quad (86)$$

and

$$2\dot{H}_1 + 12H_0H_1 + 8AH_0^2 + \bar{k}_1H_0^{-2} = 8\pi\kappa \left(\tilde{\rho}_1 - \tilde{P}_1 \right) \quad (87)$$

($\bar{k}_1 = Bk_1$). To look for a solution, we consider also for the non commutative corrections, an equation of state of the form $\tilde{P}_1 = \varpi\tilde{\rho}_1$. Then, eqs.(86) and(87) lead to:

$$4\dot{H}_1 - \frac{12}{t}(1 + \varpi)H_1 + \frac{4A}{t^2}(1 + 3\varpi) + \frac{1}{2k_1}(1 + 3\varpi)t^2 = 0 \quad (88)$$

where the general solution has the following form:

$$H_1(t) = \frac{1}{2}\dot{a}_1 = B_1 t^{3(1+\varpi)} + A \frac{(1+3\varpi)}{(4+3\varpi)} t^{-1} + \frac{1}{8k_1} \frac{(1+3\varpi)}{3\varpi} t^3 \quad (89)$$

implying that:

$$\frac{1}{2}a_1(t) = \frac{B_1}{4+3\varpi} t^{4+3\varpi} + A \frac{(1+3\varpi)}{(4+3\varpi)} \ln t + \frac{1}{32k_1} \frac{(1+3\varpi)}{3\varpi} t^4 + B_2 \quad (90)$$

and consequently, the noncommutative universe radius $R(t)$ takes the form:

$$R(t) = \frac{B}{t} \exp \eta^2 \left(\frac{B_1}{4+3\varpi} t^{4+3\varpi} + A \frac{(1+3\varpi)}{(4+3\varpi)} \ln t + \frac{1}{32k_1} \frac{(1+3\varpi)}{3\varpi} t^4 + B_2 \right) \quad (91)$$

It is worth to mention that the difference between the conventional Friedman equation and its LTB generalization, is that all the quantities in the LTB case depend in addition to the time t , on the r coordinate. Thus in the presence of inhomogeneities, the values of the Hubble rate and the matter density can vary at every spatial point. As a consequence, the inhomogeneities are of two physically different kinds: inhomogeneities in the matter distribution, and inhomogeneities in the expansion rate. In our case, we have considered only the latter case.

3. Results and Conclusions

Throughout this work, we conclude that the noncommutative cosmological dynamics differs from the classical one and the noncommutativity of the geometry plays an important role in the origin and evolution of our universe. Despite the smallness of the noncommutativity parameter, the resulted effects at high energies are very important. Various models are obtained depending on the free integration parameters of the noncommutative LTB equations. In fact and independently of the various parameters, the Non commutative universe radius $R(t)$ tends to infinity when $t \rightarrow 0$. In fact

$$R(t) \stackrel{t \rightarrow 0}{\approx} \frac{B}{t} \exp \left(\eta^2 A \frac{(1+3\varpi)}{(4+3\varpi)} \ln t \right) \stackrel{t \rightarrow 0}{\rightarrow} +\infty \quad (92)$$

For $t \rightarrow +\infty$, $R(t)$ behaves as

$$R(t) \stackrel{t \rightarrow +\infty}{\approx} \frac{B}{t} \exp \left(\eta^2 \frac{B_1}{4+3\varpi} t^{4+3\varpi} \right) \quad (93)$$

Thus, the evolution of the universe depends on the sign of the parameter B_1 .

- $B_1 < 0$: In this case, the universe starts by a " Big Crunch " at the initial time $t = 0$ until a total collapse at an infinite time. This situations is illustrated in *fig*

(1), *fig* (2) and *fig* (3), where we have set $\eta^2 = 10^{-2}$, $B_1 = -1$, $B_2 = \bar{k}_1 = 0$, with $\varpi = 0, 0, \frac{1}{3}$ and $A = +1, -1, -1$ respectively. Notice that this model does not correspond to our actual universe but a stellar gaz in a collapsing phase.

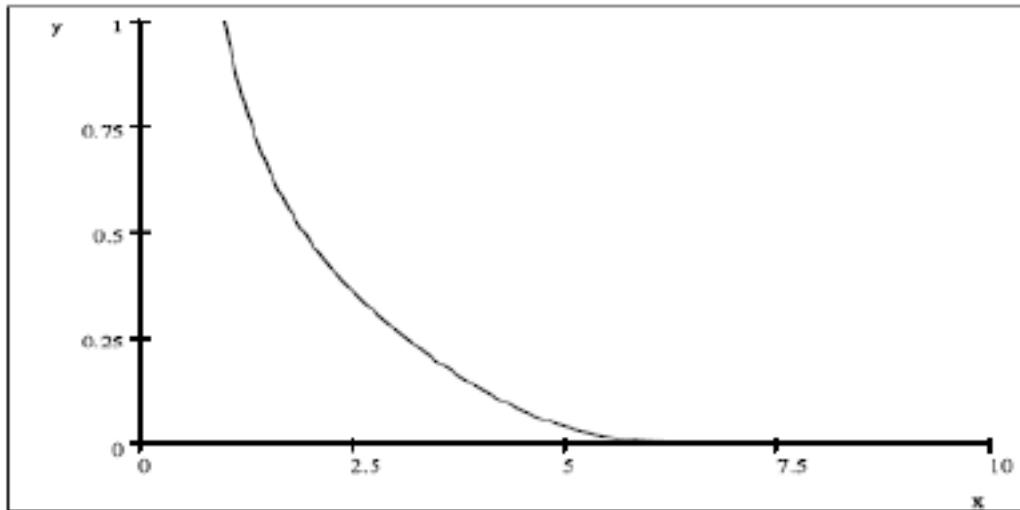


Fig. 1

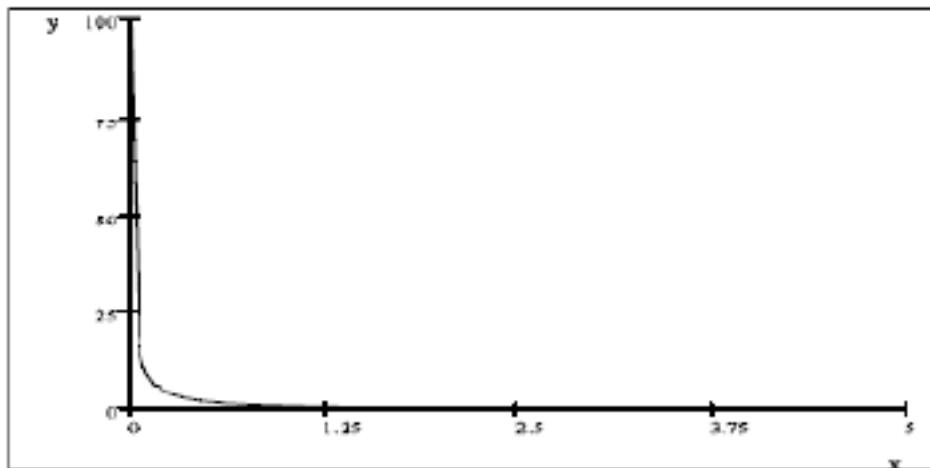


Fig. 2

- $B_1 \geq 0$: In this case, the universe starts with a " Big Crunch " at the initial time $t = 0$ where the universe radius $R(t)$ decreases until a minimal value $R(t_1)$, followed by an expansion " Big Bang " till infinity. This evolution is illustrated in *fig* (4) where we have set $\eta^2 = 10^{-2}$, $\varpi = 0$, $B_1 = +1$, $A = +1$, $B_2 = \bar{k}_1 = 0$, and in *fig* (5) where we have taken $\eta^2 = 10^{-2}$, $\varpi = \frac{1}{3}$, $B_1 = +1$, $A = +1$, $B_2 = \bar{k}_1 = 0$.

The dynamics and the origin of the universe in this model is similar to that of the cyclic model of Turok where the universe starts with a " Big Crunch " followed by a " Big Bang " in infinite cyclic sequences. Notice that in our model, the universe expansion

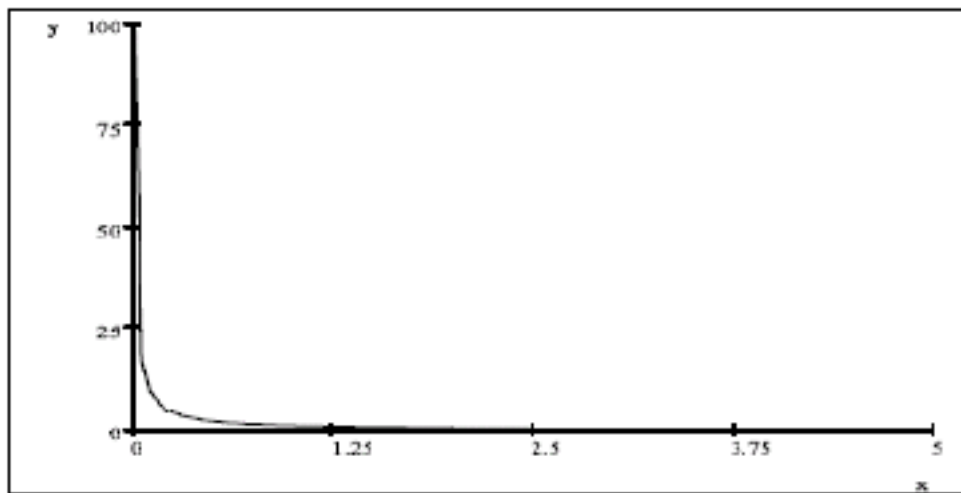


Fig. 3

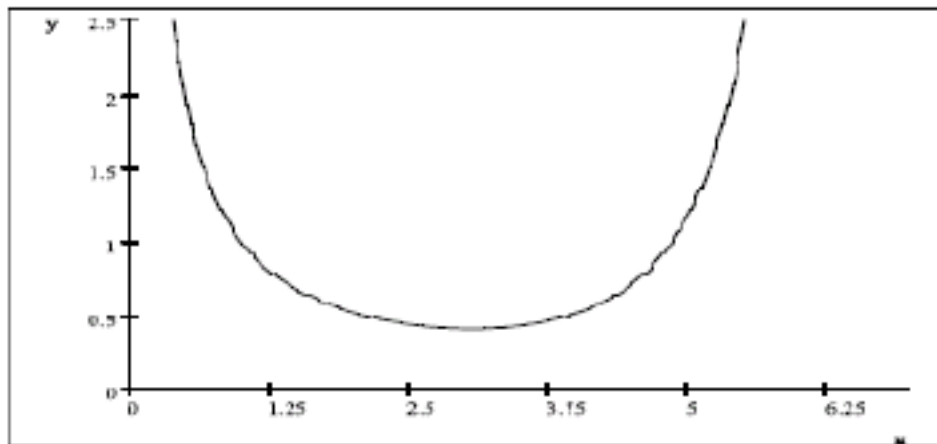


Fig. 4

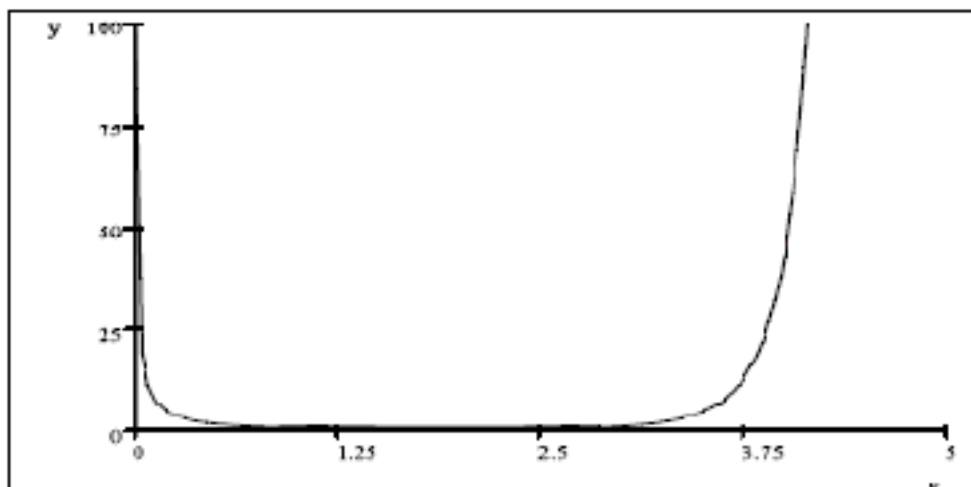


Fig. 5

continue until infinity and is determined by the noncommutativity of the space geometry. The latter is supposed related to the space-time properties at short distances and for large distances the classical solution $R_0(t) = \frac{B}{t}$ determine the dynamics and evolution of the universe. Thus after a certain time of expansion, the universe returns back to its initial state where the noncommutativity of the space geometry does not contribute. Therefore, the universe undergoes a collapse " Big Crunch " until distances where the space geometry noncommutativity becomes efficient to avoid the collapse and the universe bounce into a " Big Bang " and the cycle start over. In this model, we assume that at the origin (initial time), the universe empty from matter and the collapse is due to vacuum energy (cosmological constant or dark energy), the creation of matter and all the properties of the universe are due to the period before the " Big Bang " and not after. Thus, the dynamics and consequences of this model are the same as that of the Turok cyclic model but the origin and mechanisms of the cycles are different. In our case, it is the non commutativity which is responsible of the universe rebound and not the presence of the extra dimensions as it is the case of Turok model.

The most important thing to be mentioned here is that in this model we can give a possible explanation of the asymmetry and separation between matter and antimatter. In fact, in this model the universe starts with a collapse " Big Crunch " but the evolution depends on the sign of the parameter B_1 . For $B_1 < 0$ the collapse continues until $R(t) = 0$ at $t \rightarrow +\infty$. However, for $B_1 \geq 0$ the collapse will be prevented by the repulsive forces due to the noncommutativity of the space geometry and the universe rebound to a " Big Bang ". If we assume that the constant B_1 is related to the matter baryonic charge with $B_1 > 0$ for baryons and $B_1 < 0$ for anti-baryons, then before the matter and anti-matter creation, the universe was in a collapsing state under the gravitational action (the effects of the noncommutativity of the space geometry are negligible at large distances) until the temperature becomes fairly enough for the matter-anti-matter creation process. At this scale, the noncommutativity effects become important and as $B_1 < 0$ for anti-matter and $B_1 > 0$ for matter, the former continues its collapsing however the latter rebound under the action of the gravitational forces due to the noncommutativity of the space geometry. This mechanism gives a plausible explanation of the matter-antimatter asymmetry.

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Self Force on A Point-Like Source Coupled with Massive Scalar Field

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Abstract: The problem of determining the radiation reaction force experienced by a scalar charge moving in flat spacetime is investigated. A consistent renormalization procedure is used, which exploits the Poincaré invariance of the theory. Radiative parts of Noether quantities carried by massive scalar field are extracted. Energy-momentum and angular momentum balance equations yield Harish-Chandra equation of motion of radiating charge under the influence of an external force. This equation includes effect of particle's own field. The self force produces a time-changing inertial mass.

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1. Introduction

The problem of calculating the motion of an isolated point-like charge coupled to massive scalar field in flat spacetime is an old one which is currently receiving renewed interest. Classical equations of motion of a point particle interacting with a neutral massive vector field were first found by Bhabha [1] following a method originally developed by Dirac [2] for the case of electromagnetic field. In this method the finite force and self-force terms in the equations of motion are obtained from the conservation laws for the energy-momentum tensor of the field. It was extended by Bhabha and Harish-Chandra [3] to particles interacting with any generalized wave field and was applied to the motion of a simple pole of massive scalar field by Harish-Chandra [4].

The principal new feature of the field which carries rest mass in addition to energy and momentum is that it is nonlocal. (The field depends not only on the current state of motion of the source but on its past history.) Physically it is due to the fact that the

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massive field propagates at all speeds smaller than the velocity of light.

In [5, 6, 7, 8] a consistent theory of action at a distance was formulated in a case of point sources coupled to massive scalar or vector fields. In electrodynamics of Wheeler and Feynman [9], the interaction is assumed to be symmetric in time. The symmetric case is the only one for which the equations of motion follow from a variational principle. However, they do not contain any terms describing radiation damping. Such terms do appear if the assumption of complete absorption is applied to these equations. By this the authors [9] mean that the total advanced field of all the particles in the Universe equals their total retarded field.

The equations of motion obtained from the field-theoretical and action-at-a-distance point of view are different: the integrals over entire world line of the particle substitute for integrals over the past motion which appear in case of purely retarded fields [1, 4]. In [10, 11, 12] the total cross sections for the scattering of the various kinds of mesons by a heavy particle (nucleon) were calculated² and compared with those obtained within the Bhabha and Harish-Chandra approach. The predictions following from the two approaches should be exploited to furnish an experimental decision between the two theories. It is worth noting that in case of the retarded interactions, Havas [5] and Crownfield and Havas [13] obtain the Bhabha [1] and Harish-Chandra [4] equations.

In the present paper we calculate energy-momentum and angular momentum carried by outgoing massive scalar waves. Effective equations of motion of radiating scalar source will be obtained via the consideration of energy-momentum and angular momentum balance equations. The conservation laws are an immovable fulcrum about which tips the balance of truth regarding renormalization and radiation reaction. The verification is not a trivial matter, since the Klein-Gordon field generated by the scalar charge holds energy near the particle. This circumstance makes the procedure of decomposition of the Noether quantities into bound and radiative parts unclear.

In [14, 15, 16] Cawley and Marx study the massive scalar radiation from a point source with a prescribed world line. The authors assume that the particle accelerates only over a portion of the world line which corresponds to a finite proper time interval. They evaluate the energy-momentum which flows across a fixed three-dimensional sphere of large radius R . The radiation part of energy-momentum carried by massive scalar field was extracted which depends on R explicitly. It casts serious doubt on the validity of the result. In the present paper we apply a consistent splitting procedure which obeys the spirit of Dirac scheme of decomposition of electromagnetic potential into singular (symmetric) and regular (radiative) components [2].

Recently [17], Quinn has obtained an expression for the self-force on a point-like particle coupled to a massless scalar field arbitrarily moving in a curved spacetime. It is worth noting that in curved background massless waves propagate not just at speed of light, but at all speeds smaller than or equal to the speed of light. (It can be understood as the result of interaction between the radiation and the spacetime curvature.) Therefore, the particle may “fill” its own field, which will act on it just like an external field. In [18]

² In the present paper we shall not identify neither fields nor sources with any currently known particles.

Quinn establishes that the total work done by the scalar self-force matches the amount of energy radiated away by the particle.

Using Quinn’s general expression, Pfenning and Poisson [19] calculate the self-force experienced by a point scalar charge moving in a weakly curved spacetime. It is characterized by a generic Newtonian potential Φ which determines the small deviation of the metric $g_{\alpha\beta}$ with respect to the Minkowski values $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$. Potential Φ behaves as $-M/r$ at large distances r from the bounded mass distribution of total mass M . In contrast to the electromagnetic case, the equations of motion of scalar charge does not provide conservation of the rest mass (see also [17, 18]). In Refs.[20, 21] this phenomenon is studied for various kinds of cosmological spacetimes.

In this paper we consider the radiation reaction problem for a point particle that acts as a source for a massive scalar field in Minkowski spacetime. It is organized as follows. In Section 2. we recall the Green’s functions associated with the Klein-Gordon wave equation. Convoluting them with the point-like source, we derive the retarded scalar potential and field strengths as well as their advanced counterparts. In Section 3. we decompose the momentum 4-vector carried by massive scalar field into singular and regular parts. All diverging terms have disappeared into the procedure of mass renormalization while radiative terms survive. In analogous way we analyze the angular momentum of the Klein-Gordon scalar field. The radiative parts of Noether quantities carried by field and already renormalized particle’s individual momentum and angular momentum constitute the total energy-momentum and total angular momentum of our particle plus field system. In Section 4. we derive the effective equations of motion of radiating scalar charge via analysis of balance equations. We show that it coincides with the Harish-Chandra equation [4]. In Section 4. we discuss the result and its implications.

2. Scalar Potential and Field Strengths of A Point-Like Scalar Charge

The dynamics of a point-like charge coupled to massive scalar field is governed by the action [22, 23]

$$I_{\text{total}} = I_{\text{part}} + I_{\text{int}} + I_{\text{field}}. \quad (1)$$

Here

$$I_{\text{field}} = -\frac{1}{8\pi} \int d^4y (\eta^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} + k_0^2 \varphi^2) \quad (2)$$

is an action functional for a massive scalar field φ in flat spacetime. We shall use the metric tensor $\eta^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ and its inverse $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ to raise and lower indices, respectively. The mass parameter k_0 is a constant with the dimension of reciprocal length. The integration is performed over all the spacetime. The particle action is

$$I_{\text{part}} = -m_0 \int d\tau \sqrt{-\dot{z}^2} \quad (3)$$

where m_0 is the bare mass of the particle which moves on a world line $\zeta : \mathbb{R} \rightarrow \mathbb{M}_4$ described by relations $z^\alpha(\tau)$ which give the particle’s coordinates as functions of proper

time; $\dot{z}^\alpha(\tau) = dz^\alpha(\tau)/d\tau$. Finally, the interaction term is given by

$$I_{\text{int}} = g \int d\tau \sqrt{-\dot{z}^2} \varphi(z) \quad (4)$$

where g is scalar charge carried by a four-dimensional Dirac distribution supported on ζ : charge's density is zero everywhere, except at the particle's position where it is infinite.

The action (1) is invariant under infinitesimal transformations (translations and rotations) which constitute the Poincaré group. According to Noether's theorem, these symmetry properties yield conservation laws, i.e. those quantities that do not change with time.

Variation on field variable φ of action (1) yields the Klein-Gordon wave equation

$$(\square - k_0^2) \varphi(y) = -4\pi\rho(y), \quad (5)$$

where $\square = \eta^{\alpha\beta} \partial_\alpha \partial_\beta$ is the D'Alembert operator. We consider a scalar field satisfying eq.(5) in Minkowski spacetime with a point particle source

$$\rho(y) = g \int_{-\infty}^{+\infty} d\tau \delta^{(4)}(y - z(\tau)). \quad (6)$$

A solution to eq.(5) can be expressed as

$$\varphi(y) = \int d^4x G(y, x) \rho(x). \quad (7)$$

The relevant wave equation for the Green's function $G(y, x)$ is

$$(\square - k_0^2) G(y, x) = -4\pi\delta^{(4)}(y - x), \quad (8)$$

where $\delta^{(4)}(y - x)$ is a four-dimensional Dirac functional in \mathbb{M}_4 . The retarded Green's function [1, 22, 23]

$$G^{\text{ret}}(y, x) = \theta(y^0 - x^0) \left[\delta(\sigma) - \frac{k_0}{\sqrt{-2\sigma}} J_1(k_0\sqrt{-2\sigma})\theta(-\sigma) \right] \quad (9)$$

consists of singular part (this proportional to $\delta(\sigma)$) and smooth part (that proportional to $\theta(-\sigma)$). The former possesses support only on the past light cone of the field point y while the latter represents a function supported within the past light cone of y . By σ we denote Synge's world function in flat space-time [23]

$$\sigma(y, x) = \frac{1}{2} \eta_{\alpha\beta} (y^\alpha - x^\alpha)(y^\beta - x^\beta) \quad (10)$$

which is equal to half of the squared length of the geodesic connecting two points in \mathbb{M}_4 , namely "base point" x and "field point" y . $\theta(y^0 - x^0)$ is step function defined to be one if $y^0 > x^0$, and defined to be zero otherwise, so that $G^{\text{ret}}(y, x)$ vanishes in the past of x . $\theta(-\sigma)$ is the step function of $-\sigma(y, x)$ and J_1 is the first order Bessel's function of $k_0\sqrt{-2\sigma}$.

We substitute eq.(6) for the scalar density $\rho(x)$ in the right-hand side of eq.(7). Massive scalar waves propagate at all speeds smaller than or equal to the speed of light. Hence the retarded potential at each point y of Minkowski space \mathbb{M}_4 consists of a local term as well as non-local one. The local term is evaluated at the retarded instant $\tau^{\text{ret}}(y)$ which is determined by the intersection of the world line with the past cone of the field point y . The non-local term defines contribution from cone's interior. It reflect the circumstance that the retarded field at y is generated also by the point source during its history prior $\tau^{\text{ret}}(y)$.

Convolving the retarded Green's function (9) with the charge density (6) we construct the massive scalar field [4, 5, 23]:

$$\varphi^{\text{ret}}(y) = \frac{g}{r} - g \int_{-\infty}^{\tau^{\text{ret}}(y)} d\tau \frac{k_0 J_1[k_0 \sqrt{-(K \cdot K)}]}{\sqrt{-(K \cdot K)}} \quad (11)$$

where J_1 is the first order Bessel's function of $k_0 \sqrt{-2\sigma}$ which is rewritten as $k_0 \sqrt{-(K \cdot K)}$. By $K^\mu = y^\mu - z^\mu(\tau)$ we denote the unique timelike (or null) vector pointing from the emission point $z(\tau) \in \zeta$ to a field point $y \in \mathbb{M}_4$. The upper limit of the integral is the root of algebraic equation $\sigma(y, z(\tau)) = 0$ which satisfies causality condition $y^0 - z^0(\tau^{\text{ret}}) > 0$. By r we mean the retarded distance

$$r(y) = -\eta_{\alpha\beta}(y^\alpha - z^\alpha(\tau^{\text{ret}}))u^\beta(\tau^{\text{ret}}). \quad (12)$$

Because the speed of light is set to unity, it is also the spatial distance between $z(\tau^{\text{ret}})$ and y as measured in this momentarily comoving Lorentz frame where 4-velocity $u^\beta(\tau^{\text{ret}}) = (1, 0, 0, 0)$.

Scalar field strengths are given by the gradient of the potential (11). Let us differentiate the local term. Because y and $z(\tau^{\text{ret}})$ are linked by the light-cone mapping, a change of field point y generally comes with a change τ^{ret} . Suppose that y is displaced to the new field point $y + \delta y$. The new emission point $z(\tau^{\text{ret}} + \delta\tau^{\text{ret}})$ satisfies the algebraic equation $\sigma(y + \delta y, z(\tau^{\text{ret}} + \delta\tau^{\text{ret}})) = 0$. Expanding this to the first order of infinitesimal displacements δy and $\delta\tau^{\text{ret}}$, we obtain $K_\alpha \delta y^\alpha + r \delta\tau^{\text{ret}} = 0$, or

$$\frac{\partial \tau^{\text{ret}}}{\partial y^\alpha} = -\frac{K_\alpha}{r(y)}. \quad (13)$$

This relation allows us to differentiate the retarded distance (12) in the local Coulomb-like term involved in eq.(11).

Now we differentiate the non-local term in the potential (11). Apart from the integral

$$f_\mu^{(\theta)} = g \int_{-\infty}^{\tau^{\text{ret}}(y)} d\tau k_0^2 \frac{d}{d\Xi} \left(\frac{J_1(\Xi)}{\Xi} \right) k_0 \frac{K_\mu}{\sqrt{-(K \cdot K)}} \quad (14)$$

the gradient $f_{\text{tail},\mu} = f_\mu^{(\theta)} + f_\mu^{(\delta)}$ contains also local term

$$f_\mu^{(\delta)} = g k_0^2 \frac{J_1(\Xi)}{\Xi} \frac{K_\mu}{r} \Big|_{\tau=\tau^{\text{ret}}} \quad (15)$$

which is due to time-dependent upper limit of integral in eq.(11). Because of asymptotic behaviour of the first order Bessel's function with argument $\Xi := k_0 \sqrt{-(K \cdot K)}$ the local term $f_\mu^{(\delta)}$ is finite on the light cone where $\Xi = 0$. It diverges on the particle's trajectory only.

To simplify the non-local contribution as much as possible we use the identity

$$\frac{k_0}{\sqrt{-(K \cdot K)}} = \frac{1}{(K \cdot u)} \frac{d\Xi}{d\tau} \quad (16)$$

in the integral (14) and perform integration by parts. On rearrangement, we add it to the expression (15). The term which depends on the end points only annuls $f_\mu^{(\delta)}$. Finally, the gradient of potential (11) becomes

$$\frac{\partial \varphi^{\text{ret}}(y)}{\partial y^\mu} = -g \frac{1 + (K \cdot a)}{r^3} K_\mu + g \frac{u_\mu}{r^2} + g \int_{-\infty}^{\tau^{\text{ret}}(y)} d\tau k_0^2 \frac{J_1(\Xi)}{\Xi} \left[\frac{1 + (K \cdot a)}{r^2} K_\mu - \frac{u_\mu}{r} \right] \quad (17)$$

where $\Xi := k_0 \sqrt{-(K \cdot K)}$. As it is in the potential itself, particle's position, velocity, and acceleration in the local part are referred to the retarded instant $\tau^{\text{ret}}(y)$ while ones under the integral sign are evaluated at instant $\tau \leq \tau^{\text{ret}}(y)$. The non-local part arises from source contributions interior to the light cone. This part of field is called the "tail term". The invariant quantity

$$r = -(K \cdot u) \quad (18)$$

is an affine parameter on the time-like (null) geodesic that links y to $z(\tau)$; it can be loosely interpreted as the time delay between y and $z(\tau)$ as measured by an observer moving with the particle.

The advanced Green's function is non-zero in the past of emission point x :

$$G^{\text{adv}}(y, x) = \theta(-y^0 + x^0) \left[\delta(\sigma) - \frac{k_0}{\sqrt{-2\sigma}} J_1(k_0 \sqrt{-2\sigma}) \theta(-\sigma) \right]. \quad (19)$$

The advanced force

$$\frac{\partial \varphi^{\text{adv}}(y)}{\partial y^\mu} = -g \frac{1 + (K \cdot a)}{r^3} K_\mu + g \frac{u_\mu}{r^2} + g \int_{\tau^{\text{adv}}(y)}^{+\infty} d\tau k_0^2 \frac{J_1(\Xi)}{\Xi} \left[\frac{1 + (K \cdot a)}{r^2} K_\mu - \frac{u_\mu}{r} \right] \quad (20)$$

is generated by the point charge during its entire future history following the advanced time associated with y . Particle's characteristics in the local part are referred to the instant $\tau^{\text{adv}}(y)$.

3. Bound and Radiative Parts of Noether Quantities

In this Section we decompose the energy-momentum and angular momentum carried by massive scalar field into the bound and radiative parts. The bound terms will be absorbed

by particle's individual characteristics while the radiative terms exert the radiation reaction. We do not calculate the flows of the massive scalar field across a thin tube around a world line of the source. To extract the appropriate finite parts of energy-momentum and angular momentum we apply the scheme developed in Refs.[24, 25]. In these papers the radiation reaction problem for an electric charge moving in flat spacetime of three dimensions is considered. A specific feature of $2 + 1$ electrodynamics is that both the electromagnetic potential and electromagnetic field are non-local: they depend not only on the current state of motion of the particle, but also on its past (or future) history. The scalar potential (11) as well as the scalar field strengths (17) and (20) behave analogously.

Decomposition of Noether quantities into bound and radiative components satisfies the following conditions [24, 25]:

- proper non-accelerating limit of singular and regular parts;
- proper short-distance behaviour of regular part;
- Poincaré invariance and reparametrization invariance.

The first point means that in specific case of rectilinear uniform motion regular parts should vanish because of non-accelerating charge does not radiate. By “proper short-distance behaviour” we mean the finiteness of integrand near the coincidence limit where point of emission placed on the world line tends to the field point which also lies on ζ . (The bound parts of non-local conserved quantities in $2 + 1$ electrodynamics contain one integration over the world line while radiative ones are integrated over ζ twice.)

The scalar potential (11) and the scalar field strengths (17) and (20) contain local terms as well as non-local ones. Local part of energy-momentum carried by massive scalar field is obtained in [4, 5, 26]. It is equal to one-half of the well-known Larmor rate of radiation integrated over the world line:

$$p_{\text{loc,R}}^\mu = \frac{g^2}{3} \int_{-\infty}^{\tau} ds a^2(s) u^\mu(s). \quad (21)$$

Similarly, the local part of radiated angular momentum is equal to the one-half of corresponding quantity in classical electrodynamics [27]:

$$M_{\text{loc,R}}^{\mu\nu} = \frac{g^2}{3} \int_{-\infty}^{\tau} ds a_s^2 [z_s^\mu u_s^\nu - z_s^\nu u_s^\mu] + \frac{g^2}{3} \int_{-\infty}^{\tau} ds [u_s^\mu a_s^\nu - u_s^\nu a_s^\mu]. \quad (22)$$

There are singular terms associated with the Coulomb-like potential taken on particle's world line (see A, eq.(A.8)). Inevitable infinity is absorbed by “bare” mass within the renormalization procedure.

To find the “tail” parts of radiated Noether quantities sourced by the interior of the light cone we deal with the field defined on the world line only. Following the scheme presented in [24, 25] we build our construction upon the tail part of the field strengths (17) evaluated at point $z(\tau_1) \in \zeta$:

$$\begin{aligned} f_{\text{tail},\mu}^{\text{ret}} &= \left. \frac{\partial \varphi_{\text{tail}}^{\text{ret}}(y)}{\partial y^\mu} \right|_{y=z(\tau_1)} \\ &= g \int_{-\infty}^{\tau_1} d\tau_2 k_0^2 \frac{J_1(\xi)}{\xi} \left[\frac{1 + (q \cdot a_2)}{r_2^2} q_\mu - \frac{u_{2,\mu}}{r_2} \right]. \end{aligned} \quad (23)$$

Here $q^\mu = z_1^\mu - z_2^\mu$ defines the unique timelike 4-vector pointing from an emission point $z(\tau_2) \in \zeta$ to a field point $z(\tau_1) \in \zeta$. Index 1 indicates that particle's position, velocity, or acceleration is referred to the instant $\tau_1 \in]-\infty, \tau]$ while index 2 says that the particle's characteristics are evaluated at instant $\tau_2 \leq \tau_1$. We use the notations $\xi = k_0 \sqrt{-(q \cdot q)}$ and $r_2 = -(q \cdot u_2)$.

Next we consider the “advanced” counterpart of the expression (23):

$$f_{\text{tail},\mu}^{\text{adv}} = g \int_{\tau_1}^{\tau} d\tau_2 k_0^2 \frac{J_1(\xi)}{\xi} \left[\frac{1 + (q \cdot a_2)}{r_2^2} q_\mu - \frac{u_{2,\mu}}{r_2} \right]. \quad (24)$$

It is intimately connected with the gradient of $\varphi_{\text{tail}}^{\text{adv}}(y)$ evaluated at point $y = z(\tau_1)$. Note that the advanced force (20) is generated by the point charge during its entire future history. In (24) the domain of integration is the portion of the world line which corresponds to the interval $\tau_2 \in [\tau_1, \tau]$ where τ is the so-called “instant of observation”. This instant arise naturally in [24, 25] where an interference of outgoing waves at the plane of constant value of y^0 is investigated. Its role is elucidated in [24, Figs.2-4] and [25, Figs.1,2].

We postulate that non-local part of energy-momentum carried by outgoing radiation is one-half of work done by the retarded tail force minus one-half of work performed by the advanced one, taken with opposite sign:

$$p_{\text{tail},\mu}^{\text{R}} = -\frac{g}{2} \left(\int_{-\infty}^{\tau} d\tau_1 f_{\text{tail},\mu}^{\text{ret}} - \int_{-\infty}^{\tau} d\tau_1 f_{\text{tail},\mu}^{\text{adv}} \right). \quad (25)$$

It is obvious that the “advanced” domain of integration, $\int_{-\infty}^{\tau} d\tau_1 \int_{\tau_1}^{\tau} d\tau_2$, is equivalent to $\int_{-\infty}^{\tau} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_1$. It can be replaced by the “retarded” one, $\int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2$, via interchanging of indices “first” and “second” in the integrand. The “tail” part of energy-momentum carried by outgoing radiation becomes

$$p_{\text{tail,R}}^\mu = \frac{g^2}{2} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 k_0^2 \frac{J_1(\xi)}{\xi} \left[-\frac{1 + (q \cdot a_2)}{r_2^2} q^\mu + \frac{u_2^\mu}{r_2} - \frac{1 - (q \cdot a_1)}{r_1^2} q^\mu + \frac{u_1^\mu}{r_1} \right] \quad (26)$$

where $r_a = -(q \cdot u_a)$. It is noteworthy that all the moments are before the observation instant τ , and the retarded causality is not violated.

In the specific case of a uniformly moving source $q^\mu = u^\mu(\tau_1 - \tau_2)$ and $r_a = \tau_1 - \tau_2$ for both $a = 1$ and $a = 2$. Hence the bracketed integrands in eq. (26) is identically equal to zero. The local parts of radiation (21) and (22) vanish if $u^\mu = \text{const}$. As could be expected, nonaccelerating scalar charge does not radiate.

Now we evaluate the short-distance behaviour of the expression under the double integrals in eq.(26). Let τ_1 be fixed and $\tau_1 - \tau_2 := \Delta$ be a small parameter. With a degree of accuracy sufficient for our purposes

$$\begin{aligned} \sqrt{-(q \cdot q)} &= \Delta \\ q^\mu &= \Delta \left[u_1^\mu - a_1^\mu \frac{\Delta}{2} + \dot{a}_1^\mu \frac{\Delta^2}{6} \right] \\ u_2^\mu &= u_1^\mu - a_1^\mu \Delta + \dot{a}_1^\mu \frac{\Delta^2}{2}. \end{aligned} \quad (27)$$

Substituting these into integrands of the double integrals of eq.(26) and passing to the limit $\Delta \rightarrow 0$ yields vanishing expression. Hence the subscript “R” stands for “regular” as well as for “radiative”.

In analogous way we construct the non-local part of radiated angular momentum. First of all we introduce the torque of the retarded tail force (23) and its advanced counterpart:

$$m_{\text{tail},\mu\nu}^{\text{ret}} = z_{1,\mu} f_{\text{tail},\nu}^{\text{ret}} - z_{1,\nu} f_{\text{tail},\mu}^{\text{ret}}, \quad m_{\text{tail},\mu\nu}^{\text{adv}} = z_{1,\mu} f_{\text{tail},\nu}^{\text{adv}} - z_{1,\nu} f_{\text{tail},\mu}^{\text{adv}}. \quad (28)$$

The desired expression is equal to the one-half of integral of $m_{\text{tail},\mu\nu}^{\text{ret}}$ over the world line up to observation instant τ minus one-half of integral of $m_{\text{tail},\mu\nu}^{\text{adv}}$, taken with opposite sign:

$$M_{\text{tail,R}}^{\mu\nu} = \frac{g^2}{2} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 k_0^2 \frac{J_1(\xi)}{\xi} \left[\frac{1 + (q \cdot a_2)}{r_2^2} (z_1^\mu z_2^\nu - z_1^\nu z_2^\mu) + \frac{z_1^\mu u_2^\nu - z_1^\nu u_2^\mu}{r_2} + \frac{1 - (q \cdot a_1)}{r_1^2} (z_1^\mu z_2^\nu - z_1^\nu z_2^\mu) + \frac{z_2^\mu u_1^\nu - z_2^\nu u_1^\mu}{r_1} \right]. \quad (29)$$

In the specific case of constant velocity this expression vanishes. Substituting eqs.(27) in the integrand passing to the limit $\Delta \rightarrow 0$ leads to zero.

We postulate that bound part of energy-momentum carried by non-local part of massive scalar field is one-half of sum of work done by the retarded and the advanced tail forces:

$$p_{\text{tail},\mu}^{\text{S}}(\tau) = -\frac{g}{2} \left(\int_{-\infty}^{\tau} d\tau_1 f_{\text{tail},\mu}^{\text{ret}} + \int_{-\infty}^{\tau} d\tau_1 f_{\text{tail},\mu}^{\text{adv}} \right) \quad (30)$$

$$= -\frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \frac{q_\mu(\tau, s)}{r_\tau}.$$

The bound part of angular momentum also contains only one integration over the fragment of particle’s world line:

$$M_{\text{tail},\mu\nu}^{\text{S}}(\tau) = -\frac{g}{2} \left(\int_{-\infty}^{\tau} d\tau_1 m_{\text{tail},\mu\nu}^{\text{ret}} + \int_{-\infty}^{\tau} d\tau_1 m_{\text{tail},\mu\nu}^{\text{adv}} \right) \quad (31)$$

$$= \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \frac{z_{\tau,\mu} z_{s,\nu} - z_{\tau,\nu} z_{s,\mu}}{r_\tau}.$$

Here index τ indicates that particle’s position, velocity, or acceleration is referred to the observation instant τ while index s says that the particle’s characteristics are evaluated at instant $s \leq \tau$. We denote $r_\tau = -(q \cdot u_\tau)$.

If $u^\mu = \text{const}$ that $\xi = k_0(\tau - s)$ and $q^\mu/r_\tau = u^\mu$. Since

$$\int_{-\infty}^{\tau} ds \frac{J_1[k_0(\tau - s)]}{\tau - s} = 1, \quad (32)$$

the field generated by a uniformly moving charge contributes an amount $p_{\text{tail,S}}^\mu = -1/2g^2k_0u^\mu$ to its energy-momentum. This finding is in line with that of Appendix A where is established that if the particle is permanently at rest, the scalar meson field adds $-1/2g^2k_0$

to its energy. The bound angular momentum is $M_{\text{tail},S}^{\mu\nu} = z_0^\mu p_{\text{tail},S}^\nu - z_0^\nu p_{\text{tail},S}^\mu$ in case of uniform motion. We suppose, that the bound parts (30) and (31) of energy-momentum and angular momentum, respectively, are permanently “attached” to the charge and are carried along with it. It is worth noting that they possess the proper short-distance behaviour and, therefore, do not diverge. The “local” Coulomb-like infinity is the only divergency stemming from the pointness of the source.

There is one more question to be answered: is the choice of the force (23) the only one? If not there exists an alternative expression for radiated energy-momentum. It is interesting to apply our decomposition procedure to the massive scalar field as it is described in Refs.[14, 15, 16]. Cawley and Marx [14] remove the local Coulomb-like term from the retarded scalar potential. Using the recurrent relation $J_1(\Xi) = -dJ_0/d\Xi$ between Bessel’s function of order zero and of order one in eq.(11) yields

$$\varphi^{\text{ret}}(y) = g \int_{-\infty}^{\tau^{\text{ret}}(y)} d\tau J_0(\Xi) \frac{1 + (K \cdot a)}{r^2} \quad (33)$$

after integration by parts. The authors state that the Klein-Gordon source does not emanate massless radiation. Following their approach, we rewrite the scalar field strengths (17) as follows:

$$\frac{\partial \varphi^{\text{ret}}(y)}{\partial y^\mu} = g \int_{-\infty}^{\tau^{\text{ret}}(y)} d\tau J_0(\Xi) \left\{ -3 \frac{[1 + (K \cdot a)]^2}{r^4} K_\mu - \frac{(K \cdot \dot{a})}{r^3} K_\mu + 3 \frac{1 + (K \cdot a)}{r^3} u_\mu + \frac{a_\mu}{r^2} \right\}. \quad (34)$$

Putting the field point $z(\tau_1) \in \zeta$ and the emission point $z(\tau_2) \in \zeta$, we obtain the scalar self-field:

$$F_\mu^{\text{ret}} = g \int_{-\infty}^{\tau_1} d\tau_2 J_0(\xi) \left\{ -3 \frac{[1 + (q \cdot a_2)]^2}{r_2^4} q_\mu - \frac{(q \cdot \dot{a}_2)}{r_2^3} q_\mu + 3 \frac{1 + (q \cdot a_2)}{r_2^3} u_{2,\mu} + \frac{a_{2,\mu}}{r_2^2} \right\}. \quad (35)$$

Similarly one can construct its “advanced” counterpart which is generated by the point source during its history after τ_1 up to the observation instant τ .

Our next task is to extract the radiation part of energy-momentum carried by Cawley’s scalar field (33). Since the radiation does not propagate with the speed of light, the Larmor-like term (21) does not appear. The tail contribution to the radiation

$$\begin{aligned} p_\mu^{\text{R}} &= -\frac{g}{2} \left(\int_{-\infty}^{\tau} d\tau_1 F_\mu^{\text{ret}} - \int_{-\infty}^{\tau} d\tau_1 F_\mu^{\text{adv}} \right) \\ &= -\frac{g^2}{2} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 k_0^2 J_0(\xi) \left\{ -3 \frac{[1 + (q \cdot a_2)]^2}{r_2^4} q_\mu - \frac{(q \cdot \dot{a}_2)}{r_2^3} q_\mu \right. \\ &\quad \left. + 3 \frac{1 + (q \cdot a_2)}{r_2^3} u_{2,\mu} + \frac{a_{2,\mu}}{r_2^2} - \left[3 \frac{[1 - (q \cdot a_1)]^2}{r_1^4} q_\mu + \frac{(q \cdot \dot{a}_1)}{r_1^3} q_\mu \right. \right. \\ &\quad \left. \left. - 3 \frac{1 - (q \cdot a_1)}{r_1^3} u_{1,\mu} + \frac{a_{1,\mu}}{r_1^2} \right] \right\} \end{aligned} \quad (36)$$

is meaningful only. Let us study the short-distance behaviour. Having inserted the relations (27), we see that the double integral is ill defined because the integrand diverges at the edge $\tau_2 = \tau_1$ of the integration domain $D_\tau = \{(\tau_1, \tau_2) \in \mathbb{R}^2 : \tau_1 \in]-\infty, \tau], \tau_2 \leq \tau_1\}$. It is because the Coulomb-like divergency moves under the integral sign (cf. eqs.(11) and (33)).

In the following Section we check the formula (26) and (29) via analysis of energy-momentum and angular momentum balance equations. Analogous equations yield correct equation of motion of radiating charge in conventional 3+1 electrodynamics [28, 29] as well as in six dimensions [30]. It is reasonable to expect that conservation laws result correct equation of motion of point-like source coupled with massive scalar field where radiation back reaction is taken into account.

4. Equation of Motion of Radiating Charge

The equation of motion of radiating pole of massive scalar field was derived by Harish-Chandra [4] in 1946. (An alternative derivation was produced by Havas and Crownfield in [13].) Following the method of Dirac [2], Harish-Chandra enclosed the world line of the particle by a narrow tube, the radius of which will in the end be made to tend to zero. The author calculates the flow of energy and momentum out of the portion of the tube in presence of an external field. The condition was imposed that the flow depends only on the states at the two ends of the tube (the so-called “inflow theorem”, see [31, 3]). After integration over the tube along the world line and a limiting procedure, the equation of motion was derived. In our notation it looks as follows:

$$\begin{aligned} m_0 a_\tau^\mu - \frac{g^2}{3} (\dot{a}_\tau^\mu - a_\tau^2 u_\tau^\mu) - \frac{g^2}{2} k_0^2 u_\tau^\mu + g^2 \int_{-\infty}^{\tau} ds k_0^4 \frac{J_2(\xi)}{\xi^2} q^\mu + g^2 \frac{d}{d\tau} \left(u_\tau^\mu \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \right) \\ = g \eta^{\mu\alpha} \frac{\partial \varphi_{\text{ext}}}{\partial z^\alpha} + g \frac{d}{d\tau} (u_\tau^\mu \varphi_{\text{ext}}) \end{aligned} \quad (37)$$

where m_0 is an arbitrary constant identified with the mass of the particle and φ_{ext} is the scalar potential of the external field evaluated at the current position of the particle. $J_2(\xi)$ is the second order Bessel’s function. In this Section the Harish-Chandra equation will be obtained via analysis of energy-momentum and angular momentum balance equations.

In previous Section we introduce the radiative part $p_R = p_{\text{loc,R}} + p_{\text{tail,R}}$ of energy-momentum carried by the field. We proclaim that it alone exerts a force on the particle. We assume that the bound part, p_S , is absorbed by particle’s 4-momentum so that “dressed” charged particle would not undergo any additional radiation reaction. Already renormalized particle’s individual three-momentum, say p_{part} , together with p_R constitute the total energy-momentum of our composite particle plus field system: $P = p_{\text{part}} + p_R$. We suppose that the gradient of the external potential matches the change of P with

time:

$$\begin{aligned} \dot{p}_{\text{part}}^\mu(\tau) &= -\dot{p}_R^\mu + g\eta^{\mu\alpha} \frac{\partial\varphi_{\text{ext}}}{\partial z^\alpha} \\ &= -\frac{g^2}{3} a^2(\tau) u_\tau^\mu + \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \left[\frac{1 + (q \cdot a_s)}{r_s^2} q^\mu - \frac{u_s^\mu}{r_s} + \frac{1 - (q \cdot a_\tau)}{r_\tau^2} q^\mu - \frac{u_\tau^\mu}{r_\tau} \right] \\ &\quad + g\eta^{\mu\alpha} \frac{\partial\varphi_{\text{ext}}}{\partial z^\alpha}. \end{aligned} \quad (38)$$

The overdot means the derivation with respect to proper time τ .

Our next task is to derive expression which explain how three-momentum of “dressed” charged particle depends on its individual characteristics (velocity, position, mass etc.). We do not make any assumptions about the particle structure, its charge distribution and its size. We only assume that the particle 4-momentum p_{part} is finite. To find out the desired expression we analyze conserved quantities corresponding to the invariance of the theory under proper homogeneous Lorentz transformations. The total angular momentum, say M , consists of particle’s angular momentum $z \wedge p_{\text{part}}$ and radiative part of angular momentum carried by massive scalar field:

$$M^{\mu\nu} = z_\tau^\mu p_{\text{part}}^\nu(\tau) - z_\tau^\nu p_{\text{part}}^\mu(\tau) + M_R^{\mu\nu}(\tau). \quad (39)$$

We assume that the torque $z_\tau^\mu \partial^\nu \varphi_{\text{ext}} - z_\tau^\nu \partial^\mu \varphi_{\text{ext}}$ of the potential external force matches the change of M with time. Having differentiated (39) where the radiated angular momentum $M_R^{\mu\nu} = M_{R,\text{loc}}^{\mu\nu} + M_{R,\text{tail}}^{\mu\nu}$ is determined by eqs.(22) and (29), and inserting eq.(38) we arrive at the equality

$$u_\tau \wedge \left(p_{\text{part}} + \frac{g^2}{3} a_\tau + \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \frac{q}{r_\tau} \right) = 0. \quad (40)$$

Apart from usual velocity term, the 4-momentum of “dressed” particle contains also a contribution from field:

$$p_{\text{part}}^\mu = m u_\tau^\mu - \frac{g^2}{3} a_\tau^\mu - \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \frac{q^\mu}{r_\tau}. \quad (41)$$

The local part is the scalar analog of Teitelboim’s expression [32] for individual 4-momentum of a “dressed” electric charge in conventional electrodynamics. The integral term is then nothing but the bound part (30) of energy-momentum carried by the massive scalar field.

The expression for the scalar function $m(\tau)$ is find in B via analysis of differential consequences of conservation laws. We derive that *already renormalized* dynamical mass m depends on particle’s evolution before the observation instant τ :

$$m = m_0 + g^2 \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1[\xi(\tau, s)]}{\xi(\tau, s)} - g\varphi_{\text{ext}}. \quad (42)$$

The constant m_0 can be identified with the renormalization constant in action (3) which absorbs Coulomb-like divergence stemming from local part of potential (11). It is of great

importance that the dynamical mass, m , will vary with time: the particle will necessarily gain or lost its mass as a result of interactions with its own field as well as with the external one. The field of a uniformly moving charge contributes an amount $g^2 k_0$ to its inertial mass.

To derive the effective equation of motion of radiating charge we replace $\dot{p}_{\text{part}}^\mu$ in left-hand side of eq.(38) by differential consequence of eq.(41). We apply the formula

$$\frac{\partial}{\partial \tau} \int_{-\infty}^{\tau} ds f(\tau, s) = \int_{-\infty}^{\tau} ds \left(\frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial s} \right). \quad (43)$$

At the end of a straightforward calculations, we obtain

$$ma_\tau^\mu + \dot{m}u_\tau^\mu = \frac{g^2}{3} (\dot{a}_\tau^\mu - a_\tau^2 u_\tau^\mu) + g^2 \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \left[\frac{1 + (q \cdot a_s)}{r_s^2} q^\mu - \frac{u_s^\mu}{r_s} \right] + g\eta^{\mu\alpha} \frac{\partial \varphi_{\text{ext}}}{\partial z^\alpha} \quad (44)$$

where dynamical mass $m(\tau)$ is defined by eq.(42). The local part of the self-force is one-half of well-known Abraham radiation reaction vector while the non-local one is then nothing but the tail part of particle's scalar field strengths (17) acting upon itself (see eq.(23)). Indeed, since the massive field does not propagate with the velocity of light, the charge may “fill” its own field, which will act on it just like an external field.

Now we compare this effective equation of motion with the Harish-Chandra equation (37). The latter can be simplified substantially. Having used the recurrent relation

$$J_2(\xi) = \frac{J_1(\xi)}{\xi} - \frac{dJ_1(\xi)}{d\xi} \quad (45)$$

between Bessel functions of order two and of order one, after integration by parts we obtain

$$g^2 \int_{-\infty}^{\tau} ds k_0^4 \frac{J_2(\xi)}{\xi^2} q^\mu = \frac{g^2}{2} k_0^2 u_\tau^\mu - g^2 \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \left[\frac{1 + (q \cdot a_s)}{r_s^2} q^\mu - \frac{u_s^\mu}{r_s} \right]. \quad (46)$$

We also collect all the total time derivatives involved in Harish-Chandra equation (37). The term $m(\tau)u_\tau^\mu$ arises under the time derivative operator, where time-dependent function $m(\tau)$ is then nothing but the dynamical mass (42) of the particle. On rearrangement, the Harish-Chandra equation of motion (37) coincides with the equation (44) which is obtained via analysis of balance equations. It is in favour of the renormalization scheme for non-local theories developed in [24, 25].

To clear physical sense of the effective equation of motion (44) we move the velocity term $\dot{m}u_\tau^\mu$ to the right-hand side of this equation:

$$m(\tau)a_\tau^\mu = \frac{g^2}{3} (\dot{a}_\tau^\mu - a_\tau^2 u_\tau^\mu) + f_{\text{self}}^\mu + f_{\text{ext}}^\mu. \quad (47)$$

According to [22], the scalar potential produces the Minkowski force

$$f_{\text{ext}}^\mu = g(\eta^{\mu\alpha} + u_\tau^\mu u_\tau^\alpha) \frac{\partial \varphi_{\text{ext}}}{\partial z^\alpha} \quad (48)$$

which is orthogonal to the particle's 4-velocity. The self-force

$$f_{\text{self}}^\mu = g^2 \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \left[\frac{1 + (q \cdot a_s)}{r_s^2} (q^\mu - r_\tau u_\tau^\mu) - \frac{u_s^\mu + (u_s \cdot u_\tau) u_\tau^\mu}{r_s} \right] \quad (49)$$

is constructed analogously from the tail part of gradient (17) of particle's own field (11) supported on the world line ζ . The own field contributes also to particle's inertial mass $m(\tau)$ defined by eq.(42).

Conclusions

In the present paper, we find the radiative parts of energy-momentum and angular momentum carried by massive scalar field coupled to a point-like source. Scrupulous analysis of energy-momentum and angular momentum balance equations yields the Harish-Chandra equation of motion of radiating scalar pole. This equation includes the effect of particle's own field as well as the influence of an external force.

To remove divergences stemming from the pointness of the particle we apply the regularization scheme originally developed for the case of electrodynamics in flat spacetime of three dimensions [24, 25]. It summarizes a scrupulous analysis of energy-momentum and angular momentum carried by non-local electromagnetic field of a point electric charge. The simple rule allows us to identify that portion of the radiation which arises from source contributions interior to the light cone.

Energy-momentum and angular momentum balance equations for radiating scalar pole constitute system of ten linear algebraic equations in variables $p_{\text{part}}^\mu(\tau)$ and their first time derivatives $\dot{p}_{\text{part}}^\mu(\tau)$ as the functions of particle's individual characteristics (velocity, acceleration, charge etc.). The system is degenerate, so that solution for particle's 4-momentum includes arbitrary scalar function, $m(\tau)$, which can be identified with the dynamical mass of the particle. Besides renormalization constant, the mass includes contributions from particle's own field as well as from an external field.

This is a special feature of the self force problem for a scalar charge. Indeed, the time-varying mass arises also in the radiation reaction for a pointlike particle coupled to a massless scalar field on a curved background [17]. The phenomenon of mass loss by scalar charge is studied in [20, 21]. Similar phenomenon occurs in the theory which describe a point-like charge coupled with massless scalar field in flat spacetime of three dimensions [33]. The charge loses its mass through the emission of monopole radiation.

A Energy-momentum of the scalar massive field of uniformly moving particle

The simplest scalar field is generated by an unmoved source placed at the coordinate origin. Setting $z = (t, 0, 0, 0)$ and $u = (1, 0, 0, 0)$ in eq.(33), one can derive the static potential [1, 14]:

$$\varphi(y) = g \frac{\exp(-k_0 r)}{r} \quad (\text{A.1})$$

where $r = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$ is the distance to the charge. It is the well-known Yukawa field.

In this Appendix we calculate the energy-momentum

$$p_{\text{sc}}^\nu(\tau) = \int_{\Sigma} d\sigma_{\mu} T^{\mu\nu} \quad (\text{A.2})$$

carried by the scalar massive field due to a uniformly moving pointlike source g . The stress-energy tensor \hat{T} is given by [14, 15, 16]

$$4\pi T_{\mu\nu} = \frac{\partial\varphi}{\partial y^{\mu}} \frac{\partial\varphi}{\partial y^{\nu}} - \frac{\eta_{\mu\nu}}{2} \left(\eta^{\alpha\beta} \frac{\partial\varphi}{\partial y^{\alpha}} \frac{\partial\varphi}{\partial y^{\beta}} + k_0^2 \varphi^2 \right) \quad (\text{A.3})$$

and Σ is an arbitrary space-like three-surface.

It is convenient to choose the simplest plane $\Sigma_t = \{y \in \mathbb{M}_4 : y^0 = t\}$ associated with unmoving observer. We start with the spherical coordinates

$$y^0 = s + r, \quad y^i = r n^i \quad (\text{A.4})$$

where $n^i = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$ and s is the parameter of evolution. To adopt them to the integration surface Σ_t we replace the radius r by the expression $t - s$. On rearrangement, the final coordinate transformation $(y^0, y^1, y^2, y^3) \mapsto (t, s, \phi, \theta)$ looks as follows:

$$y^0 = t, \quad y^i = (t - s)n^i. \quad (\text{A.5})$$

The surface element is given by

$$d\sigma_0 = (t - s)^2 ds d\Omega \quad (\text{A.6})$$

where $d\Omega = \sin\theta d\theta d\phi$ is an element of solid angle.

After trivial calculation one can derive the only non-trivial component of energy-momentum (A.2) is

$$\begin{aligned} p_{\text{sc}}^0 &= \frac{1}{4\pi} \int_{-\infty}^t ds (t - s)^2 \int d\Omega \frac{1}{2} \left[\sum_i \left(\frac{\partial\varphi}{\partial y^i} \right)^2 + k_0^2 \varphi^2 \right] \\ &= \frac{g^2}{2} \left[k_0 \exp[-2k_0(t - s)] + \frac{k_0 \exp[-2k_0(t - s)]}{t - s} \right]_{s \rightarrow -\infty}^{s \rightarrow t} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{g^2}{2\varepsilon} - \frac{g^2}{2} k_0 \end{aligned} \quad (\text{A.7})$$

where ε is positively valued small parameter.

Having performed Poincaré transformation, the combination of translation and Lorentz transformation, we find the energy-momentum carried by massive scalar field of uniformly moving charge:

$$p_{\text{sc}}^{\mu} = \lim_{\varepsilon \rightarrow 0} \frac{g^2}{2\varepsilon} u^{\mu} - \frac{g^2}{2} k_0 u^{\mu}. \quad (\text{A.8})$$

The divergent Coulomb-like term is absorbed by the “bare” mass m_0 involved in action integral (1) while the finite term contributes to the particle’s individual 4-momentum (41).

B Derivation of the dynamical mass

The scalar product of particle 4-velocity on the first-order time-derivative of particle 4-momentum (38) is as follows:

$$\begin{aligned} (\dot{p}_{\text{part}} \cdot u_\tau) &= \frac{g^2}{3} a_\tau^2 + \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \left[\frac{1 + (q \cdot a_s)}{r_s^2} (q \cdot u_\tau) - \frac{(u_s \cdot u_\tau)}{r_s} + \frac{(q \cdot a_\tau)}{r_\tau} \right] \\ &+ g \frac{d\varphi_{\text{ext}}}{d\tau}. \end{aligned} \quad (\text{B.1})$$

Since $(u \cdot a) = 0$, the scalar product of particle acceleration on the particle 4-momentum (41) does not contain the scalar function m :

$$(p_{\text{part}} \cdot a_\tau) = -\frac{g^2}{3} a_\tau^2 - \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \frac{(q \cdot a_\tau)}{r_\tau}. \quad (\text{B.2})$$

Summing up (B.1) and (B.2) we obtain the non-local expression:

$$\frac{d}{d\tau} (p_{\text{part}} \cdot u_\tau) = \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi)}{\xi} \frac{\partial}{\partial s} \left[\frac{(q \cdot u_\tau)}{r_s} \right] + g \frac{d\varphi_{\text{ext}}}{d\tau}. \quad (\text{B.3})$$

We rewrite the expression under the integral sign as the following combination of partial derivatives in time variables:

$$k_0^2 \frac{J_1(\xi)}{\xi} \frac{\partial}{\partial s} \left(\frac{(q \cdot u_\tau)}{r_s} \right) = \frac{\partial}{\partial s} \left(k_0^2 \frac{J_1(\xi)}{\xi} \frac{(q \cdot u_\tau)}{r_s} \right) - \frac{\partial}{\partial \tau} \left(k_0^2 \frac{J_1(\xi)}{\xi} \right). \quad (\text{B.4})$$

This circumstance allows us to integrate the expression (B.3) over τ :

$$\begin{aligned} (p_{\text{part}} \cdot u_\tau) &= -m_0 + \frac{g^2}{2} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \left[\frac{\partial}{\partial \tau_2} \left(k_0^2 \frac{J_1(\xi)}{\xi} \frac{(q \cdot u_1)}{r_2} \right) - \frac{\partial}{\partial \tau_1} \left(k_0^2 \frac{J_1(\xi)}{\xi} \right) \right] \\ &+ g\varphi_{\text{ext}} \\ &= -m_0 - \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1(\xi(\tau, s))}{\xi(\tau, s)} + g\varphi_{\text{ext}}. \end{aligned} \quad (\text{B.5})$$

To integrate the second term in between the square brackets we substitute $\int_{-\infty}^{\tau} d\tau_2 \int_{\tau_2}^{\tau} d\tau_1$ for $\int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2$. The external potential is referred to the observation instant τ .

Alternatively, the scalar product of 4-momentum (41) and 4-velocity is as follows:

$$(p_{\text{part}} \cdot u_\tau) = -m + \frac{g^2}{2} \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1[\xi(\tau, s)]}{\xi(\tau, s)}. \quad (\text{B.6})$$

Having compared these expressions we obtain:

$$m = m_0 + g^2 \int_{-\infty}^{\tau} ds k_0^2 \frac{J_1[\xi(\tau, s)]}{\xi(\tau, s)} - g\varphi_{\text{ext}}. \quad (\text{B.7})$$

We suppose that the renormalization constant m_0 already absorbs the Coulomb-like infinity which arises in eq.(A.8).

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New Jarlskog Determinant from Physics above the GUT Scale

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Abstract: We study the Planck scale effects on Jarlskog determinant. Quantum gravitational (Planck scale) effects lead to an effective $SU(2) \times U(1)$ invariant dimension-5 Lagrangian involving neutrino and Higgs fields, which give rise to additional terms in neutrino mass matrix on electroweak symmetry breaking. We assume that gravitational interaction is flavor blind and compute the Jarlskog determinant due to Planck scale effects. In the case of neutrino sector, the strength of CP violation is measured by Jarlskog determinant. We applied our approach to study Jarlskog determinant due to the Planck scale effects.

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1. Introduction

The origin of CP violation is still mystery in particle physics. Recent advance to neutrino physics observation mainly of astrophysical observation suggested the existence of tiny neutrino mass. The experiments and observation have shown evidence for neutrino oscillation. The solar neutrino deficit has long been observed [1-4] the atmospheric neutrino anomaly has been found [5-7] and currently almost confirmed by IMB [8], and hence indicates that neutrinos are massive and there is mixing in lepton sector. Since there is a mixing in lepton sector, this indicates to imagine that there occurs CP violation in lepton sector. Several physicists have considered whether we can see CP violation effect in lepton sector through long baseline neutrino oscillation experiments. The neutrino oscillation probability, in general depends on six parameters two independent mass square difference Δ_{21} and Δ_{31} , three mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ and one CP violating phase δ .

CP violation arise as three or more generation [9, 10]. CP violation in neutrino

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oscillation is interesting because it relates directly to CP phase parameter in the mixing for $n > 3$ degenerate neutrino. We can write down the compact formula for the difference of transition probability between conjugate channel.

$$\Delta P(\alpha, \beta) = P(\nu_\mu \rightarrow \nu_e) - P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e), \quad (1)$$

where

$$(\alpha, \beta) = (e, \mu), (\mu, \tau), (\tau, e).$$

The main physical goal in future experiments are the determination of the unknown parameter θ_{13} and upper bound $\sin^2 2\theta_{13} < 0.01$ is obtained for the ref [11]. In particular, the observation of δ is quite interesting for the point of view that δ related to the origin of the matter in the universe. The determination of δ is the final goal of the future experiments. We get the analytical expression for $\Delta P(\alpha, \beta)$ using the usual form of the MNS matrix parametrization [12].

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{23}s_{13} \end{pmatrix}, \quad (2)$$

where c and s denoted the cosine and sine of the respective notation, thus $\Delta P(\alpha, \beta)$ in vacuum can be written as

$$\Delta P(\alpha, \beta) = 16J (\sin\Delta_{21}\sin\Delta_{32}\sin\Delta_{31}). \quad (3)$$

Here α and β denote different neutrino or anti-neutrino flavour where

$$\Delta_{ij} = 1.27 \left(\frac{\Delta_{ij}}{eV^2} \right) \left(\frac{L}{Km} \right) \left(\frac{1GeV}{E} \right), \quad (4)$$

$\Delta_{ij} = (m_i^2 - m_j^2)$ is the difference of i^{th} and j^{th} vacuum mass square eigenvalue, E is the neutrino energy and L is the travel distance and the well known Jarlskog determinant [13], J is the standard mixing parametrization is given by

$$\begin{aligned} J &= Im (U_{e1}U_{e2}^*U_{\mu 1}^*U_{\mu 2}) \\ &= \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta, \end{aligned} \quad (5)$$

and the asymmetry parameter suggested by Cabibbo [14], as an alternative to measure CP violation in the lepton sector

$$A_{cp} = \frac{\Delta P}{P(\nu_\mu \rightarrow \nu_e) - P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)}, \quad (6)$$

The purpose of this paper is to study the Planck scale effects on Jarlskog Determinant. In Sec-2, we discuss the neutrino mixing angle due to Planck scale effects. In Sec-3, we give the conclusions.

2. Neutrino Mixing Angle and Mass Square Differences Above the GUT Scale

To calculate the effects of perturbation on neutrino observables. The calculation developed in an earlier paper [15]. A natural assumption is that unperturbed (0^{th} order mass matrix) is given by

$$\mathbf{M} = U^* \text{diag}(M_i) U^\dagger, \quad (7)$$

where, $U_{\alpha i}$ is the usual mixing matrix and M_i , the neutrino masses is generated by Grand unified theory. Most of the parameters related to neutrino oscillation are known, the major expectation is given by the mixing elements U_{e3} . We adopt the usual parameterizations.

$$\frac{|U_{e2}|}{|U_{e1}|} = \tan\theta_{12} \quad (8)$$

$$\frac{|U_{\mu 3}|}{|U_{\tau 3}|} = \tan\theta_{23} \quad (9)$$

$$|U_{e3}| = \sin\theta_{13} \quad (10)$$

In term of the above mixing angles, the mixing matrix is

$$U = \text{diag}(e^{if_1}, e^{if_2}, e^{if_3}) R(\theta_{23}) \Delta R(\theta_{13}) \Delta^* R(\theta_{12}) \text{diag}(e^{ia_1}, e^{ia_2}, 1). \quad (11)$$

The matrix $\Delta = \text{diag}(e^{\frac{1\delta}{2}}, 1, e^{-\frac{i\delta}{2}})$ contains the Dirac phase. This leads to CP violation in neutrino oscillation a_1 and a_2 are the so called Majorana phase, which effects the neutrinoless double beta decay. f_1 , f_2 and f_3 are usually absorbed as a part of the definition of the charge lepton field. Planck scale effects will add other contribution to the mass matrix that gives the new mixing matrix can be written as [15]

$$U' = U(1 + i\delta\theta),$$

where $\delta\theta$ is a hermitian matrix that is first order in μ [16,17]. The first order mass square difference $\Delta M_{ij}^2 = M_i^2 - M_j^2$, get modified [16,17] as

$$\Delta M_{ij}^{\prime 2} = \Delta M_{ij}^2 + 2(M_i \text{Re}(m_{ii}) - M_j \text{Re}(m_{jj})). \quad (12)$$

The change in the elements of the mixing matrix, which we parameterized by $\delta\theta$ [15], is given by

$$\delta\theta_{ij} = \frac{i\text{Re}(m_{jj})(M_i + M_j) - \text{Im}(m_{jj})(M_i - M_j)}{\Delta M_{ij}^{\prime 2}}. \quad (13)$$

The above equation determines only the off diagonal elements of matrix $\delta\theta_{ij}$. The diagonal element of $\delta\theta_{ij}$ can be set to zero by phase invariance. Using Eq(12), we can calculate neutrino mixing angle due to Planck scale effects,

$$\frac{|U'_{e2}|}{|U'_{e1}|} = \tan\theta'_{12} \quad (14)$$

$$\frac{|U'_{\mu3}|}{|U'_{\tau3}|} = \tan\theta'_{23} \quad (15)$$

$$|U'_{e3}| = \sin\theta'_{13} \quad (16)$$

As one can see from the above expression of mixing angle due to Planck scale effects, depends on new contribution of mixing $U' = U(1 + i\delta\theta)$. To see the mixing angle due to Planck scale effects [14,16] only θ_{13} and θ_{12} mixing angle have small deviation due to Planck scale effects.

3. Jarlskog Determinant Due to Planck Scale Effects

Note from eq(14), that the correction term depends crucially on the type of neutrino mass spectrum. For a hierarchical or invert hierarchial spectrum the correction is negligible. Hence we consider a degenerate neutrino spectrum and take the common neutrino mass to 2 eV, which is the upper limit from the tritium decay experiment [18]. Let us compute Jarlskog determinant due to new mixing due to Planck scale effects

$$\begin{aligned} J' &= \text{Im} \left(U'_{e1} U'^*_{e2} U'^*_{\mu1} U'_{\mu2} \right) \\ &= \text{Im} \left((U_{e1} + i(U_{e2}\delta\theta^*_{12} + U_{e3}\delta\theta_{13})) (U_{e2} - i(U_{e1}^*\delta\theta^*_{12} + U_{e3}^*\delta\theta_{13})) \right. \\ &\quad \left. ((U_{\mu1}^* - i(U_{\mu2}\delta\theta_{12} + U_{\mu3}\delta\theta_{13})) (U_{\mu2} + i(U_{\mu1}\delta\theta_{12} + U_{\mu3}\delta\theta^*_{23})) \right) \end{aligned} \quad (17)$$

We simplified Jarlskog determinant due to new mixing matrix

$$\begin{aligned} J' &= \text{Im} (U_{e1}U_{e2}^*U_{\mu1}^*U_{\mu2}) + \text{Im}(i(U_{\mu1}U_{\mu2})(|U_{e2}|^2\delta\theta^*_{12} + U_{e2}U_{e3}\delta\theta_{13} - |U_{e1}|^2\delta\theta^*_{12} - U_{e1}U_{e3}^*\delta\theta^*_{23}) \\ &\quad + \text{Im}(i(U_{e1}^*U_{e2})(|U_{\mu1}|^2\delta\theta_{12} + U_{\mu1}^*U_{\mu3}\delta\theta^*_{23} - |U_{\mu2}|^2\delta\theta_{12} - U_{\mu2}U_{\mu3}^*\delta\theta_{13})) \\ &= J + \Delta J \end{aligned}$$

In terms of mixing angle, we can write Jarlskog determinant in terms of mixing parameter due to Planck scale effects

$$J' = \frac{1}{8} \sin 2(\theta_{12} + \epsilon_{12}) \sin 2(\theta_{23} + \epsilon_{23}) \sin 2(\theta_{13} + \epsilon_{13}) \cos(\theta_{13} + \epsilon_{13}) \sin \delta, \quad (18)$$

We define the percentage change in Jarlskog determinant due to Planck scale effects

$$P = \frac{J^*}{J_{GUT}} \times 100 \quad (19)$$

The Majorana phases a_1 and a_2 have a non-trivial effect on the Planck scale corrections. We show the results as contour plots of J' in the $a_1 - a_2$ plane. In our calculation, we used best fit values of mixing angles, $\theta_{12} = 34^\circ$, $\theta_{23} = 45^\circ$, $\theta_{13} = 10^\circ$. We considered non zero value of CP phase and we took $\delta = 45^\circ, 90^\circ$.

In Fig. 1 and Fig. 3 for $\theta_{13} = 10^\circ$. For the value, which is the upper limit coming for CHOOZ experiments, note that there is reasonable range of Majorana phases, where Jarlskog determinant change 5% only, this change due to two mixing angles. In this paper, we studied Planck scale effects the Jarlskog determinant. MNS matrix and Jarlskog determinant, which is signal for CP violation in neutrino oscillation. We have obtained, due to Planck scale effects, two mixing angle θ_{12} and θ_{13} extra contributes to Jarlskog determinant.

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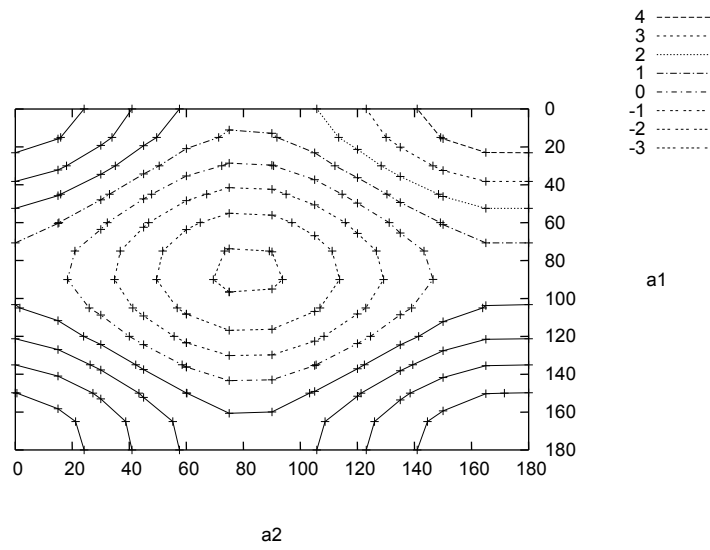


Fig. 1 J' due to Planck scale effects as function of Majorana phase for $\delta = 45^\circ$.

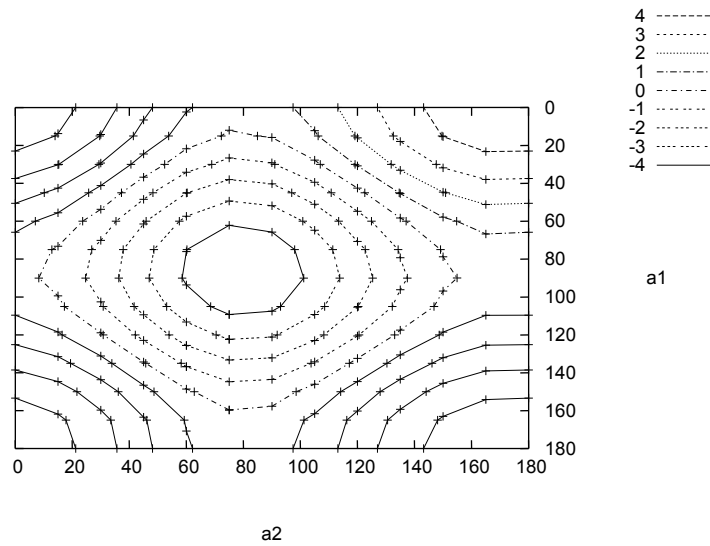


Fig. 2 J' due to Planck scale effects as function of Majorana phase for $\delta = 90^\circ$.

Derivation of the Rabi Equation by Means of the Pauli Matrices

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Abstract: The general quantum-mechanical properties of two-level systems are very relevant for the study of different physical subjects of great interest, as, for instance, magnetic resonance, nuclear physics, molecular dynamics, masers, neutrino oscillations, quantum computation, etc. In this work we present a nonperturbative approach to the study of the oscillations among two quantum states when an interaction term is considered. First, we show that using a simple parametrization for the superposition of the two states, it is possible to derive easily the Rabi equation. Next, we show that the Pauli matrices can be conveniently used to write the Hamiltonian operator and to derive the Rabi equation, directly applying the time evolution operator to the initial state of the system.

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1. Introduction

A great variety of physical processes belonging to the areas of quantum computation, condensed matter, atomic and molecular physics [1], nuclear and particle physics [2, 3], can be conveniently studied in terms of two-level quantum systems. In this case, one of the most interesting effects is represented by the oscillations between the two energy levels [1], as for example, $\nu_e - \nu_\mu$ flavour neutrino oscillations. Most textbooks of quantum mechanics deal with this problem with a standard methodology [1, 4]. We propose a pedagogically new, nonperturbative procedure that should allow the students to understand autonomously and in more detail the previously mentioned two-level physical

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processes.

First, in Sect. 2, by using a simple reparametrization of the superposition of two quantum states, we solve the eigenvalue equation of the problem. We suggest, as an exercise, to use perturbation theory to derive the approximate eigenvalues when a *weak* interaction is considered.

Then, in Sect. 3, by means of the previous results, we easily derive the Rabi equation that determines the transition amplitude between the two states. At the end of this Section we make a comparison with the result given by the perturbation theory.

Finally, in Sect. 4, we show that the Hamiltonian operator of the system can be conveniently written in terms of the Pauli matrices. Then, by means of their properties, the Rabi equation is directly obtained by applying the time evolution operator of the system. In this concern, we point out that, pedagogically, the knowledge of the properties of the Pauli matrices represents a useful and general tool for the study of theoretical physics.

2. The Eigenvalue Equation

Let us consider a quantum-mechanical system that can be described by the following total Hamiltonian

$$\hat{H} = \hat{H}_o + \hat{H}' \quad (1)$$

Conventionally, \hat{H}_o and \hat{H}' will be defined respectively as the *free* and the *interaction* Hamiltonian of the system. In the present work we make the hypothesis that the free system is characterized by the presence of only two accessible eigenstates (normalized to unity), denoted as $|1\rangle$ and $|2\rangle$, satisfying the standard eigenvalue equation

$$\hat{H}_o|1\rangle = E_1|1\rangle, \quad \hat{H}_o|2\rangle = E_2|2\rangle \quad (2)$$

where E_1 and E_2 represent the free energy eigenvalues of the system.

Given that only the two previously introduced states are accessible, we assume that they represent a *complete set* of states for the system. (In real cases, it means that we neglect the coupling with other states). In consequence, we can write the free and the interaction Hamiltonians in the following form

$$\hat{H}_o = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|, \quad \hat{H}' = \gamma[|1\rangle\langle 2| + |2\rangle\langle 1|] \quad (3)$$

The *real* factor γ represents the strength of the interaction. Note that the expression for the interaction term \hat{H}' automatically takes into account the hermiticity of the Hamiltonian operator.

Let us now study the *complete* eigenvalue equation

$$\hat{H}|\alpha\rangle = E_\alpha|\alpha\rangle \quad (4)$$

where $|\alpha\rangle = |a\rangle$, $|b\rangle$ and E_α represent respectively the eigenstates and eigenvalues of the total Hamiltonian \hat{H} . These eigenstates $|a\rangle$ and $|b\rangle$ can be written in terms of the

free eigenstates $|1\rangle$ and $|2\rangle$ as

$$|\alpha\rangle = C_{\alpha 1}|1\rangle + C_{\alpha 2}|2\rangle \quad (5)$$

We first observe that, being the eigenstates $|a\rangle$ and $|b\rangle$ normalized to unity, the decomposition coefficients must satisfy the following condition

$$|C_{\alpha 1}|^2 + |C_{\alpha 2}|^2 = 1 \quad (6)$$

for $\alpha = a, b$. In consequence, we can introduce a parametrization of the coefficients $C_{\alpha i}$ in terms of two angles θ_α , that is

$$\begin{aligned} C_{a1} &= \cos \theta_a & C_{a2} &= \sin \theta_a \\ C_{b1} &= \cos \theta_b & C_{b2} &= \sin \theta_b \end{aligned} \quad (7)$$

Furthermore, the orthogonality of the two eigenstates, that is $\langle b|a\rangle = 0$, with standard trigonometric handlings, gives

$$\cos(\theta_a - \theta_b) = 0 \quad (8)$$

that is satisfied by

$$\theta_b = \theta_a + \frac{\pi}{2} \quad (9)$$

It means that the coefficients of eq. (7) depend on *only one* parameter, that is $\theta = \theta_a$, as

$$\begin{aligned} C_{a1} &= \cos \theta & C_{a2} &= \sin \theta \\ C_{b1} &= -\sin \theta & C_{b2} &= \cos \theta \end{aligned} \quad (10)$$

In consequence, the decomposition of eq. (5) can be syntetically written in the following matrix form

$$\begin{pmatrix} |a\rangle \\ |b\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} \quad (11)$$

In the special case $\theta = 0$, one has $|a\rangle = |1\rangle$ and $|b\rangle = |2\rangle$.

We now turn to solve the complete eigenvalue equation (4). By means of the decomposition of eq. (5) and using eq. (3), it takes the form

$$\begin{aligned} E_1 \cos \theta_\alpha |1\rangle + E_2 \sin \theta_\alpha |2\rangle + \gamma \cos \theta_\alpha |2\rangle + \gamma \sin \theta_\alpha |1\rangle = \\ E_\alpha \cos \theta_\alpha |1\rangle + E_\alpha \sin \theta_\alpha |2\rangle \end{aligned} \quad (12)$$

Then, multiplying by $\langle 1|$ and $\langle 2|$, one obtains the following system of equations

$$\begin{pmatrix} E_1 - E_\alpha & \gamma \\ \gamma & E_2 - E_\alpha \end{pmatrix} \begin{pmatrix} \cos \theta_\alpha \\ \sin \theta_\alpha \end{pmatrix} = 0 \quad (13)$$

Solving this system, one finds two eigenvalues corresponding to two *orthogonal* eigenvectors. We can associate the first solution to E_a, θ_a and the second one to E_b, θ_b . Standard algebra gives

$$\begin{aligned} E_a &= E_1(1 + R)/2 + E_2(1 - R)/2 \\ E_b &= E_1(1 - R)/2 + E_2(1 + R)/2 \end{aligned} \quad (14)$$

with

$$R = \sqrt{1 + 4\gamma^2/(E_1 - E_2)^2} \quad (15)$$

Note that the two eigenvalues satisfy the condition $E_a + E_b = E_1 + E_2$. Furthermore, if the interaction is absent, that is $\gamma = 0$, one has $E_a = E_1$, $E_b = E_2$.

As an exercise, the reader can study the case of a *weak* interaction. Introducing the adimensional parameter $\lambda = \gamma/(E_1 - E_2)$, one can make a Taylor expansion in powers of λ for the eigenvalues of eq. (14). In this expansion, the term of order λ^0 gives the free eigenvalues and the first corrective term is of order λ^2 . The reader should also calculate the latter term using the time independent perturbation theory, taking \hat{H}' as the perturbative operator. In this concern, it is important to note that such term is obtained at the *second order* of the perturbation theory, being vanishing the first order contribution.

Finally, to find the angles θ_α that parametrize the eigenvectors, we first consider the equation given by the second line of the matrix in eq. (13). We select $\alpha = a$ and replace the explicit expression of E_a given by eq. (14). The result is

$$\tan \theta_a = 2\gamma/[(E_1 - E_2)(1 + R)] \quad (16)$$

Then, the angle θ_b can be easily found by means of the condition (9) and the expression of the state $|b\rangle$ can be obtained from eq. (11) recalling that $\theta = \theta_a$. As an exercise, the reader can obtain that result directly from eq. (13).

3. Time Evolution and Rabi Equation

We now turn to study the time evolution of the system. We consider the case in which the interaction term \hat{H}' (see eqs. (2) and (3)) is time independent. If at $t = 0$ the system is in the state $|\Psi(t = 0)\rangle$, at the time t the system is in the state

$$|\Psi(t)\rangle = \exp[-i\hat{H}t/\hbar]|\Psi(t = 0)\rangle \quad (17)$$

where $\hat{U}(t) = \exp[-i\hat{H}t/\hbar]$ represents the time evolution operator [4].

Let us suppose that at $t = 0$ the system is *prepared* in an eigenstate of the free Hamiltonian, say, for definiteness, in the state $|1\rangle$, so that $|\Psi(t = 0)\rangle = |1\rangle$. Inverting eq. (11), we can also write that state as a superposition of the eigenstates of the total Hamiltonian, in the form

$$|\Psi(t = 0)\rangle = |1\rangle = \cos \theta |a\rangle - \sin \theta |b\rangle \quad (18)$$

Using eq. (17) its time evolution is

$$|\Psi(t)\rangle = \cos\theta \exp[-iE_a t/\hbar]|a\rangle - \sin\theta \exp[-iE_b t/\hbar]|b\rangle \quad (19)$$

Multiplying the previous equation by $\langle 2|$, we obtain the probability amplitude of finding the system (initially in $|1\rangle$) in the state $|2\rangle$ at the time t . We have

$$\langle 2|\Psi(t)\rangle = \cos\theta \exp[-iE_a t/\hbar]\langle 2|a\rangle - \sin\theta \exp[-iE_b t/\hbar]\langle 2|b\rangle \quad (20)$$

By means of eq. (11) we know that $\langle 2|a\rangle = \sin\theta$ y $\langle 2|b\rangle = \cos\theta$, so that the transition probability amplitude of eq. (20) can be finally written as

$$\langle 2|\hat{U}(t)|1\rangle = \langle 2|\Psi(t)\rangle = F(\theta)G(E_a, E_b) \quad (21)$$

where

$$F(\theta) = \cos\theta \sin\theta \quad G(E_a, E_b) = \exp[-iE_a t/\hbar] - \exp[-iE_b t/\hbar] \quad (22)$$

With some algebraic handlings and by using eq. (16), we obtain

$$F(\theta) = \tan\theta/(\tan^2\theta + 1) = \gamma/(2\sqrt{\gamma^2 + (E_1 - E_2)^2/4}) \quad (23)$$

Analogously, if we replace in $G(E_a, E_b)$ the expression (14) for the eigenvalues E_a, E_b , we find

$$G(E_a, E_b) = 2i \exp[-it(E_1 + E_2)/2\hbar] \sin[t\sqrt{[(E_1 - E_2)^2 + 4\gamma^2]/4\hbar^2}] \quad (24)$$

Finally, the transition probability from the state $|1\rangle$ at $t = 0$ to the state $|2\rangle$ at the time t , is [4]

$$P_{12} = |\langle 2|\hat{U}(t)|1\rangle|^2 = \frac{\gamma^2}{\gamma^2 + (E_1 - E_2)^2/4} \sin^2 \left[\sqrt{\frac{(E_1 - E_2)^2 + 4\gamma^2}{4\hbar^2}} t \right] \quad (25)$$

that is the so-called Rabi equation. Note that the transition probability is represented by an oscillating function of the time.

We now turn to compare the previous result with the prediction of the so-called time dependent perturbation theory at the first order. We treat the free term \hat{H}_0 as the *unperturbed* Hamiltonian, while the interaction term \hat{H}' is considered as the perturbation. One has the following well-known expression [4]

$$P_{12}^{pert} = \frac{4\gamma^2}{(E_1 - E_2)^2} \sin^2 \left[\frac{(E_1 - E_2)}{2\hbar} t \right] \quad (26)$$

where from eq. (3) we have taken $\langle 2|\hat{H}'|1\rangle = \gamma$. By using the adimensional parameter λ , introduced in the previous section, the reader can easily check that expanding the exact equation (25) up to the first order in λ^2 (that is equivalent to consider the first order in λ for the amplitude), one obtains the perturbative result of eq. (26).

4. Use of the Pauli Matrices for the Time Evolution Operator

Given that the Hilbert space for our system is spanned by *two* independent states, we can advantageously use the formalism of the Pauli matrices. First, we introduce the spinors

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, to represent the states $|1\rangle$ and $|2\rangle$, respectively. In consequence, with standard handlings, the total Hamiltonian, by means of eqs.(1) and (3), is written as

$$\hat{H} = \begin{pmatrix} E_1 & \gamma \\ \gamma & E_2 \end{pmatrix} \quad (27)$$

Given that the identity matrix and three Pauli matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (28)$$

form a *complete* set of matrices in the 2×2 space, we can represent the Hamiltonian matrix (27) in the following way

$$\hat{H} = h_o \mathbf{1} + \mathbf{h} \boldsymbol{\sigma} \quad (29)$$

with

$$h_o = \frac{1}{2}(E_1 + E_2), \quad h_1 = \gamma, \quad h_2 = 0, \quad h_3 = \frac{1}{2}(E_1 - E_2) \quad (30)$$

Recalling that the two eigenvalues of $\mathbf{h} \boldsymbol{\sigma}$ are, in general, $\pm |\mathbf{h}| = \pm \sqrt{h_1^2 + h_2^2 + h_3^2}$, using eq. (30), the reader can directly find from eq. (29) the eigenvalues of \hat{H} obtaining exactly the same result given in eqs. (14) and (15). We now study, with the formalism of the Pauli matrices, the time evolution operator, introduced in Sect. 3. It is given by

$$\hat{U}(t) = \exp \left[-\frac{i\hat{H}t}{\hbar} \right] = \exp \left[-\frac{it}{\hbar} (h_o \mathbf{1} + \mathbf{h} \boldsymbol{\sigma}) \right] = \exp \left[-\frac{it}{\hbar} h_o \mathbf{1} \right] \exp \left[-\frac{it}{\hbar} \mathbf{h} \boldsymbol{\sigma} \right] \quad (31)$$

In the last expression, the first exponential simply gives a phase factor

$$\exp \left[-\frac{it}{\hbar} h_o \mathbf{1} \right] = \exp \left[-i \frac{(E_1 + E_2)}{2\hbar} t \right] \mathbf{1} \quad (32)$$

where we have used the explicit expression of h_o .

On the other hand, for the second exponential, an interesting handling is possible. We expand this term in a Taylor series of $\mathbf{h} \boldsymbol{\sigma}$. Then, recalling the standard properties of the *powers* of the Pauli matrices

$$(\boldsymbol{\sigma} \mathbf{a})^{2n} = (\mathbf{a}^2)^n, \quad (\boldsymbol{\sigma} \mathbf{a})^{2n+1} = (\mathbf{a}^2)^n \boldsymbol{\sigma} \mathbf{a} \quad (33)$$

we can write the time evolution equation (31) in the form

$$\hat{U}(t) = \exp \left[-i \frac{(E_1 + E_2)}{2\hbar} t \right] \times \left[\sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(-\frac{t^2}{\hbar^2} \mathbf{h}^2 \right)^n + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(-\frac{it}{\hbar} \right)^{2n+1} (\mathbf{h}^2)^n \mathbf{h}\boldsymbol{\sigma} \right] \quad (34)$$

Finally, summing up the two power series we obtain

$$\hat{U}(t) = \exp \left[-i \frac{(E_1 + E_2)}{2\hbar} t \right] \left[\cos \left(\frac{t|\mathbf{h}|}{\hbar} \right) \mathbf{1} - i \sin \left(\frac{t|\mathbf{h}|}{\hbar} \right) \frac{\mathbf{h}\boldsymbol{\sigma}}{|\mathbf{h}|} \right] \quad (35)$$

We can now easily calculate the transition amplitude from the state $|1\rangle$, at the initial time $t = 0$, to the state $|2\rangle$, at the final time t , making use of the time evolution operator in the form of eq. (35). This amplitude is given by the matrix element $A_{21} = \langle 2|\hat{U}(t)|1\rangle$, with the spinorial representation of the states $|1\rangle$ and $|2\rangle$ introduced at the beginning

of this section, being $\hat{U}(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix}$. The matrix elements of $\hat{U}(t)$ are found

by replacing explicitly the Pauli matrices in eq. (35). After some algebraic handlings, we find that A_{12} can be put in the *same form* of eq. (21), that is $A_{12} = F(\theta)G(E_a, E_b)$, so that the Rabi equation (25) for the transition probability is obtained as in Sect. 3. This second derivation is recommendable because it is based on the algebraic properties of the Pauli matrices and on the series expansion of quantum mechanical operators. These techniques are, in our opinion, of fundamental importance for the study of many other problems in quantum mechanics.

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Non-interacting Spin-1/2 Particles in Non-commuting External Magnetic Fields

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Abstract: We obtain, in one dimension, all the energy levels of a system of non-interacting spin-1/2 particles in non-commuting external magnetic fields. Examples of how to incorporate interactions as perturbations are given for the Ising model in two orthogonal fields and for the XZ model in two orthogonal fields.

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1. Introduction

Hamiltonian spin models involving two external non-commuting magnetic fields are being increasingly studied these days [1, 2, 3]. It is almost always the case that such models cannot be solved exactly, in the presence of spin interactions. Interaction is usually due to either nearest neighbour exchange or next nearest neighbour exchange or both. As is common practice, approximate calculations can be done by first starting with an exactly solvable model and then introducing the interactions as perturbations. This is done in this paper. We start by considering the simple model described by the Hamiltonian

$$H = -h_x \sum_{i=1}^N S_i^x - h_z \sum_{i=1}^N S_i^z, \quad (1)$$

where S_i^x and S_i^z are spin-1/2 operators on the i th lattice site of a one-dimensional chain of N particles. h_x and h_z are the external transverse and longitudinal magnetic fields, respectively, measured in units where the splitting factor and Bohr magneton are unity.

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It will often be convenient to write

$$H = \sum_{i=1}^N H_i,$$

where

$$H_i = -h_x S_i^x - h_z S_i^z.$$

The model (1) is exactly solvable and in this work we will obtain all the energy levels and the corresponding states. The main import of the result will be that interactions can be included as perturbations on the model (1), this approach being effective in obtaining accurate results for models incorporating two external fields.

2. The One-particle Model

Since the N spin-1/2 particles described by (1) are non-interacting, all results can be obtained from the Hamiltonian for a single particle. We drop the site subscripts in (1) and write H_ε for the one spin system and write

$$H_\varepsilon = -h_x S^x - h_z S^z,$$

where S^x and S^z are simply the spin-1/2 operators

$$S^x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in a basis with S^z diagonal and with $\hbar = 1$. We can denote the basis states in the S^z basis by the set $\{|\uparrow\rangle, |\downarrow\rangle\}$ and in the S^x basis by $\{|\rightarrow\rangle, |\leftarrow\rangle\}$, so that

$$\begin{aligned} S^z |\uparrow\rangle &= \frac{1}{2} |\uparrow\rangle \\ S^z |\downarrow\rangle &= -\frac{1}{2} |\downarrow\rangle \\ S^x |\rightarrow\rangle &= \frac{1}{2} |\rightarrow\rangle \\ S^x |\leftarrow\rangle &= -\frac{1}{2} |\leftarrow\rangle. \end{aligned}$$

S^x is obtained from S^z by the similarity transformation

$$S^x = P S^z P^{-1},$$

where P is the unitary matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = P^{-1}.$$

Thus, the S^x basis states and the S^z basis states are related by

$$\begin{aligned} |\rightarrow\rangle &= \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \\ |\leftarrow\rangle &= \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle), \end{aligned}$$

with the inverse relation

$$\begin{aligned} |\uparrow\rangle &= \frac{1}{\sqrt{2}} (|\rightarrow\rangle + |\leftarrow\rangle) \\ |\downarrow\rangle &= \frac{1}{\sqrt{2}} (|\rightarrow\rangle - |\leftarrow\rangle). \end{aligned}$$

The Hamiltonian of the one particle system in the S^z basis is

$$H_\varepsilon = -\frac{1}{2} \begin{pmatrix} h_z & h_x \\ h_x & -h_z \end{pmatrix}.$$

The normalized eigenstates of H_ε are

$$|\varepsilon_0\rangle = \frac{h_x |\downarrow\rangle + (h_z + \sqrt{h_x^2 + h_z^2}) |\uparrow\rangle}{\sqrt{h_x^2 + (h_z + \sqrt{h_x^2 + h_z^2})^2}},$$

with eigenenergy

$$\varepsilon_0 = -\frac{\sqrt{h_x^2 + h_z^2}}{2},$$

and

$$|\varepsilon_1\rangle = \frac{h_x |\downarrow\rangle + (h_z - \sqrt{h_x^2 + h_z^2}) |\uparrow\rangle}{\sqrt{h_x^2 + (h_z - \sqrt{h_x^2 + h_z^2})^2}},$$

with eigenenergy

$$\varepsilon_1 = +\frac{\sqrt{h_x^2 + h_z^2}}{2}.$$

Now that we have obtained the eigenstates and the corresponding energies of the one-particle Hamiltonian, we return to the general Hamiltonian (1).

3. Energy Levels and Degeneracies

From the results of the previous section and the fact that the particles are non-interacting, it is clear that there are $N + 1$ energy levels. We obtain them presently.

3.1 Ground State

The ground state $|E_0\rangle$ of the system of non-interacting N spin-1/2 particles is the direct product state

$$\begin{aligned} |E_0\rangle &= |\varepsilon_0\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle \\ &= (|\varepsilon_0\rangle)^N . \end{aligned} \quad (2)$$

The ground state energy of the system is found as follows

$$\begin{aligned} H |E_0\rangle &= \sum_i H_i |E_0\rangle \\ &= (H_1 |\varepsilon_0\rangle) |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle \\ &\quad + |\varepsilon_0\rangle (H_2 |\varepsilon_0\rangle) |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle \\ &\quad + |\varepsilon_0\rangle |\varepsilon_0\rangle (H_3 |\varepsilon_0\rangle) |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle \\ &\quad + \cdots \cdots \cdots \\ &\quad + |\varepsilon_0\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle (H_N |\varepsilon_0\rangle) . \end{aligned}$$

Since $H_i |\varepsilon_0\rangle = \varepsilon_0 |\varepsilon_0\rangle$ and there are N terms in the above sum, we thus find that

$$H |E_0\rangle = N\varepsilon_0 |E_0\rangle .$$

The ground state energy of the model is therefore

$$\begin{aligned} E_0 &= N\varepsilon_0 \\ &= -\frac{N}{2} \sqrt{h_x^2 + h_z^2} . \end{aligned}$$

We shall have more to say about the ground state, in the next section, but presently we obtain the other energy levels.

3.2 Excited States

A first excited state (FES) of the model is

$$|E_1\rangle^1 = |\varepsilon_0\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle |\varepsilon_1\rangle .$$

We see at once that the FES is N -fold degenerate, since $|\varepsilon_1\rangle$ can occur anywhere in the direct product state. The superscript 1 was affixed in anticipation. The remaining $N - 1$ states that are degenerate with $|E_1\rangle^1$ are

$$\begin{aligned} |E_1\rangle^2 &= |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle |\varepsilon_1\rangle |\varepsilon_0\rangle \\ |E_1\rangle^3 &= |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_1\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \\ &\quad \cdots \cdots \cdots \\ |E_1\rangle^N &= |\varepsilon_1\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle |\varepsilon_0\rangle . \end{aligned}$$

The energy of the first excited state is found from

$$\begin{aligned}
 H |E_1\rangle^1 &= \sum_i H_i |E_1\rangle^1 \\
 &= (H_1 |\varepsilon_0\rangle) |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle |\varepsilon_1\rangle \\
 &\quad + |\varepsilon_0\rangle (H_2 |\varepsilon_0\rangle) |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle |\varepsilon_1\rangle \\
 &\quad + \cdots \cdots \cdots \\
 &\quad + |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle (H_N |\varepsilon_1\rangle) \\
 &= (N-1)\varepsilon_0 |E_1\rangle^1 + \varepsilon_1 |E_1\rangle^1 \\
 &= [(N-1)\varepsilon_0 + \varepsilon_1] |E_1\rangle^1.
 \end{aligned}$$

From which it follows that

$$\begin{aligned}
 E_1 &= (N-1)\varepsilon_0 + \varepsilon_1 \\
 &= (N-2)\varepsilon_0.
 \end{aligned}$$

In general, a k th excited state has k $|\varepsilon_1\rangle$ factors in the direct product state. Since there are (N, k) ways of arranging the k $|\varepsilon_1\rangle$ factors, this means that the k th excited state has degeneracy $g(E_k)$, given by

$$g(E_k) = \binom{N}{k} = \frac{N!}{k!(N-k)!}.$$

The degenerate energy E_k is easily found by the above scheme to be

$$\begin{aligned}
 E_k &= (N-k)\varepsilon_0 + k\varepsilon_1 \\
 &= (N-2k)\varepsilon_0.
 \end{aligned}$$

We see that only the ground state $|E_0\rangle$ and the N th excited state $|E_N\rangle$ are non-degenerate.

4. An Explicit Expression for the Ground State

We recall from equation (2) that the ground state is the direct product state

$$|E_0\rangle = (|\varepsilon_0\rangle)^N, \tag{3}$$

where

$$|\varepsilon_0\rangle = \frac{h_x |\downarrow\rangle + \left(h_z + \sqrt{h_x^2 + h_z^2}\right) |\uparrow\rangle}{\sqrt{h_x^2 + \left(h_z + \sqrt{h_x^2 + h_z^2}\right)^2}}.$$

Performing the binomial expansion suggested by equation (3), we have that

$$|E_0\rangle = \frac{\sum_{m=0}^N h_x^{N-m} \left(h_z + \sqrt{h_x^2 + h_z^2}\right)^m |S_m\rangle}{\left[h_x^2 + \left(h_z + \sqrt{h_x^2 + h_z^2}\right)^2\right]^{N/2}}, \quad (4)$$

where $|S_m\rangle$ is the linear combination of the (N, m) states with m spins up (in the total S^z basis), that is, the linear combination of all states with total $S^z = m - N/2$.

We can check the trivial limits of (4):

(1) $h_z = 0$

$$H = -h_x \sum_i S_i^x$$

Putting $h_z = 0$ in (4) gives

$$\begin{aligned} |E_0\rangle &= \frac{\sum_{m=0}^N h_x^{N-m} h_x^m |S_m\rangle}{(2h_x^2)^{N/2}} \\ &= \frac{1}{(\sqrt{2})^N} \sum_{m=0}^N |S_m\rangle \\ &= \frac{1}{(\sqrt{2})^N} (|\uparrow\rangle + |\downarrow\rangle)^N \\ &= (|\rightarrow\rangle)^N \\ &= |\rightarrow\rightarrow\rightarrow \dots \rightarrow\rangle \end{aligned}$$

(2) $h_x = 0$

that is

$$H = -h_z \sum_i S_i^z$$

Here the ground state is trivially the all spins up, non-degenerate ferromagnetic state

$$|E_0\rangle = |\uparrow\uparrow\uparrow \dots \uparrow\rangle$$

with energy

$$E_0 = -Nh_z/2$$

Putting $h_x = 0$ in in (4), we see that the only non-vanishing term in the sum occurs when $m = N$ and so,

$$\begin{aligned} |E_0\rangle &= \frac{(2h_z)^N}{(2h_z)^N} |S_N\rangle \\ &= |\uparrow\uparrow\uparrow \dots \uparrow\rangle . \end{aligned}$$

5. Example Applications

In this section, we will consider two models with interaction, the interactions will be treated as perturbations. Specifically we will obtain the ground state energy of the Ising model in two fields and also the XZ model in non-commuting fields to second order in nearest neighbour exchange interactions. We will also compute first order corrections to the energy of the first excited state for each model.

We first note that for the one-particle system

$$\begin{aligned} S_z |\varepsilon_0\rangle &= \frac{1}{2} \frac{-h_x |\downarrow\rangle + (h_z + \sqrt{h_x^2 + h_z^2}) |\uparrow\rangle}{\sqrt{h_x^2 + (h_z + \sqrt{h_x^2 + h_z^2})^2}} \\ &= |\alpha\rangle \end{aligned}$$

and

$$\begin{aligned} S_z |\varepsilon_1\rangle &= \frac{1}{2} \frac{-h_x |\downarrow\rangle + (h_z - \sqrt{h_x^2 + h_z^2}) |\uparrow\rangle}{\sqrt{h_x^2 + (h_z - \sqrt{h_x^2 + h_z^2})^2}} \\ &= |\beta\rangle, \end{aligned}$$

so that we have the matrix elements

$$\begin{aligned} \langle \varepsilon_0 | S^z | \varepsilon_0 \rangle &= \langle \varepsilon_0 | \alpha \rangle = \frac{1}{2} \frac{h_z}{\sqrt{h_x^2 + h_z^2}} \\ \langle \varepsilon_1 | S^z | \varepsilon_0 \rangle &= \langle \varepsilon_1 | \alpha \rangle = -\frac{1}{2} \frac{h_x}{\sqrt{h_x^2 + h_z^2}} = \langle \varepsilon_0 | \beta \rangle \\ \langle \varepsilon_1 | S^z | \varepsilon_1 \rangle &= \langle \varepsilon_1 | \beta \rangle = -\frac{1}{2} \frac{h_z}{\sqrt{h_x^2 + h_z^2}}. \end{aligned} \tag{5}$$

Furthermore,

$$S_x |\varepsilon_0\rangle = \frac{1}{2} \frac{h_x |\uparrow\rangle + (h_z + \sqrt{h_x^2 + h_z^2}) |\downarrow\rangle}{\sqrt{h_x^2 + (h_z + \sqrt{h_x^2 + h_z^2})^2}}$$

and

$$S_x |\varepsilon_1\rangle = \frac{1}{2} \frac{h_x |\uparrow\rangle + (h_z - \sqrt{h_x^2 + h_z^2}) |\downarrow\rangle}{\sqrt{h_x^2 + (h_z - \sqrt{h_x^2 + h_z^2})^2}},$$

so that we have also the following matrix elements:

$$\begin{aligned} \langle \varepsilon_0 | S^x | \varepsilon_0 \rangle &= \langle \varepsilon_0 | \lambda \rangle = \frac{1}{2} \frac{h_x}{\sqrt{h_x^2 + h_z^2}} = -\langle \varepsilon_1 | \alpha \rangle \\ \langle \varepsilon_1 | S^x | \varepsilon_0 \rangle &= \langle \varepsilon_1 | \lambda \rangle = \frac{1}{2} \frac{h_z}{\sqrt{h_x^2 + h_z^2}} = \langle \varepsilon_0 | S^x | \varepsilon_1 \rangle = \langle \varepsilon_0 | \delta \rangle \\ \langle \varepsilon_1 | S^x | \varepsilon_1 \rangle &= \langle \varepsilon_1 | \delta \rangle = -\frac{1}{2} \frac{h_x}{\sqrt{h_x^2 + h_z^2}} = \langle \varepsilon_1 | \alpha \rangle = -\langle \varepsilon_0 | \lambda \rangle \end{aligned}$$

We are now ready to compute the energy corrections.

5.1 Perturbation Expansion of the Ground State Energy of the Ising Model in Non-Commuting Fields

The Ising model in non-commuting magnetic fields is described by the Hamiltonian

$$\begin{aligned} H_I &= -j \sum_i S_i^z S_{i+1}^z - h_x \sum_i S_i^x - h_z \sum_i S_i^z \\ &= V + H, \end{aligned} \quad (6)$$

where

$$V = -j \sum_i S_i^z S_{i+1}^z$$

and H is as given in (1).

The nearest neighbour exchange interaction j is assumed positive, so that we have a ferromagnetic model. Furthermore, we assume a periodic boundary condition, so that $S_{N+i}^z = S_i^z$. For weak interaction, V can be treated as a perturbation.

5.1.1 First Order Energy Correction

The first order correction to the ground state energy of the Hamiltonian (6) is given by

$$\begin{aligned} \Delta E_{0I}^{(1)} &= \langle E_0 | V | E_0 \rangle \\ &= -j \langle E_0 | \sum_i S_i^z S_{i+1}^z | E_0 \rangle. \end{aligned}$$

Now,

$$\begin{aligned} \sum_i S_i^z S_{i+1}^z | E_0 \rangle &= S_1^z S_2^z | E_0 \rangle + S_2^z S_3^z | E_0 \rangle + \cdots + S_N^z S_1^z | E_0 \rangle \\ &= |\alpha\rangle |\alpha\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots |\varepsilon_0\rangle \\ &\quad + |\varepsilon_0\rangle |\alpha\rangle |\alpha\rangle |\varepsilon_0\rangle \cdots \cdots |\varepsilon_0\rangle \\ &\quad + |\varepsilon_0\rangle |\varepsilon_0\rangle |\alpha\rangle |\alpha\rangle \cdots \cdots |\varepsilon_0\rangle \\ &\quad + \cdots \cdots \cdots \\ &\quad + |\alpha\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots |\varepsilon_0\rangle |\alpha\rangle. \end{aligned} \quad (7)$$

Multiplying from the left by $\langle E_0 |$, we have

$$\begin{aligned} \langle E_0 | \sum_i S_i^z S_{i+1}^z | E_0 \rangle &= N (\langle \varepsilon_0 | \alpha \rangle)^2 \\ &= \frac{N}{4} \frac{h_z^2}{h_x^2 + h_z^2}. \end{aligned}$$

We therefore have that the first order correction to the ground state energy of the Ising model in non-commuting magnetic fields is

$$\Delta E_{0_I}^{(1)} = \frac{-jN}{4} \frac{h_z^2}{h_x^2 + h_z^2}.$$

5.1.2 Second Order Correction

The second order correction to the ground state energy of the Ising model in non-commuting fields has the form

$$\Delta E_{0_I}^{(2)} = \sum_E \frac{|\langle E | V | E_0 \rangle|^2}{E_0 - E}. \quad (8)$$

From the form of (7) and the fact that $\langle \varepsilon_0 | \varepsilon_1 \rangle = 0$ it is clear that only states with $E = E_1$ and those with $E = E_2$ can contribute to the sum in (8). We consider them in turns.

When $E = E_1$, we observe that $|E_1\rangle^1 = |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots |\varepsilon_0\rangle |\varepsilon_1\rangle$ gives

$$\begin{aligned} & {}^1\langle E_1 | \sum_i S_i^z S_{i+1}^z | E_0 \rangle \\ &= \langle \varepsilon_1 | \langle \varepsilon_0 | \langle \varepsilon_0 | \cdots \cdots \langle \varepsilon_0 | \varepsilon_0 \rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots |\varepsilon_0\rangle |\alpha\rangle |\alpha\rangle \\ & \quad + \langle \varepsilon_1 | \langle \varepsilon_0 | \langle \varepsilon_0 | \cdots \cdots \langle \varepsilon_0 | \alpha \rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots |\varepsilon_0\rangle |\varepsilon_0\rangle |\alpha\rangle \\ &= 2 \langle \varepsilon_1 | \alpha \rangle \langle \varepsilon_0 | \alpha \rangle. \end{aligned}$$

We have similar results from the remaining $N - 1$ states that are degenerate with $|E_1\rangle^1$. The $E = E_1$ states therefore contribute

$$N \frac{(2 \langle \varepsilon_1 | \alpha \rangle \langle \varepsilon_0 | \alpha \rangle)^2 (-j)^2}{2\varepsilon_0} = \frac{Nj^2}{8\varepsilon_0} \frac{h_x^2 h_z^2}{(h_x^2 + h_z^2)^2}$$

to the sum in (8). We have used (5) to substitute for $\langle \varepsilon_1 | \alpha \rangle$ and $\langle \varepsilon_0 | \alpha \rangle$ and we have also used the fact that $E_0 - E_k = 2k\varepsilon_0$.

As for the states with $E = E_2$, only N (those with two consecutive $|\varepsilon_1\rangle$ factors) of the $(N, 2)$ states contribute to the sum. Their contribution is

$$N \frac{(\langle \varepsilon_1 | \alpha \rangle)^2 (-j)^2}{4\varepsilon_0} = \frac{Nj^2}{64\varepsilon_0} \frac{h_x^4}{(h_x^2 + h_z^2)^2}.$$

The second order correction to the ground state energy of the Ising model in non-commuting fields is therefore

$$\Delta E_{0_I}^{(2)} = \frac{Nj^2}{8\varepsilon_0} \frac{h_x^2 h_z^2}{(h_x^2 + h_z^2)^2} + \frac{Nj^2}{64\varepsilon_0} \frac{h_x^4}{(h_x^2 + h_z^2)^2}.$$

5.2 Perturbation expansion of the ground state energy of the XZ model in non-commuting fields.

The XZ model in one dimension, in the presence of non-commuting external fields is described by the Hamiltonian

$$H_{xz} = -j_1 \sum_{i=1}^N S_i^x S_{i+1}^x - j_2 \sum_{i=1}^N S_i^z S_{i+1}^z - h_x \sum_{i=1}^N S_i^x - h_z \sum_{i=1}^N S_i^z,$$

where $j_1 > 0$ and $j_2 > 0$ are the nearest neighbour exchange interaction energy. Again we assume a periodic boundary condition, so that $S_{N+i}^x = S_i^x$ and $S_{N+i}^z = S_i^z$. For weak nearest neighbour interactions, after calculations similar to that in the previous section, we obtain the following corrections to the ground state energy:

$$\Delta E_{0xz}^{(1)} = -\frac{N}{4} \frac{1}{h_x^2 + h_z^2} (j_1 h_x^2 + j_2 h_z^2)$$

and

$$\begin{aligned} \Delta E_{0xz}^{(2)} &= \frac{N}{8\varepsilon_0} \frac{h_x^2 h_z^2}{(h_x^2 + h_z^2)^2} (j_1^2 + j_2^2) \\ &+ \frac{N}{64\varepsilon_0} \frac{1}{(h_x^2 + h_z^2)^2} (j_1^2 h_x^4 + j_2^2 h_z^4). \end{aligned}$$

The ground state energy of the XZ model in non-commuting fields, to second order in interactions j_1 and j_2 is thus

$$\begin{aligned} E_{0xz} &= -\frac{N}{2} \sqrt{h_x^2 + h_z^2} - \frac{N}{4} \frac{1}{h_x^2 + h_z^2} (j_1 h_x^2 + j_2 h_z^2) \\ &+ \frac{N}{8\varepsilon_0} \frac{h_x^2 h_z^2}{(h_x^2 + h_z^2)^2} (j_1^2 + j_2^2) \\ &+ \frac{N}{64\varepsilon_0} \frac{1}{(h_x^2 + h_z^2)^2} (j_1^2 h_x^4 + j_2^2 h_z^4). \end{aligned}$$

5.3 First Order Correction to the Energy of the First Excited State (FES) of the Ising Model in Non-Commuting Fields

The first excited state, in the absence of interactions, is N -fold degenerate, with the N states as given in (3.2). We therefore use degenerate perturbation theory to find the first order correction to the first excited state energy of the Ising model in non-commuting fields.

$$\begin{aligned} \sum_i S_i^z S_{i+1}^z |E_1\rangle^1 &= |\alpha\rangle |\alpha\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_1\rangle \\ &+ |\varepsilon_0\rangle |\alpha\rangle |\alpha\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle |\varepsilon_1\rangle \\ &+ \cdots \cdots \cdots \\ &+ |\varepsilon_0\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle |\alpha\rangle |\beta\rangle \\ &+ |\alpha\rangle |\varepsilon_0\rangle |\varepsilon_0\rangle \cdots \cdots \cdots |\varepsilon_0\rangle |\varepsilon_0\rangle |\beta\rangle. \end{aligned}$$

We therefore have that

$$\begin{aligned} & {}^1\langle E_1 | \sum_i S_i^z S_{i+1}^z | E_1 \rangle^1 \\ &= (N-2) (\langle \varepsilon_0 | \alpha \rangle)^2 + 2 \langle \varepsilon_0 | \alpha \rangle \langle \varepsilon_1 | \beta \rangle \\ &= \frac{h_z^2}{h_x^2 + h_z^2} \left(\frac{N}{4} - 1 \right). \end{aligned}$$

A similar calculation for the remaining $N-1$ states shows that the diagonal elements of the $N \times N$ perturbation matrix V , are equal, being given by:

$$\begin{aligned} V_{ii} &= {}^i\langle E_1 | \left(-j \sum_i S_i^z S_{i+1}^z \right) | E_1 \rangle^i \\ &= \frac{-j h_z^2}{h_x^2 + h_z^2} \left(\frac{N}{4} - 1 \right) = f. \end{aligned} \quad (9)$$

We can also compute V_{21} thus

$$\begin{aligned} V_{21} &= {}^2\langle E_1 | \left(-j \sum_i S_i^z S_{i+1}^z \right) | E_1 \rangle^1 \\ &= -j \langle \varepsilon_0 | \langle \varepsilon_1 | \langle \varepsilon_0 | \cdots \cdots \langle \varepsilon_0 | \varepsilon_0 \rangle \cdots \cdots | \alpha \rangle | \beta \rangle \\ &= -j \langle \varepsilon_1 | \alpha \rangle \langle \varepsilon_0 | \beta \rangle \\ &= \frac{-j}{4} \frac{h_x^2}{h_x^2 + h_z^2}. \end{aligned}$$

In fact, for $m \neq n$, we have

$$\begin{aligned} V_{mn} &= {}^m\langle E_1 | \left(-j \sum_i S_i^z S_{i+1}^z \right) | E_1 \rangle^n \\ &= \frac{-j}{4} \frac{h_x^2}{h_x^2 + h_z^2} \delta_{m,n\pm 1} \\ &= V_{nm} = g. \end{aligned} \quad (10)$$

The $N \times N$ perturbation matrix V is therefore

$$V = \begin{pmatrix} f & g & 0 & 0 & 0 & 0 & \cdots & g \\ g & f & g & 0 & 0 & 0 & \cdots & 0 \\ 0 & g & f & g & 0 & 0 & \cdots & 0 \\ 0 & 0 & g & f & g & 0 & \cdots & 0 \\ 0 & 0 & 0 & g & f & g & & \vdots \\ 0 & 0 & 0 & & \ddots & & & \\ \vdots & & & & & g & f & g \\ g & 0 & 0 & 0 & \cdots & 0 & g & f \end{pmatrix}$$

where f and g are as given in (9) and (10). Clearly f and g are negative for $N > 4$. The smallest eigenvalue of V is $2g + f$ and is non-degenerate, so that the correction to the first excited state energy, to first order in interaction j is

$$\begin{aligned}\Delta E_{1_I}^{(1)} &= 2g + f \\ &= \frac{-j}{2} \frac{h_x^2}{h_x^2 + h_z^2} - \frac{j h_z^2}{h_x^2 + h_z^2} \left(\frac{N}{4} - 1 \right).\end{aligned}$$

The nearest neighbour interactions thus lift the degeneracy in the first excited state of the model. The energy of the first excited state of the Ising model in non-commuting fields, to first order in j is therefore

$$E_{1_I} = \left(1 - \frac{N}{2} \right) \sqrt{h_x^2 + h_z^2} - \frac{j}{2} \frac{h_x^2}{h_x^2 + h_z^2} - \frac{j h_z^2}{h_x^2 + h_z^2} \left(\frac{N}{4} - 1 \right).$$

5.4 First Order Correction to the Energy of the First Excited State (FES) of the XZ Model in Non-Commuting Fields

A completely analogous treatment to the above gives the energy of the first excited state of the XZ model in non-commuting fields, to first order in the interactions j_1 and j_2 as

$$\begin{aligned}E_{1_{XY}} &= - \left(\frac{N}{2} - 1 \right) \sqrt{h_x^2 + h_z^2} \\ &\quad - \frac{1}{2} \frac{j_2 h_x^2 + j_1 h_z^2}{h_x^2 + h_z^2} - \left(\frac{N}{4} - 1 \right) \frac{j_1 h_x^2 + j_2 h_z^2}{h_x^2 + h_z^2}.\end{aligned}$$

Conclusion

We have obtained all the energy levels of a system of N non-interacting spin-1/2 particles in non-commuting external magnetic fields, in one dimension. The energy of the k th level is $E_k = (N - 2k)\varepsilon_0$, where $\varepsilon_0 = -\sqrt{(h_x^2 + h_z^2)}/2$ and $k = 0, 1, 2, \dots, N$. The k th energy level was found to be $(N, k) = N!/ [k!(N - k)!]$ -fold degenerate. An explicit expression for the ground state

was also derived (equation (4)). It is not difficult to give similar explicit expressions for the excited states. A scheme for obtaining the energies of the excited states was highlighted in section 3.2.

Since most real systems do interact, examples of how the interaction terms of an Hamiltonian can be included by Rayleigh-Schrödinger perturbation was demonstrated by finding energy corrections to the ground state energy of the Ising model in non-commuting fields, as well as to the ground state energy of the XZ model in non-commuting fields. Corrections to the first excited state energy were also calculated.

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An Open Question: Are Topological Arguments Helpful in Setting Initial Conditions for Transport Problems and Quantization Criteria/Quantum Computing for Density Wave Physics?

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Abstract: The tunneling Hamiltonian is a proven method to treat particle tunneling between different states represented as wavefunctions in many-body physics. Our problem is how to apply a wave functional formulation of tunneling Hamiltonians to a driven sine-Gordon system. We apply a generalization of the tunneling Hamiltonian to charge density wave (CDW) transport problems in which we consider tunneling between states that are wavefunctionals of a scalar quantum field. We present derived I-E curves that match Zenier curves used to fit data experimentally with wavefunctionals congruent with the false vacuum hypothesis. The open question is whether the coefficients picked in both the wavefunctionals and the magnitude of the coefficients of the driven sine Gordon physical system should be picked by topological charge arguments that in principle appear to assign values that have a tie in with the false vacuum hypothesis first presented by Sidney Coleman. Our supposition is that indeed this is useful and that the topological arguments give evidence as to a first order phase transition which gives credence to the observed and calculated I-E curve as evidence of a quantum switching phenomena in density wave physics, one which we think with further development would have applications to quantum computing, via quantum coherent phase evolution, as outlined in this paper.

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1. Introduction

This paper's main result has a very strong convergence with the slope of graphs of electron-positron pair production representations. The newly derived results include a threshold electric field explicitly as a starting point without an arbitrary cut off value for the start of the graphed results, thereby improving on both the Zener plots and Lin's generalization of Schwingers 1950 electron-positron nucleation values results for low dimensional systems. The similarities in plot behavior of the current values after the threshold electric field values argue in favor of the Bardeen pinning gap paradigm. We conclude with a discussion of how these results can be conceptually linked to a new scheme of exact evolution of the dynamics of quantum ϕ^4 field theory in 1+1 dimensions. In several dimensions, we find that the Gaussian wavefunctionals would be given in the form given by Lu. Lu's integration given below is a two dimensional Gaussian wave functional. The analytical result we are working with is a one-dimensional version of a ground-state wave functional of the form

$$|0 \rangle = N \cdot \exp \left\{ - \int_{x,y} [(\phi_x - \varphi) \cdot f_{xy} \cdot (\phi_y - \varphi)] \cdot dx \cdot dy \right\} \quad (1)$$

Lu's Gaussian wave functional is for a non-perturbed, Hamiltonian as given in Eq. (17b) below

$$H_O = \int_x \left[\frac{1}{2} \cdot \Pi_x^2 + \frac{1}{2} \cdot (\partial_x \phi_x)^2 + \frac{1}{2} \cdot \mu^2 \cdot (\phi_x - \varphi)^2 - \frac{1}{2} \cdot I_0(\mu) \right] \cdot dx \cdot dy \quad (2)$$

We should note that Lu intended the wavefunctional given in Eq. (17a) to be a test functional, much as we would do for finding an initial test functional, using a simple Gaussian in computing the ground state energy of a simple Harmonic oscillator variational derivative. calculation. We may obtain a 'ground state' wave functional by taking the one dimensional version of the integrand given in Eq. (17a). This means have [1, 2??] a robust Gaussian. Lu' Gaussian wave functional set

$$\frac{\partial^2 \cdot V_E}{\partial \cdot \phi^a \cdot \partial \cdot \phi^b} \propto f_{xy} \quad (3)$$

Here, we call V_E a (Euclidian-time style) potential, with subscripts a and b referring to dimensionality; and ϕ_x an 'x dimension contribution' of alternations of 'average' phase φ , as well as ϕ_y an 'y dimension contribution' of alternations of 'average' phase φ . This average phase is identified in the problem we are analyzing as ϕ_C

This leads to writing the new Gaussian wavefunctional to be looking like

$$\Psi \equiv c \exp(-\alpha \cdot \int dx [\phi - \phi_C]^2) \quad (4)$$

Making this step from Eq. (17a) to Eq. (18) involves recognizing, when we go to one-dimension, that we look at a washboard potential with pinning energy contribution from

$D \cdot \omega_P^2$ in one- dimensional CDW systems

$$\frac{1}{2} \cdot D \cdot \omega_P^2 \cdot (1 - \cos \phi) \approx \frac{1}{2} \cdot D \cdot \omega_P^2 \cdot \left(\frac{\phi^2}{2} - \frac{\phi^4}{24} \right) \quad (5)$$

The fourth-order phase term is relatively small, so we look instead at contributions from the quadratic term and treat the fourth order term as a small perturbing contribution to get our one dimensional CDW potential, for lowest order, to roughly look like Eq. (19b). In addition, we should note that the c is due to an error functional-norming procedure, discussed below; α is proportional to one over the length of distance between instaton centers

Figure 1 below represents the constituent components of a S-S' pair; the phase value, ϕ_C , is set to represent a configuration of phase in which the system evolves to/from in the course of the S-S' pair evolution. This leads to

$$c_1 \cdot \exp \left(-\alpha_1 \cdot \int d\tilde{x} [\phi_F]^2 \right) \cong \Psi_{initial} \quad (6)$$

As well as

$$c_2 \cdot \exp \left(-\alpha_2 \cdot \int d\tilde{x} [\phi_T]^2 \right) \cong \Psi_{final} \quad (7)$$

$$f_{xy} \xrightarrow{\text{reduction-to-one-dim}} \delta(x-y)/L^{1+\delta+} \quad (8)$$

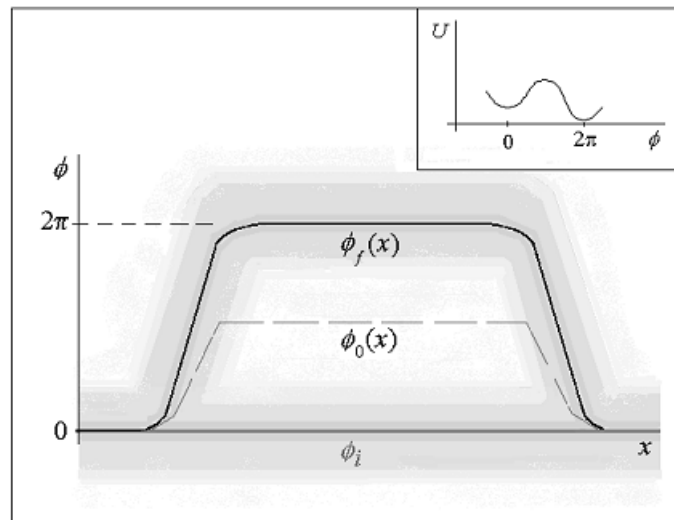


Fig. 1 Evolution from an initial state $\Psi_i[\phi]$ to a final state $\Psi_f[\phi]$ for a double-well potential (inset) in a 1-D model, showing a kink-antikink pair bounding the nucleated bubble of true vacuum. The shading illustrates quantum fluctuations about the initial and final optimum configurations of the field, while $\phi_0(x)$ represents an intermediate field configuration inside the tunnel barrier. The upper right hand side of this figure is how the fate of the false vacuum hypothesis gives a difference in energy between false and true potential vacuum values.

Whereas in multi dimensional treatments, we have

$$f_{xy} \approx \frac{\partial V_{eff}}{\partial r} \quad (9)$$

In the current vs. applied electric field derivation results, we identify the $\Psi_i[\phi]$ as the initial wave function at the left side of a barrier and $\Psi_f[\phi]$ as the final wave function at the right side of a barrier. This can most easily be seen in the following diagram of how the S-S' pair structure arose in the first place, as given by Fig. 2:

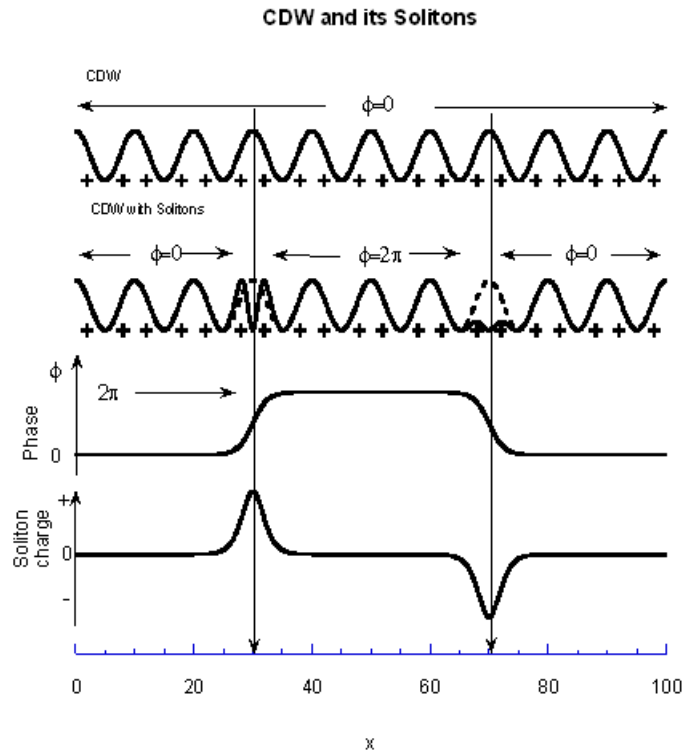


Fig. 2 The above figures represents the formation of soliton-anti soliton pairs along a ‘chain’ . The evolution of phase is spatially given by $\phi(x) = \pi \cdot [\tanh b(x - x_a) + \tanh b(x_b - x)]$

The tunneling Hamiltonian incorporates wavefunctionals whose Gaussian shape keeps much of the structure as represented by Fig. 2. Following the false vacuum hypothesis, we have a false vacuum phase value $\phi_F \equiv \langle \phi \rangle_1 \cong$ *very small value*, as well as having in CDW, a final true vacuum $\phi_T \cong \phi_{2\pi} \equiv 2 \cdot \pi + \varepsilon^+$. This led to Gaussian wavefunctionals with a simplified structure. For experimental reasons, we need to have (if we set the charge equal to unity, dimensionally speaking)

$$\alpha \approx L^{-1} \equiv \Delta E_{gap} \equiv V_E(\phi_F) - V_E(\phi_T) \quad (10)$$

This is equivalent to the situation as represented by Fig. 3

$$I \propto \tilde{C}_1 \cdot \left[\cosh \left[\sqrt{\frac{2 \cdot E}{E_T \cdot c_V}} - \sqrt{\frac{E_T \cdot c_V}{E}} \right] \right] \cdot \exp \left(-\frac{E_T \cdot c_V}{E} \right) \quad (11)$$

The current expression is a great improvement upon the phenomenological Zener current expression, where G_P is the limiting Charge Density Wave (CDW) conductance.

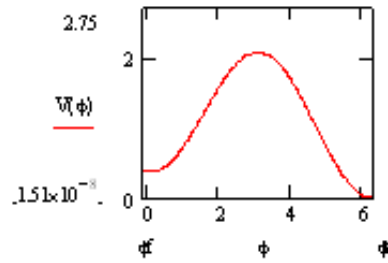


Fig. 3 Fate of the false vacuum representation of what happens in CDW. This shows how we have a difference in energy between false and true vacuum values. This eventually leads to a current along the lines of..???

$$I \propto G_P \cdot (E - E_T) \cdot \exp\left(-\frac{E_T}{E}\right) \quad \text{if } E > E_T \tag{12}$$

$$0 \quad \text{otherwise}$$

Fig. 4 illustrates to how the pinning gap calculation improve upon a phenomenological curve fitting result used to match experimental data. The most important feature here is that the theoretical equation takes care of the null values before thre threshold is reached by itself. I.e. we do not need to set it to zero as is done arbitrarily in Eqn (12). This is in any case tied in with a tilted zenier.

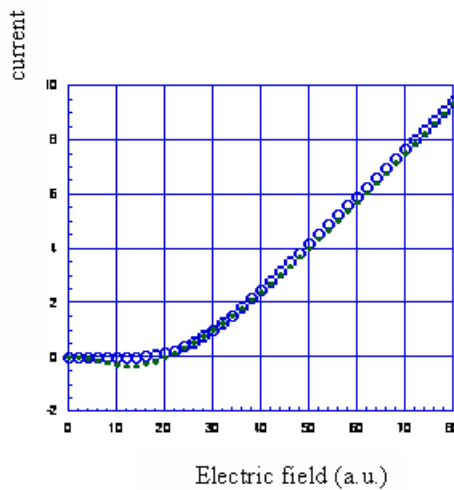


Fig. 4 Experimental and theoretical predictions of current values versus applied electric field. The dots represent a Zenier curve fitting polynomial, whereas the blue circles are for the $S - S'$ transport expression derived with a field theoretic version of a tunneling Hamiltonian.

So then, we have $L \propto E^{-1}$.. When we consider a Zener diagram of CDW electrons with tunneling only happening when $e^* \cdot E \cdot L > \varepsilon_G$ where e^* is the effective charge of each condensed electron and ε_G being a pinning gap energy, we find , assuming that x is the de facto distance between an instanton pair and a measuring device.

In the current vs. applied electric field derivation results, we identify the $\Psi_i[\phi]$ as the initial wave function at the left side of a barrier and $\Psi_f[\phi]$ as the final wave function

at the right side of a barrier. Note that Tekman⁵ extended the tunneling Hamiltonian method to encompass more complicated geometries. We notice that when the matrix elements T_{kq} are small, we calculate the current through the barrier using linear response theory. This may be used to describe coherent Josephson-like tunneling of either Cooper pairs of electrons or boson-like particles, such as super fluid He atoms. In this case, the supercurrent is linear with the effective matrix element for transferring a pair of electrons or transferring a single boson, as shown rather elegantly in Feynman's derivation of the Josephson current-phase relation. This means a current density proportional to $|T|$ rather than $|T|^2$ since tunneling, in this case, would involve coherent transfer of individual (first-order) bosons rather than pairs of fermions. Note that the initial and final wave functional states were in conjunction with a pinning gap formulation of a variation of typical band calculation structures. This also lead us to after much work to make the following scaling rule which showed up in the linkage between the tilted zenier band model and the distance to a measuring device, which we call x , representing the distance between a modeled instanton structure in density wave physics and a measuring device.

$$\frac{L}{x} \cong c_v \cdot \frac{E_T}{E} \quad (13)$$

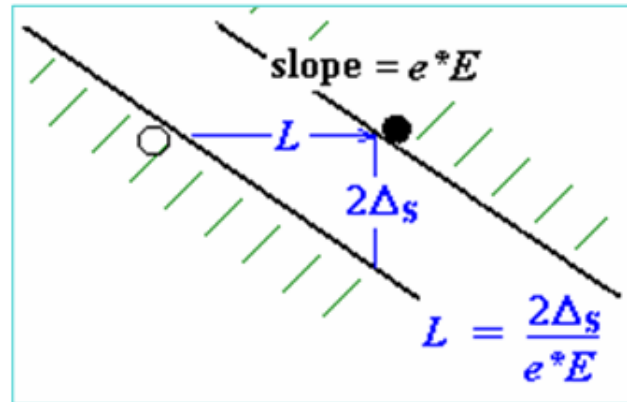


Fig. 5 This is a representation of ‘Zener’ tunneling through pinning gap with band structure tilted by applied E field

2. Comparison With LIN’s Generalization

In a 1999, Qiong-gui Lin proposed a general rule regarding the probability of electron-positron pair creation in $D+1$ dimensions, with D varying from one to three, leading, in the case of a pure electric field, to

$$w_E = (1 + \delta_{d3}) \cdot \frac{|e \cdot E|^{(D+1)/2}}{(2 \cdot \pi)^D} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{(D+1)/2}} \cdot \exp\left(-\frac{n \cdot \pi \cdot m^2}{|e \cdot E|}\right) \quad (14)$$

When D is set equal to three, we get (after setting $e^2, m \equiv '1'$)

$$wIII(E) = \frac{|E|^2}{(4 \cdot \pi^3)} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \exp\left(-\frac{n \cdot \pi}{|E|}\right) \quad (15)$$

which, if graphed gives a comparatively flattened curve compared w.r.t. to what we get when D is set equal to one (after setting $e^2, m \equiv 1$)

$$wI(E) = \frac{|E|^1}{(2 \cdot \pi^1)} \cdot \sum_{n=1}^{\infty} \frac{1}{n^1} \cdot \exp\left(-\frac{n \cdot \pi}{|E|}\right) \equiv -\frac{|E|}{2 \cdot \pi} \cdot \ln\left[1 - \exp\left(-\frac{\pi}{E}\right)\right] \quad (16)$$

which is far more linear in behavior for an e field varying from zero to a small numerical value. We see these two graphs in Fig. 6.

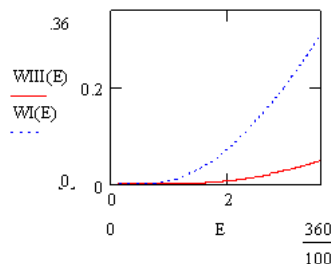


Fig. 6 Two curves representing probabilities of the nucleation of an electron-positron pair in a vacuum. $wI(E)$ is a nearly-linear curve representing a 1+1 dimensional system, whereas the second curve is for a 3 + 1 dimensional physical system and is far less linear

This is indicating that, as dimensionality drops, we have a steady progression toward linearity. The three-dimensional result given by Lin is merely the Swinger result observed in the 1950s. When I have $D = 1$ and obtain behavior very similar to the analysis completed for the SS' current argument just presented, the main difference is in a threshold electric field that is cleanly represented by our graphical analysis. This is a major improvement in the prior curve fitting exercised used in 1985 to curve-fit data.

3. Conclusion- and Linkage to Exact Dynamics of ϕ^4 Field Theory in 1+1 Dimensions

We restrict this analysis to ultra fast transitions of CDW; this is realistic and in sync with how the wavefunctionals used are formed in part by the fate of the false vacuum hypothesis.

Additionally, we explore the remarkable similarities between what we have presented here and Lin's expansion of Schwinger's physically significant work in electronpositron pair production. That is, the pinning wall interpretation of tunneling for CDW permits construction of $I-E$ curves that match experimental data sets; beforehand these were merely Zener curve fitting polynomial constructions.

Having obtained the $I-E$ curve similar to Lin's results gives credence to a pinning gap analysis of CDW transport, with the main difference lying in the new results giving a definitive threshold field effect, whereas both the Zenier curve fit polynomial and Lin's

results are not with a specifically delineated threshold electric field. v The derived result does not have the arbitrary zero value cut off specified for current values below given by Miller et al¹ in 1985 but gives this as a result of an analytical derivation. This assumes that in such a situation that the electric field is below a given threshold value.

We used the absorption of a Peierls gap Δ' term as clearly demonstrated in a numerical simulation paper I wrote to help form solitons (anti-solitons) used in my Gaussian wave functionals for the reasons stated in my. IJMPB article. This is new physics which deserves serious further investigation. It links our formalism formally with a JJ (Josephon junction) approach, and provides analogies worth pursuing in a laboratory environment. The also stunning development is that the plotting of Eqn (16) ties in with the electron-positron plots as given by Lin in Fig (6) for low dimensional systems, which conveniently fits with identification of a S-S' pair with different 'charge centers'

How does this relate to what we can think of concerning questions we raise about phase transitions ?

Question 1) Can we use a topological model for phase transitions concerning the change from a false vacuum to a real vacuum? The answer to this one lies in our interpretation of $\alpha \approx L^{-1} \equiv \Delta E_{gap} \equiv V_E(\phi_F) - V_E(\phi_T)$, and in the congruency of Eqn. (16) with the I-E plot given by Eqn. (11). If there is a close analytical match up between Eqn. (11) and Eqn. (16), the topological match up is probably not necessary, and we can de facto stick with analytical derivation of current with its direct match up with electron – positron models. If we cannot get a close to exact congruency between Eqn. (11) and (16) then perhaps we should think of the Bogomolnyi inequality approach mentioned in several publications by this author to explain the inherent breaking of symmetry presented above

Question 2) What about linkage to quantum switching devices implied by the abrupt I-E curve turn on implied by **Fig 4** above? My intuition is that this can be answered if or not there is a phase transition. I.e. if the Bogomolnyi inequality can to be used directly to get $\alpha \approx L^{-1} \equiv \Delta E_{gap} \equiv V_E(\phi_F) - V_E(\phi_T)$ exactly configured as a bridge of the ϕ^4 field theory in 1+1 dimension to Gaussian wave functionals, we will have quantum switching and ultra fast data transitions along requirements needed for phase transitions. Please see the following references as to how this is modeled

Question 3) What about direct analogies as to how we can solve a ϕ^4 field theory in 1+1 dimensions exactly in terms of real time evolution, without invoking the Bogomolnyi inequality? This is in terms of taking a real time evolution of the phase is a way to fill in detail alluded to in **Fig 1** , making use of the following

$$\begin{aligned}
 \Psi_f [\phi(\mathbf{x})] |_{\phi \equiv \phi_{Cf}} &= \\
 c_f \cdot \exp \left\{ - \int d\mathbf{x} \alpha [\phi_{Cf}(\mathbf{x}) - \phi_0(\mathbf{x})]^2 \right\} &\rightarrow \\
 c_2 \cdot \exp \left(-\alpha_2 \cdot \int d\tilde{x} [\phi_T]^2 \right) &\cong \Psi_{final},
 \end{aligned} \tag{17a}$$

and

$$\begin{aligned}
 \Psi_i [\phi(\mathbf{x})] |_{\phi \equiv \phi_{Ci}} &= \\
 c_i \cdot \exp \left\{ -\alpha \int d\mathbf{x} [\phi_{Ci}(\mathbf{x}) - \phi_0]^2 \right\} &\rightarrow \\
 c_1 \cdot \exp \left(-\alpha_1 \cdot \int d\tilde{x} [\phi_F]^2 \right) &\equiv \Psi_{initial},
 \end{aligned} \tag{17b}$$

This will represent a kink, anti kink combination with the kink given to us as part of Coopers presentation of a symplectic algorithm of updating the operator equations of quantum evolution of a potential system he writes as

$$V[\phi] = -\frac{m^2}{2} \cdot \phi^2 + \frac{\lambda}{4} \cdot \phi^4 \leftrightarrow \frac{1}{2} \cdot D \cdot \omega_P^2 \cdot (1 - \cos \phi) \approx \frac{1}{2} \cdot D \cdot \omega_P^2 \cdot \left(\frac{\phi^2}{2} - \frac{\phi^4}{24} \right) \tag{18}$$

which has a kink solution of the form $\phi[x] = \frac{m}{\sqrt{\lambda}} \cdot \tanh \frac{mx}{\sqrt{2}}$. A kink-anti kink structure so implied by the Gaussian wave functional is stated by Cooper, quoting Moncrief to have an evolution given by a symplectic evolution equation, as given below assuming an averaging procedure we can write as

$$y_i \sim \int_{V_i} dx \phi[x, t] / \Delta V_i \approx \text{average of } \phi[x, t] \text{ in a ball about } x_i \text{ of volume } \Delta V_i \tag{19a}$$

And

$$\frac{dy_i}{dt} \equiv \pi_i[t] \tag{19b}$$

And

$$\frac{d\pi_i}{dt} \equiv \frac{1}{a^2} \cdot [y_{i+1} + y_{i-1} - 2y_i] - \lambda y_i^3 + m^2 y_i = F[y_i] \tag{19c}$$

This is assuming that we spatially discretize a Hamiltonian density via

$$\int dx \rightarrow a \cdot \sum_i \tag{19d}$$

Following a field theory replacement of $\hat{x} \rightarrow \phi_{op}[x, t]$, and a discretized time structure given by $t = j \in$

This leads to the possibility of looking at a quantum foam evolution as given in **Fig 1** via the following symplectic structures , with i the ‘spatial component along a chain’ , and j the ‘time component’ along a chain. Eqn. (19e) and Eqn (19f) are materially no different than having energy course through a wave lattice as seen in ocean swells accommodating an energy pulse through the water.

$$y_i [j + 1] = y_i [j] + \epsilon \cdot \pi_i [j] + \frac{\epsilon^2}{2} \cdot F (y_i [j]) \quad (19e)$$

$$\pi_i [j + 1] = \pi_i [j] + \frac{\epsilon}{2} \cdot (F (y_i [j]) + F (y_{i+1} [j])) \quad (19f)$$

A proper understanding of this evolution dynamic should permit a more mature quantum foam interpretation of false vacuum nucleation. This is, of course independent of the datum that adjacent chains in themselves interacting are a necessary condition for the formation of instantons in the first place, as given by the following appendix entry presentation as to a necessary condition for the formation of a kink (anti kink) in the first place

4. Appendix: Formation of the Instanton Via Adjacent Chains

What role does the multi chain argument play as far as formulation of the soliton – instanton ? Why add in the Pierls gap term in the first place?

Answer

First of all, we add the following term, based upon the Pierls gap to an analysis of how an instanton evolves

$$H = \sum_n \left[\frac{\Pi_n^2}{2 \cdot D_1} + E_1 [1 - \cos \phi_n] + E_2 (\phi_n - \Theta)^2 + \Delta' \cdot [1 - \cos (\phi_n - \phi_{n-1})] \right] \quad (1)$$

$\Pi_n = \left(\frac{\hbar}{i} \right) \cdot \partial / \partial \phi_n$ which then permits us to write

$$U \approx E_1 \cdot \sum_{l=0}^{n+1} [1 - \cos \phi_l] + \frac{\Delta'}{2} \cdot \sum_{l=0}^n (\phi_{l+1} - \phi_l)^2 \quad (2)$$

which allowed using $L = T - U$ a Lagrangian based differential equation of

$$\ddot{\phi}_i - \omega_0^2 [(\phi_{i+1} - \phi_i) - (\phi_i - \phi_{i-1})] + \omega_1^2 \sin \phi_i = 0 \quad (3)$$

with

$$\omega_0^2 = \frac{\Delta'}{m_e \cdot l^2} \quad (4)$$

$$\omega_1^2 = \frac{E_1}{m_e \cdot l^2} \quad (5)$$

where we assume the chain of pendulums, each of length l , leads to a kinetic energy

$$T = \frac{1}{2} \cdot m_e \cdot l^2 \cdot \sum_{j=0}^{n+1} \dot{\phi}_j^2 \quad (6)$$

To get this, we make the following approximation. This has

$$\Delta' (1 - \cos [\phi_n - \phi_{n-1}]) \rightarrow \frac{\Delta'}{2} \cdot [\phi_n - \phi_{n-1}]^2 + \text{very small H.O.T.s.} \quad (7)$$

and then consider a nearest neighbor interaction behavior via

$$V_{n.n.}(\phi) \approx E_1 [1 - \cos \phi_n] + E_2 (\phi_n - \Theta)^2 + \frac{\Delta'}{2} \cdot (\phi_n - \phi_{n-1})^2 \quad (8)$$

Here, we set $\Delta' \gg E_1 \gg E_2$, so then this is leading to a dimensionless Sine–Gordon equation we write as

$$\frac{\partial^2 \phi(z, \tau)}{\partial \tau^2} - \frac{\partial^2 \phi(z, \tau)}{\partial z^2} + \sin \phi(z, \tau) = 0 \quad (9)$$

Punch line. Without the Pierls term added in, we do not get a Sine Gordon equation. No Instanton formulation.

Please consult the following four references [3-6] as to the derivation and the significance of Eqn. (11) of this document. Called the ‘Diamond’ in TcSAM by Dr. Patrick Xie who pointed out the significance of this derivation to the Author during the Author’s Dissertation defense in December 2001.

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Self-Organization and Emergence in Neural Networks

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Abstract: The interest for neural networks stems from the fact that they appear as universal approximators for whatever kind of nonlinear dynamical system of arbitrary complexity. Because nonlinear systems, are studied mainly to model self-organization and emergence phenomena, there is the hope that neural networks can be used to investigate these phenomena in a simpler way, rather than resorting to nonlinear mathematics, with its unsolvable problems. In this contribution we will deal with the conditions granting for the occurring of self-organization and emergence phenomena in neural networks. We will present a number of arguments supporting the claim that emergence is strictly connected to the presence of quantum features in neural networks models.

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1. Introduction

In the last thirty years the study of self-organization and emergence phenomena has become one of the main topics of actual scientific research. Such a circumstance is due to two main reasons:

- (1) *theoretical* ; the development of research domains such as the *Theory of Dissipative Structures* (Nicolis & Prigogine, 1977) or the *Synergetics* (Haken, 1978, 1983), together with the advances in *Dynamical Systems Theory* (Guckenheimer & Holmes, 1983; Glendinning, 1994), gave new tools for modelling (and whence understanding and controlling) the emergence of macroscopic phenomena or structures from

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microscopic fluctuations; in turn, this fact seemed to open the possibility for building a theory of emergence in systems which were traditionally considered as not susceptible of a mathematical analysis, such as the biological, psychological, social and economical ones; thus, within scientific world began to be diffused the hope of solving, once for all, the problems connected to a number of celebrated dichotomies, such as *mind* vs. *body*, *substance* vs. *function*, *collective* vs. *individual*, *automatic* vs. *controlled*, and so on;

- (2) *technological*; the acknowledgement that a number of important physical phenomena, such as superconductivity and superfluidity, were nothing but particular cases of emergence, and the search for new technologies, such as the ones connected to *high-temperature superconductivity*, showed that emergence is not only an affair for philosophers, mathematicians, or psychologists, but even for engineers; besides, in more recent times the development of *agent-based software* and of *nanotechnology* forced engineers to deal with systems which, owing to purely physical reasons, cannot be *designed* nor *controlled* from outside; whence, in these contexts *self-design* and *self-control* appear as the only feasible strategies; in order to implement them, however, we need a reliable theory of emergence.

A formidable problem in studying self-organization and emergence is due to the fact that the available mathematical tools are very complex and their application often runs into insurmountable difficulties. The very explosion, in the last fifteen years, of *neural network models* seemed to offer a new way to cope with such a problem. Namely we can approximate whatever dynamical system, complex as it may be, through a suitable neural network which, in ultimate analysis, is nothing but a set of interconnected units, each one implementing a very simple input-output transfer function. The latter system should be much easier to study, mathematically or through computer simulations, than the original dynamical system approximated by the neural network. Thus, we could use neural networks to investigate the occurring of self-organization and emergence phenomena by resorting to methods which should be simpler than the traditional ones.

In this paper we will try to determine up to which point such a claim is tenable. To this end we will present a short review about neural networks and the behaviours they can exhibit. After introducing suitable definitions of self-organization and emergence we will, then, present a number of arguments supporting the claim that most neural networks behaviours *cannot* be qualified as emergent. The same arguments will be used to support a conjecture, according to which emergent phenomena can occur only within a new, non-traditional, class of neural networks, which will be named *quantum neural networks*. An attempt to characterize these latter, and their connection with *quantum computers*, will be contained in the final part of this paper.

2. What are Neural Networks?

Shortly speaking, the term *neural network* (cfr. Bishop, 1995; Rojas, 1996) denotes a generic system composed by *units* and *interconnections* between these units. The units

are devices endowed with one *output line* and a number of *input lines*; they produce an *output activation* as response to suitable input signals, according to a suitable transfer function, whereas the interconnections vehiculate and modulate the signals flowing from the output of a unit to the input of another unit. As regards the possible choices of unit transfer functions and of interconnection modulatory actions, they are virtually unlimited. However, chiefly in the early times of neural networks development, most people constrained such a variety of choices by requiring that the behaviours of single units and of interconnections should be *neural-like*, that is approximating, in some way, the known behaviour of biological neurons and synapses. Practically, it is very difficult to state what this requirement should mean. Actually there exists a number of different models of units and of interconnections, ranging from a complete lack of adherence to biological neuron operation, to a moderate imitation of some gross features of such an operation, to a detailed representation of many its microfeatures (see, on this latter category, Arbib, 1998; Koch, 1998; O'Reilly & Munakata, 2000).

We remind here that, in almost the totality of implementations, the operation of an interconnection line is described as equivalent to the multiplication of transmitted signal amplitude by a suitable real number, the so-called *connection weight* associated to the line under consideration. In a sense, such a weight can be viewed as a rough counterpart of the *synaptic efficiency* in biological neural networks. As a consequence of such a choice, the net input (sometimes referred to as *activation potential*) received at time t by a given unit, labeled, for convenience, by the index i , will be given by:

$$\text{net}_i = \sum_k w_{ik} x_k(t) - s, \quad (1)$$

where $x_k(t)$ denotes the output activation of the k -th unit, w_{ik} is the connection weight associated to the line connecting the output of the k -th unit to the input of the i -th unit, whereas s is a suitable (optional) threshold value.

In the following we will list some of the most popular choices regarding the transfer functions of neural units. We will start from a case in which the biological interpretation is completely lacking, represented by a unit, named *RBF unit* (*RBF* stands for *Radial Basis Function*), acting as a feature detector, whose output u , in response to a suitable input vector v_i , is given by:

$$u = G(|p_i - v_i|/\sigma), \quad (2)$$

where p_i is a suitable prototype feature vector, associated to the unit, σ is a parameter characterizing the unit sensitivity, usually called *variance*, whereas the symbol $|p_i - v_i|$ denotes the modulus of the distance between the two vectors. The function G is the usual Gauss function defined by:

$$G(y) = \exp(-y^2). \quad (3)$$

The second example is more adherent to the biological realm, as it takes into account the all-or-none behaviour of a number of real neurons. Its discretized-time version can be represented by the *McCulloch-Pitts unit*, whose output is given by:

$$u_i(t+1) = H[net_i(t)], \quad (4)$$

where H is the usual Heaviside function. Another possible realization is given by the *sigmoidal unit*:

$$u_i(t+1) = F[net_i(t)], \quad (5)$$

where F denotes the usual sigmoidal function:

$$F(y) = 1/[1 + \exp(-y)]. \quad (6)$$

The latter has the advantage of being everywhere continuous and differentiable. A continuous-time version of (5) is given by the differential equation:

$$dx_i/dt = -x_i + F[net_i(t)]. \quad (7)$$

A more realistic continuous-time model is given by the *Freeman unit*, describing the average behaviour of populations of biological neurons (cfr. Freeman, Yao & Burke, 1988; Whittle, 1998; Freeman, 2000). Such a kind of neural unit is characterized by an activation law of the form:

$$y = f(x), \quad Lx = u, \quad (8)$$

where y denotes the neuron firing rate, x the current within cell body, u the external input pulse rate, and L is a differential operator of the form:

$$L = D_{tt} + aD_t + b, \quad D_t = d/dt, \quad (9)$$

in which the two parameters a and b are to be chosen in such a way that the polynomial $s^2 + a s + b$ have two negative real roots. The low-frequency approximation (time derivative vanishing) of Freeman unit activation law gives rise, by introducing a discretized time scale, and a suitable rescaling of measure units, to the previously shown activation law (5) of artificial sigmoidal neurons.

A neural network is, of course, characterized also by an *architecture*, that is by the spatial arrangement of the units it contains. To this regard two main classes of networks are to be distinguished: the ones, which we will denote as *virtual*, in which the “coordinates” associated to each unit are to be considered merely as *labels*, devoid of any spatial meaning, and the ones, which we will denote as *spatial*, in which these coordinates denote the location of the unit in a physical space. As regards the former class of networks, it is evident that we cannot introduce notions such as “distance between units”, or “neighboring units”, whereas these concepts have a precise meaning within the latter class. In both cases a neural network is usually described as a set of *layers of units*. Of course, the concept of “geometrical shape” of a layer is endowed with a meaning only for spatial networks. Usually people distinguishes between *input layers*, to which belong only the units having at least an input line coming directly from the outside, *output layers*, to

which belong only the units whose output line points directly to the outside, and *intermediate*, or *hidden*, *layers*, containing all units (when present) not belonging to the input or output layer. In some network models input and output layers coincide. As regards connection lines, a distinction is made between *feedforward connections*, which connect one layer to another along the direction from the input to the output layer, *feedback*, or *recurrent connections*, connecting one layer to another along the opposite direction, from the output to the input layer, and *lateral connections*, connecting units lying within the same layer. An important subclass of networks is constituted by spatial networks, in which lateral connections exist only between neighboring units. This subclass includes the *Cellular Neural Networks* (Chua & Yang, 1988), the *Neuronic Cellular Automata* (Pessa, 1991), and a number of *lattice models* currently used in theoretical physics, such as the celebrated Ising model and all spin glass models.

3. Neural Networks as Universal Approximators

There are many reasons for considering neural networks as *universal approximators* of whatever kind of system (here and hereafter the word “system” is to be meant as equivalent to “model of a given system”). The first one is due to the fact that whatever input-output behaviour can be reproduced, with arbitrary degree of approximation, by a suitable neural network. Such a circumstance is grounded on one of the most beautiful theorems of mathematics, proved by Kolmogorov in 1957. In short such a theorem (for a proof see, e.g., Sprecher, 1964) states that a continuous function of n variables can always be represented through a finite composition of functions of a single variable and of the operation of addition. In more precise terms, if we have a function of such a kind, to be denoted by $f(x_1, x_2, \dots, x_n)$, and suitably restricted in such a way as to be a map from $[0, 1]^n$ to $[0, 1]$, there exist a function u of one argument, $2n + 1$ functions of one argument ϕ_k ($k = 1, \dots, 2n + 1$), and n constants w_r ($r = 1, \dots, n$) such that the following relation holds:

$$f(x_1, x_2, \dots, x_n) = \sum_k u\left[\sum_r w_r \phi_k(x_r)\right]. \quad (10)$$

A short look to (10) allows us to identify in a natural way both u and ϕ_k with the transfer functions of suitable neural units, whereas the w_r can be viewed as connection weights, so that the right hand member of this formula can be interpreted as the description of a suitable neural network. In a word, Kolmogorov theorem asserts that every continuous function of n variables can be implemented through a finite neural network. Such a result can be easily generalized to more general continuous maps from suitable intervals of \mathbf{R}^n to suitable intervals of \mathbf{R}^m , which are defined simply through systems of continuous functions of n variables.

From a practical point of view, however, the application of Kolmogorov theorem runs into considerable difficulties, due to two main circumstances:

- (1) there is no algorithm to find the concrete form of u and ϕ_k , as well as the values of

- the w_r in the most general case of an arbitrary choice of the functional form of f ;
- (2) not every choice of u and ϕ_k is allowable, as the latter should have the property of being *neural-like*, that is corresponding to some transfer function concretely used in neural network models.

These circumstances forced the researchers to shift the attention from the *exact* reproduction of a map through a neural network to the problem of the *approximation* of such a map through the transfer function of a suitable neural architecture. Even in this case, however, Kolmogorov theorem turns out to be of invaluable help, as it lets us to individuate a special class of functions, the ones implemented in an exact way through a well specified architecture made by suitably chosen neural-like units. Once given such a class of functions, one can take advantage of the results and of the methods of Functional Analysis (see, e.g., Rudin, 1973; 1987) in order to know what are the approximation capabilities of such a class with respect to more general functional classes. In this way, for example, Hornik, Stinchcombe and White (1989) were able to prove that multilayered feedforward neural networks containing at least a layer of sigmoidal units can act as universal approximators of a whatever input-output transfer function, with arbitrary degree of approximation, provided the number of units and the connection weights be properly chosen. Analogous results were obtained by other authors (cfr. Park & Sandberg, 1993) for multilayered feedforward networks containing RBF units.

The approximation capabilities of neural networks, however, go far beyond input-output relationships. Namely, these systems can mimic, with any degree of approximation, even the behaviour of *classical continua*, whose dynamics is described by partial differential equations. The easiest way for reproducing the behaviour of continuous media is to make resort to the aforementioned Cellular Neural Networks (CNN). These latter can be described (cfr. Chua & Roska, 1993) as systems of nonlinear units arranged in one or more layers on a regular grid. The CNN differ from other neural networks since the interconnections between units are local and translationally invariant. The latter property means that both the type and the strength of the connection from the i -th to the j -th unit depend only on the relative position of j with respect to i . At every time instant to each unit of a CNN are associated two values: its (internal) *state*, denoted by $v_i^m(t)$, and its *output*, denoted by $u_i^m(t)$. Here the index m denotes the layer to which the unit belongs and the index i denotes the spatial location of the same unit within the layer. The general form of the dynamical laws ruling the time evolution of these functions is:

$$dv_i^m(t)/dt = -g[v_i^m(t)] + \sum_q \sum_k a_k^{qm} [u_{i+k}^q(t), u_i^m(t); \mathbf{P}_{ak}^{qm}] \quad (11a)$$

$$u_i^m(t) = f[v_i^m(t)]. \quad (11b)$$

In these formulae the function g describes the *inner dynamics* of a single unit, whereas f denotes the *output function* (often of sigmoidal type). Besides, the sum on the index q runs on all layer indices, and the sum on the index k runs on all values such that $i+k$ lies in the neighborhood of i . Finally, the symbol \mathbf{P}_{ak}^{qm} denotes a set of suitable constant

parameter values entering into the explicit expression of the *connection function* a_k^{qm} . Such parameters, whose values are independent from i , are referred to as *connection weights* or *templates*.

In the case of a single-layered CNN, in absence of inner dynamics, and when the output function coincides with the identity function, the (11) can be reduced to the simpler form:

$$dv_i(t)/dt = \sum_k a_k [v_{i+k}(t), v_i(t); \mathbf{P}_{ak}]. \quad (12)$$

Let us now show, through a simple example, how a continuum described by partial differential equations can be approximated by a suitable CNN. To this end, let us choose a 1-dimensional medium ruled by the celebrated Korteweg-De Vries equation:

$$\partial\varphi/\partial t + \varphi(\partial\varphi/\partial x) + \delta^2(\partial^3\varphi/\partial x^3) = 0, \quad (13)$$

whose solution describes traveling solitary waves. Here δ^2 is a suitable parameter. If we introduce a discretization of spatial coordinates, based on a fixed space step Δx , the field function $\varphi(x, t)$ is replaced by a set of time-dependent functions $v_i(t)$ ($i = 1, 2, \dots, N$), and (13) is replaced, in turn, by the set of ordinary differential equations:

$$dv_i/dt = -v_i[(v_{i+1} - v_{i-1})/\Delta x] - \delta^2[(v_{i+2} - 2v_{i+1} + 2v_{i-1} - v_{i-2})/(\Delta x)^3]. \quad (14)$$

If we compare (14) with (12), it is easy to see that this set of equations describes the dynamics of a CNN, whose connection function is given by:

$$a_k = w_k v_{i+k} + r_k v_i v_{i+k}. \quad (15)$$

In such a CNN the neighborhood of the i -th unit goes from the $(i+2)$ -th unit to the $(i-2)$ -th unit, so that the index k can assume only the values $+2, +1, 0, -1, -2$. A direct inspection of (14) shows that the values of connection weights w_k and r_k are given by:

$$w_2 = -\delta^2/(\Delta x)^3, w_1 = 2\delta^2/(\Delta x)^3, w_0 = 0, w_{-1} = -2\delta^2/(\Delta x)^3, w_{-2} = \delta^2/(\Delta x)^3,$$

$$r_2 = 0, r_1 = -1/\Delta x, r_0 = 0, r_{-1} = 1/\Delta x, r_{-2} = 0.$$

A procedure like the one described in this example allows us to find a precise relationship of equivalence between a classical continuous medium and a CNN (see also Roska *et al.*, 1995).

A more complex, and still unanswered, question is whether neural networks are or not universal approximators of quantum fields. To this regard, we remind that from a long time physicists stressed a close analogy between the lattice version of Euclidean formulation of Quantum Field Theory (QFT) and the statistical mechanics of a system of spin (that is, neural-like) particles, coupled via next-neighbor interactions (cfr. Mézard,

Parisi & Virasoro, 1987; Montvay & Münster, 1997). The existence of such an analogy constitutes, of course, a strong motivation for studying neural networks as possible implementations of QFT. Roughly speaking, neuronal units activations should correspond to fermionic source fields and connection weights should represent bosonic gauge fields mediating the interactions between fermions. Suitable restrictions on the possible neuronal interconnections (limited to neighboring units) should grant for the fulfillment of requirements such as microscopic causality. Besides, taking both a thermodynamical and a continuum limit on neural networks descriptions, we should recover the usual quantum field-theoretical ones. However, despite the attractiveness of such a picture, the detailed specification of a precise equivalence relationship between neural networks and QFT is still lacking. On the contrary, there are a number of no-go theorems (cfr. Svozil, 1986; Meyer, 1996), which evidence how the search for such an equivalence be a very difficult affair.

Along with the results suggesting that neural networks can be used as universal approximators of systems of very general nature, seemingly having little to do with neural-like behaviours, there are other results, complementing the previous ones, evidencing how neural networks behaviours can also be described in non-neural terms. In most cases such results came from attempts to derive the *macroscopic*, that is *global*, features of neural network dynamics starting from the knowledge of its *microscopic*, neural-like, features. To this regard we can roughly individuate two different approaches: a *statistical* one, and a *field-like* one. The statistical approach tries to extract from microscopic neural dynamics, by resorting to methods of Probability Theory and Statistical Mechanics, dynamical laws ruling the time evolution of suitable average quantities, such as the average network activation, its variance, and so on. Such an approach was fostered by S. Amari and his school (cfr. Amari, 1972; Amari *et al.*, 1977; Amari, 1983), but received a number of important contributions by other authors (cfr. Rössler & Varela, 1987; Clark, 1988). As example of results obtained within this approach we can quote the macroscopic law of evolution for the average output activity $u(t)$, as deriving from the microdynamics of a network of reciprocally interconnected binary units, described by:

$$x_i(t+i) = F[\sum_j w_{ij} x_j(t)], \quad (16)$$

where w_{ij} denotes the (fixed) connection weights, and $F(z) = 1$ if $z > 0$, whereas $F(z) = -1$ if $z < 0$; in this case Amari was able to derive from (16) that the time evolution of $u(t)$ is ruled by the following integral equation:

$$u(t+1) = [2/(2\pi)^{1/2}] \int_0^{Vu(t)} \exp(-y^2/2) dy, \quad (17)$$

where:

$$V = W(N)^{1/2} / \sigma_w,$$

and W , σ_w denote, respectively, the mean and the variance of connection weights.

As regards the other approach, the field-like one, it is based on some sort of continuum limit, taken on a neural network, which reduces this latter to a space-time dependent

activation field. Between the first contributions to this approach we remind the celebrated Wilson-Cowan model of macroscopic behavior of neural assemblies (cfr. Wilson & Cowan, 1973), based on the pair of integrodifferential equations:

$$\mu \partial E / \partial t = -E + (1 - r_e E) F_e[\alpha \mu (\rho_e E \otimes \beta_{ee} - \rho_i I \otimes \beta_{ie} + P)] \quad (18a)$$

$$\mu \partial I / \partial t = -I + (1 - r_i I) F_i[\alpha \mu (\rho_e E \otimes \beta_{ei} - \rho_i I \otimes \beta_{ii} + Q)], \quad (18b)$$

where $E(x, t)$, $I(x, t)$ are, respectively, the average activities of excitatory and inhibitory neurons, $\rho_e(x, t)$ and $\rho_i(x, t)$ are their densities, $\beta_{ee}(x)$, $\beta_{ie}(x)$, $\beta_{ei}(x)$, $\beta_{ii}(x)$ denote coefficients (it would be better to call them "fields") of hetero- or autoinfluence of excitatory and inhibitory neurons, α , μ , r_e , r_i are suitable positive parameters, whereas $P(x, t)$ and $Q(x, t)$ are suitable external inputs. Besides, F_e and F_i denote activation functions (of sigmoidal type) of their arguments, and the symbol \otimes denotes the spatial convolution integral operator. The equations (18.a) and (18.b), or their generalizations or specializations, exhibit a very rich bifurcation phenomenology, including spatial and temporal pattern formation (cfr. Ermentrout & Cowan, 1980; Ermentrout, 1998), and rhythm or wave generation (cfr. Osovets *et al.*, 1983, Paluš *et al.*, 1992, Jirsa *et al.*, 1994, Pessa, 1996; Haken, 1996; Jirsa & Haken, 1997; Robinson *et al.*, 1998; Frank *et al.*, 2000).

Summarizing, we can assert that neural networks are endowed with two fundamental properties: on one hand, they act as discrete universal approximators of most systems, even if described by continuous variables, and, on the other hand, even if characterized by a discretized structure, from a macroscopic point of view they behave like continuous systems of a very general nature. Just such a circumstance was the cause for the interest of an ever growing number of researchers in behavioural features of neural networks: the study of neural networks appears as, more or less, equivalent to the building of a *General Systems Theory* (in the sense already proposed by Von Bertalanffy, 1968).

4. What Neural Networks Can Do

The most interesting ability exhibited by neural networks is that they can *learn from experience*. All their properties, including the approximation capabilities described in the previous paragraph, can be acquired without resorting to a preventive careful design of their structure, but only submitting them to a suitable set of examples, that is of input patterns. Of course, suitably chosen learning laws are also needed, but these latter are far simpler than any conceivable set of symbolic rules specifying in detail how to obtain the wanted performances without large errors.

It was just such an ability which, at first sight, suggested to researchers, such as the ones supporting the connectionist approach (see, to this regard, McClelland & Rumelhart, 1986; Smolensky, 1988), that neural networks should be considered as the prototypes of systems exhibiting *emergent behaviors*. Namely, if we compare the (generally random) structure of connection weights of a network before a learning session with the same structure after the learning occurred, we will find that this latter cannot be longer considered as random (even if it can appear as very complex), because it will be organized

in a so fine way as to give rise to well-structured network behaviors, whose occurrence, on the other hand, would be impossible before the learning process. Notwithstanding, such a structure is not the product of the action of a designer, but results only from the combination of a number of suitable input patterns and of the choice of a simple learning law. What is to be underlined is that, generally, the knowledge of these ingredients isn't enough *per se* for forecasting (at least from a mathematical point of view) the detailed features of organized network behaviors taking place after the learning. Thus, these features can rightly be considered as *emergent*, in the same sense in which we call emergent a new ability (such as doing algebra or understanding a foreign language) acquired by a student after reading a good textbook or attending the lectures of a good teacher. Such arguments, however, are formulated in rather intuitive terms, and we need a deeper analysis of effective learning capabilities of neural networks before dealing with the question whether neural networks exhibit or not emergence. A short review of the analysis so far done on learning processes in neural networks will be just the subject of this paragraph.

We remark, to this regard, that a survey of the field of neural network learning is a very complicated task, owing to the huge number of learning methods so far proposed. In general these methods consist in algorithms which dictate the modifications to be done on connection weights (or on other parameters) as a function of suitable external influences. They can be grouped into the five fundamental categories described below.

- *Supervised learning*, based on the presentation to the network of examples, each one constituted by a pattern pair: the *input pattern*, and the corresponding *wanted, or correct, output pattern*. The modification of connection weights is determined by the *output error*, that is by a suitable measure of the difference existing between the wanted output pattern and the output pattern effectively produced by the network. This kind of learning is largely, and most often successfully, used to obtain neural networks realizing a given input-output transfer function.
- *Unsupervised learning*, in which each example consists only in an input pattern, and the concept itself of correct output pattern is devoid of any meaning. The modification of connection weights at a given instant of time is determined by the local pattern presented in input at the same time, and by the result of the competition between suitable *categorization units*. This kind of learning is very useful to *clusterize* (that is, group into categories) a complex data set.
- *Reinforcement learning*, in which we have the presentation to the network of *time sequences of examples*, each one constituted by an *input pattern*, by the corresponding *output pattern (action or decision)*, and, eventually, by the *reinforcement signal* received from the environment as a consequence of the output pattern produced. The modification of connection weights aims to maximize the total amount of positive reinforcement (the *reward*) received as a consequence of the whole behavioral sequence. Such a type of learning is very useful to design systems able to adapt to a rapidly changing, unpredictable, and noisy environment.
- *Rote learning*, in which the connection weights are modified in a brute way, according to a fixed law, under the direct influence of input patterns, or of suitable input-output

associations. Such a kind of learning is useful to design *associative memories*, in which external patterns (or input-output associations) are stored, and subsequently recalled following the presentation of suitable cues.

- *Evolutionary learning*, in which the connection weights, and most often the whole network architecture change as a function of the *fitness value* associated to suitable behavioral features of the network itself. Such a value is related to a given measure of the efficiency of the network in dealing with problems posed by external environment, a measure which, obviously, was chosen by the designer on the basis of his needs or of his goals.

Each kind of learning is associated to a number of preferred architectural implementations, even if there's no necessary connection between a learning process and its network realization. Besides, resorting to suitable prototypical architectures, some kinds of learning were studied from a theoretical point of view, in order to characterize them with respect to the convergence of adopted learning laws. Most studies dealt with supervised learning, both by using methods of optimization theory (cfr. Bartlett & Anthony, 1999; Bianchini *et al.*, 1998; Frasconi, Gori & Tesi, 1997) and from a statistical mechanics perspective (Sollich & Barber, 1997; Saad, 1998; Mace & Coolen, 1998; Marinaro & Scarpetta, 2000). A consistent number of other studies, traditionally done within the context of *spin-glass physics*, was devoted to rote learning (Domany, Van Hemmen & Schulten, 1996; Kamp & Hasler, 1990; Amit, 1989). A smaller number of papers and books dealt with unsupervised learning (cfr. Hinton & Sejnowski, 1999; Cottrell, Fort & Pagès, 1997). Both reinforcement learning and evolutionary learning, even if extensively studied under many respects, appear actually as very difficult topics when one is interested to assess their convergence features.

The results so far obtained from these studies can be, in very broad terms, thus summarized:

- (1) the dynamics of most popular supervised learning algorithms (such as, e.g., the celebrated *error-backpropagation*) can be described both from a microscopic and a macroscopic point of view; this latter, in particular, allows for a prediction of the macroscopic features of the attractors of such a dynamics, and of their dependence on the values of control parameters related to network architecture and/or to learning law;
- (2) we still lack a method for avoiding the existence of *local minima* in error surfaces, holding for every possible kind of training set; in other words, even if a neural network can, in principle, approximate with a precision whatsoever every possible input-output relationship, we still lack a supervised learning algorithm able to grant for the reaching of such an approximation, or, at least, able to forecast, for *whatever kind* of training set, whether such a goal will be attainable or not through a learning from experience; the solution of such a problem, on the other hand, appears to be NP-complete;
- (3) we have actually no exact method at disposal to predict in advance the detailed outcome of a retrieval dynamics, triggered by a suitable cue pattern, following a

rote learning; a qualitative statistical prediction of the macroscopic features of such a dynamics is, however, possible, provided we work in the thermodynamical limit;

- (4) in the case of unsupervised learning implemented through *Kohonen's Self-Organizing Maps* (Kohonen, 1995), the only one so far studied, there are methods to predict the asymptotic state of connection weights and of the form of clusters obtained at the end of learning procedure, provided the input pattern statistical distribution and the neighborhood function be known in advance (see also Ritter & Schulten, 1988; Erwin, Obermayer & Schulten, 1992); these methods, however, work only when input and categorization layer are 1-dimensional; in the multidimensional cases no general method is available.

These results evidence how the general theory of learning in neural networks be still in a very unsatisfactory state: the best thing neural networks are able to do is to learn but, unfortunately, we don't know *why*, *how*, and *when* they learn. Our incapability of controlling them is reflected in a number of hard conceptual problems that the neural network models so far proposed are unable to solve. Between these problems we will quote the following ones:

- ***The Catastrophic Interference Problem***, related to supervised learning (and partly to rote learning), consisting in the fact that the learning of new input-output relationships gives rise to a dramatic decay of performance relative to previously learned relationships (cfr. McCloskey & Cohen, 1989; Ratcliff, 1990);
- ***The inability to represent complex symbolic structures***, due to the fact that the vectors used to represent activation patterns of neural units are unsuitable to code the complex structural relationships occurring within a data set (or within human knowledge), such as, e.g., tree structures;
- ***The Binding Problem***, stemming from the fact that a neural network cannot learn from experience alone to correctly bind into a unique, holistic and global, entity a number of different local features *independently* from the very nature of the features themselves;
- ***The Grounding Problem***, consisting in the fact that a neural network is unable to connect higher-level symbolic representations with lower-level space-time distributions of physical signals coming from the environment (cfr. Harnad, 1990).

All solutions to these problems so far proposed (see, to make some quotations, Pessa & Penna, 1994; French, 1999; Nenov & Dyer, 1993, 1994; Medsker, 1995; Chalmers, 1990; Shastri & Ajjanagadde, 1993; Van der Velde, 1995; Wilson *et al.*, 2001; Pessa & Terenzi, 2002) were based on neural architectures which went far beyond the standard complexity of most popular neural networks and, in a way or in another, were strictly inspired by biological or cognitive findings. On the other hand, such a circumstance seems to point to a possible connection between our inability in predicting and controlling neural network behaviors, when grounding only on mathematical arguments, and the fact that neural networks themselves could be particular prototypes of systems exhibiting *emergent behaviors*. In other words, the results so far obtained on neural network learning show that

the outcomes of the latter process are, in many cases, unpredictable, even if compatible with mathematical descriptions of neural network themselves, just as it occurs for the outcomes of a human learning process. The only way for designing neural networks endowed with high-level behavioral features would, therefore, be the one of imitating, as closely as possible, the design principles underlying biological or cognitive architectures. A further discussion of this topic, however, calls for a more precise definition of what we mean by using the word ‘emergence’.

5. What is Emergence?

The topic of emergence is rather old within the history of ideas of twentieth century: it was proposed for the first time already in 1923 within the context of the doctrine of “emergent evolutionism”, advanced by C.L. Morgan (cfr. Morgan, 1923). However, this subject began to become popular only in the Eighties, owing to the explosion of the research on deterministic chaos. This is due to the fact that chaotic deterministic behavior appears as “unpredictable” by definition, and “new” with respect to the expectancies of physicists even if it can be produced by using very simple deterministic computational rules, such as the one of Feigenbaum’s logistic map (Feigenbaum, 1983):

$$x_{n+1} = 4\lambda x_n(1 - x_n), \quad 0 < \lambda < 1, \quad 0 < x_0 < 1. \quad (19)$$

Who could think of the sequence of numbers generated by such simple recursion formula as a chaotic one? However, this is what happens when $\lambda > 0.892\dots$. Thus it is spontaneous to qualify such phenomena as “emergent”.

Despite the interest for emergence triggered by studies on chaotic deterministic behaviors, the need for a more precise definition of the concept of “emergence” arose only in the last decade, when researchers became to build Artificial Life models. Within this context it was coined the new expression “emergent computation”, to denote the appearance of new computational abilities resulting from the cooperative action of a number of individual agents interacting according to simple rules (cfr. Forrest, 1990; Langton, 1989, 1994; Langton *et al.*, 1992). An important conclusion resulting from the debate on emergence within Artificial Life researchers is that the concept of “emergence” cannot be defined in a absolute, objective, way, but is rather bound in an indissoluble way to the existence of an *observer*, already equipped with a model of the system under study, in turn designed to attain some goals of the observer itself. Then, a model behavior can be said to be *emergent* if it doesn’t belong to the category of behaviors which were the goals of the design of the model itself. According to this definition, chaotic behaviors produced by (19) are emergent, as the elementary mathematics contained in this formula is normally used for the goal of producing only simple, deterministic, predictable behaviors.

A further important contribution to this topic was given by Crutchfield (1994), who introduced a distinction between three kinds of emergence:

- e.1) *intuitive*, corresponding to the rough identification of “emergence” with “novelty”;
- e.2) *pattern formation*, in which a pattern is said to be “emergent” when it occurs

as a non-trivial consequence of the model structure adopted, even if it could be forecast in advance on the basis of a sophisticated mathematical analysis of the model itself; this is the case, for example, of some macroscopic models of self-organization making use of bifurcation phenomena such as the ones we will discuss below;

e.3) intrinsic emergence, in which not only the occurrence of a particular behavior cannot be predicted in advance (even if it is compatible with model assumptions), but such an occurrence gives rise, in turn, to a deep modification of system's structure, in such a way as to require the formulation of a new model of the system itself; this is the case, e.g., of the emergence of new computational abilities (see Sipper, 1997; Suzuki; 1997).

For many years the only kind of emergence taken into consideration by model builders was pattern formation, usually labelled as *Self-Organization*, for reasons we will explain later. All models of self-organization are described in terms of suitable macrovariables, whose values depend on time and, eventually, on other independent variables. The focus is on time evolution of these macrovariables, ruled by suitable *evolution equations*, containing a number of *parameters*, and associated to given *initial* or *boundary conditions*. It is easy to understand that the most important forecasting one can derive from these models deals with the number and the type of *attractors* of dynamical evolution undergone by macrovariables. In turn, these features of attractors depend on the actual values of parameters entering into evolution equations and (in a more complicated way) on initial or boundary conditions. Such a dependency can be studied by resorting to mathematical methods of *dynamical systems theory* (cfr. Glendinning, 1994; Guckenheimer & Holmes, 1983).

Within this framework, the most studied phenomenon has been the one of *bifurcation*. This term denotes a change of the number or of the type of attractors as a consequence of a change of parameter values. In the most simple cases, a bifurcation takes place when the value of a parameter, the so-called *bifurcation* (or *critical*) *parameter*, crosses a *critical value*, which thus appears as a separator between two structurally different states of affairs: the one corresponding to values of bifurcation parameter lesser than critical value, and the other corresponding to values of bifurcation parameter greater than critical value. Such a circumstance suggests a close analogy between bifurcation phenomena and *phase transitions* taking place within physical systems. Namely the two different states of affairs, before and after critical value, can be viewed as analogous to different *phases* of matter, the critical value itself being viewed as analogous to the *critical point* of a phase transition. We shall see later, however, that such an analogy breaks down when we take into account the fact that the values of macrovariables undergo unavoidable fluctuations, due both to the limited sensitivity of our measuring instruments, and to the coupling between the system and a noisy environment. Despite that, most researchers, starting from Prigogine (cfr. Nicolis & Prigogine, 1977), identified bifurcation phenomena with *self-organization* phenomena, i.e. with sudden changes in the structural properties of system behavior (typically from a disordered behavior to an ordered one) taking place when a parameter crosses a critical value.

We remind, to this regard, that bifurcation phenomenology is very rich, so rich as to offer a potential description of most *pattern formation* processes. Usually mathematicians distinguish between three different classes of bifurcations phenomena:

b.1) *subtle bifurcations*, in which the number of attractors is kept constant, but their type undergoes a change; the most celebrated example of such a class is *Hopf's bifurcation*, in which an attractor, having the structure of an equilibrium point, is changed, once a parameter crosses a critical value, into a limit cycle, i.e. a periodic motion; such a type of bifurcation plays a capital role in describing the birth of *self-oscillatory* phenomena;

b.2) *catastrophic bifurcations*, in which the number of attractors changes; an elementary example of such a class of bifurcations is given by the unidimensional motion of a mass point in a force field characterized by the potential:

$$U(x) = k(x^2/2) + \lambda x^4 + \alpha, \quad (20)$$

where $\lambda > 0$, $\alpha > 0$ are parameters whose value is kept constant, whereas k is chosen as bifurcation parameter, and whence we suppose that its value can change as a consequence of external influences; straightforward mathematical considerations show that, when $k > 0$, the potential $U(x)$ will have *only one* minimum, corresponding to a stable equilibrium point, located in $x = 0$, and having the value $U(0) = \alpha$; on the contrary, when $k < 0$, the potential $U(x)$ will have *two* equivalent minima, located in $x_1 = (|k|/4\lambda)^{1/2}$, and $x_2 = -(|k|/4\lambda)^{1/2}$, and corresponding to two stable equilibrium points, in which the value of $U(x)$ is given by:

$$U(x_1) = U(x_2) = \alpha + (5k^2/16\lambda);$$

thus the *critical value* $k = 0$ will mark the passage from a situation characterized by only one attractor to a situation characterized by two attractors;

b.3) *plosive bifurcations*, in which the attractor size undergoes a sudden, discontinuous, change as the bifurcation parameter crosses the critical value; a very simple example of a plosive bifurcation is given by the system obeying the nonlinear differential equation:

$$dx/dt = \alpha x \{x^2 - (\alpha + 1)x + [(\alpha^2 + 1)/4]\}, \quad (21)$$

where α denotes the bifurcation parameter; the equilibrium points (attractors) of (21) are defined by the condition:

$$dx/dt = 0, \quad (22)$$

which implies:

$$\alpha x \{x^2 - (\alpha + 1)x + [(\alpha^2 + 1)/4]\} = 0; \quad (23)$$

a straightforward algebra then shows that, when $\alpha < 0$, the equation (23) has one and only one real solution, given by $x = 0$; thus, a linearization of (21) around such a solution gives the following equation ruling the behavior of a small perturbation ξ of the zero solution:

$$d\xi/dt = \alpha[(\alpha^2 + 1)/4]\xi; \quad (24)$$

as the coefficient of ξ in the right-hand member of (24) is negative for $\alpha < 0$; we can conclude that the attractor (equilibrium point) corresponding to $x = 0$; is *stable* for $\alpha < 0$; we can now observe that, when $\alpha = 0$ (the critical value), the equation (23) has two real roots: again $x = 0$, and the further root $x = 1/2$ (with double multiplicity); further, when $\alpha > 0$, (23) has three real roots, i.e. the previous root $x = 0$ and the other two roots:

$$x = [\alpha + 1 \pm (2\alpha)^{1/2}]/2; \quad (25)$$

a linear stability analysis around each of these three roots shows that, when $\alpha > 0$, both roots $x=0$ and $x = [\alpha + 1 + (2\alpha)^{1/2}]/2$ correspond to *unstable* equilibrium points, whereas the third root $x = [\alpha + 1 - (2\alpha)^{1/2}]/2$ represents the only one *stable* equilibrium point; thus, the bifurcation taking place for $\alpha=0$ gives rise to a sudden change from a stable equilibrium state, corresponding to $x = 0$, to a new stable equilibrium state, corresponding to $x = 1/2$, or lying near to this value for α positive and small enough; such a sudden change of size of the attractor is an example of a plosive bifurcation.

We remind further that mathematicians introduced another classification of bifurcations into two categories: *local* and *global* (cfr. Ott, 1993). A local bifurcation gives rise to changes in attractor structure only in a small neighborhood of phase space of the system under study. On the contrary, a global bifurcation results from a connection between distant attractor, and gives rise to sudden structural changes of large domains of phase space. The different categories of bifurcations we sketched before are object of intensive study by mathematicians. We still lack, however, a general theory of bifurcation covering all possible phenomenologies. Whereas for particular types of bifurcation there exist well know algorithms (implemented even though computer programs for symbolic manipulation) which let us forecast their phenomenology in a detailed way, in other cases we are still forced to resort only to numerical simulations. Anyway, mathematicians and theoretical physicists evidenced how nonlinear systems can always be described by suitable *canonical forms*, valid near bifurcation points. In this context between the most useful tools we will quote Haken's *Slaving Principle*, wich lies at the hearth of Synergetics (cfr. Haken, 1983).

Despite the attractiveness of the concept of bifurcation and of the underlying theory, in the past times it was not so popular between neural network modellers. For this reason the topic of emergence in neural networks, even if restricted only to pattern formation, was somewhat neglected. There have been, however, notable exceptions. We remind, to this regard, the neural field theories, previously quoted and inspired by the pioneering model of Wilson and Cowan (cfr. Ermentrout & Cowan, 1980; see also Pessa, 1988), whose prototype is the "Field theory of self-organizing neural nets" put forward by Amari (cfr Amari, 1983). This approach experimented a resurgence from the end of the Eighties, determined mostly by the fast development of the so-called Statistical Field Theory (Mézard, Parisi & Virasoro, 1987; Parisi, 1988). Following the suggestions of

this latter, a number of researchers tried to describe the dynamics of a neural network as a Markovian process ruled by a suitable master equation (see, to this regard, Rössler & Varela, 1987; Clark, 1988; Coolen & Ruijgrok, 1988; Coolen, 1990). From the latter, as is well known from the theory of stochastic processes (cfr. Van Kampen, 1981), it is possible to derive a partial differential equation, that is the Fokker-Planck equation (or its generalizations), describing the time evolution of the probability density of microscopic network states. By using a suitable mathematical machinery it is possible to derive, in turn, from this equations a number of differential equations (total or partial) ruling the time (or space-time) evolution of a number of given macrovariables, of course defined through moments of probability density. These equations should be identified with the ones to which one should apply the methods of bifurcation theory previously described.

Such an approach has a number of advantages, the most relevant one being to make clear that pattern formation, or self-organization, is a phenomenon which takes place only at a macroscopic level. This suggests that such a form of emergence is not intrinsic if we study it starting already from a macroscopic point of view, but it *could* appear as intrinsic if studied from a microscopic point of view. This would be the case, for instance, if the statistical fluctuations of the microvariables could trigger the occurrence of collective effects involving a large number of neural units. To answer these questions, however, is not so easy, owing to the remarkable mathematical difficulties involved in the derivation of the macroscopic equations from the microscopic equations describing the behavior of the single neural units. As a matter of fact, this approach has been applied so far only to a limited number of cases. Between them, we will quote the celebrated Hopfield associative memory model based on rote learning (Hopfield, 1982), the most extensively studied from this side (cfr. Domany, Van Hemmen & Schulten, 1996), the online supervised learning by a multiplayer perceptron through the celebrated *error-backpropagation* rule (cfr. Saad, 1998), and the online supervised learning in RBF networks (cfr. Marinaro & Scarpetta, 2000).

We remind here that, in more recent times, a completely new procedure was introduced, in order to translate every classical description of entities, interacting through suitable reaction mechanisms (like particles, molecules, or chemical species), into a quantum-field theoretical description. Without entering into details on such a procedure (see, e.g., Lee & Cardy, 1995), we will remark that it shows the existence of a previously unsuspected link between classical and quantum theories of self-organization. The importance of such a connections stems from the fact that the latter, as we shall see in the next paragraphs, appear as the only ones able to exhibit intrinsic emergence. As regards the reformulation of neural network dynamics in terms of reactions between different chemical species, this can be done in many different ways. In the following we will limit ourselves to present one possible way, proposed many years ago by Schnakenberg (1976).

To this regard let us consider, for simplicity, a neural network with fixed connection weights, whose units (N will denote their total number) can have only a finite number K of possible activation levels. The total number of possible network states will therefore be N^K . To each one of these possible states we will associate, at every time instant t

, its probability of occurrence $p_i(t)$, with the index i running from 1 to N^K . The time evolution of $p_i(t)$ will be ruled by the master equation:

$$dp_i(t)/dt = \sum_j [\langle i|T|j \rangle p_j(t) - \langle j|T|i \rangle p_i(t)], \quad (26)$$

where the symbol $\langle i|T|j \rangle$ denotes the transition rate (in turn depending on connection weight values) from the state j to the state i . Let us now introduce N^K different chemical species, with the symbol c_i denoting the concentration of the i -th species, whose reciprocal reactions obey the rate equations:

$$dc_i(t)/dt = \sum_j [\langle i|T|j \rangle c_j(t) - \langle j|T|i \rangle c_i(t)]. \quad (27)$$

If we introduce a set of probabilities $p_i(t)$, associated to the concentrations c_i and given by:

$$p_i(t) = c_i(t) / \sum_j c_j(t), \quad (28)$$

it is easy to see that these latter obey just the master equation (26), describing the neural network dynamics.

From the definitions previously proposed, it is clear, however, that the most interesting models of emergent phenomena should exhibit intrinsic emergence. As regards models endowed with such a property, we will subdivide them into two main classes:

m.1) ideal models of emergence, characterized by:

- (1) the identification of macroscopic phenomena with the ones corresponding to infinite volume limit (thermodynamical limit),
- (2) the possibility of deriving microscopic dynamical equations from a suitable general maximum (or minimum) principle, or from a suitable variational principle;

m.2) not ideal models of emergence, characterized by:

- (1) the existence of a finite, predefined and fixed, volume into which the system is contained,
- (2) the derivation of microscopic dynamical equations only from phenomenological arguments.

A very interesting, but still unsolved, question is whether models belonging to class *m.2*) can be reformulated or not in such a way as to be classified as belonging to class *m.1*). The interest in such a question stems from the fact that it is very difficult to find criteria letting us decide whether intrinsic emergence exhibited by a not ideal model results from a particular, fortunate, choice of system's volume (and whence of boundary conditions), and of the form of microdynamical equations, or from general principles underlying the model itself. It is to be underlined, to this regard, that the introduction of a finite volume within a model of emergence, even if it appears, at first sight, as natural and realistic, could be devoid of any meaning. Let us clarify this circumstance within the context of natural or artificial neural networks. We start by remarking that each volume measure, within our

system, is possible only owing to the presence of a suitable measuring device (physical or ideal), characterized by an elementary volume V_0 . The latter can be considered as the *unit volume* (at least as regards the particular measuring device introduced). At the same time V_0 defines the *minimal volume*, under which no further volume measure is possible. Thus, if we have a given spatial domain of volume V , its measure, according to the particular measuring device described above, is obtained by specifying the number of times, denoted by v , in which V_0 enters into V (according to a suitable measuring procedure). Whence we will have that:

$$v = V/V_0. \quad (29)$$

When V_0 tends to zero, by keeping fixed V , obviously v tends to infinity. But such a circumstance is just the one occurring in natural or artificial neural networks. Namely in this case our uncertainty about the locations of natural or artificial neurons is vanishing or very small, because we consider them as lying in fixed sites, whose spatial position is known in advance with almost unlimited precision. This is equivalent to make V_0 tending to zero, so that, even if the volume V occupied by the system is zero, its measured volume practically tends to infinity, making the infinite volume limit a more realistic choice than the one of limiting ourselves only to finite volumes.

The considerations about emergence, made in this paragraph, should, however, be supplemented by suitable exhibitions of concrete models of emergence, ideal or not ideal. In the following we will shortly present a possible framework for building ideal models of intrinsic emergence: the one of quantum theory.

6. Quantum Models

The history of quantum models of cognitive processing began in 1967 with the publication of the seminal paper by Ricciardi and Umezawa on the usefulness of application of methods of Quantum Field Theory (QFT) to study cognitive processes taking place as collective effects in assemblies of brain neurons (Ricciardi & Umezawa, 1967). This paper gave rise to the building of a number of interesting models of cognitive processes (mainly of memory operation), based on QFT, and proposed mainly by Umezawa and his pupils (Stuart, Takahashi & Umezawa, 1978, 1979; Jibu & Yasue, 1995; Vitiello, 1995). Starting from very different premises, in the same period a number of other researchers (cfr. Stapp, 1993) tried to apply the methods of quantum theory to the study of consciousness. Both lines of research produced, as a consequence, an ever growing interest for the applications of models based on quantum theory to the description of a number of features of cognitive system operation (cfr. Pribram, 1993; Nanopoulos, 1995; Jibu & Yasue, 1997).

Why should quantum models be so attractive for neural networks modellers? Essentially because they offer convincing, and powerful, mechanisms to explain the occurrence of long-range orderings (i.e. macroscopic effects) within sets of locally interacting microscopic entities. And these mechanisms are the only ones, so far known, able to fully

describe the sudden structural changes taking place in complex system, and called, in physics, *phase transitions*. Such changes, as a matter of fact, appear to underlie the very nature of cognitive development in humans and animals: learning, insight, acquisition of new cognitive skills, understanding a language or a mathematical theory, abstraction, creative thinking, are all examples of deep restructurations (often of global nature) of cognitive system, which appear as unexplainable within conventional approach to cognitive science, based on symbolic processing paradigm. Quantum models offer, on one hand, a general theoretical framework for describing these changes through a more convenient formal language, and, on the other hand, a tool for investigating what features of the changes themselves could be taken into consideration, and eventually forecast and controlled. Thus, these models should constitute the very hearth of a new "quantum" cognitive science, much more powerful than classical cognitive science born in the Seventies. The arguments supporting these claims are complex and require some technical knowledge coming from theoretical physics. For this reason we will devote the next paragraph of this paper to a discussion of the advantages of quantum models over classical ones, in order to examine the validity of the thesis, according to which only quantum models can describe cognitive processes as collective effects emergent from the cooperative interaction of microcognitive units, and only quantum models can describe structural changes and phase transitions. Within this paragraph, instead, we will limit ourselves to a short listing of some features of quantum models.

To this regard, we will begin by reminding that quantum models can be subdivided into two main categories: the ones making use of *quantum mechanics* (QM), and the ones making use of *Quantum Field theory* (QFT). Even if the domain of application of both types of models is virtually unlimited, historically they were applied almost exclusively to description of behavioral features of subatomic particles. Such a circumstance influenced so heavily the concrete development of these models, that today is practically impossible to mention their features without making reference to the concept of *particle*. Of course, this isn't a *logical* necessity, but rather a *practical* necessity. It prevented, however, from an application of quantum models to domains different from particle physics (such as the one of cognitive processing), in which the concept of particle appears as useless. In recent times the situation is slightly changing, but we still need a more general formulation of quantum models, able to exploit their logical potentialities without being forced to describe every process as due to "interactions" between suitable "particles".

Let's now turn to QM. It deals (for a textbook see, for example, San Fu Tuan, 1994) with systems constituted by a finite, and fixed, number of particles, contained within a finite, and fixed, volume of space (even if infinite volumes are allowed). Physical quantities characterizing these particles, however, cannot be all measured simultaneously with arbitrary precision. In the most simple and popular form, such a constraint is expressed, for a single particle, through the celebrated *Heisenberg uncertainty principle*:

$$\Delta p \Delta q \geq h/4\pi, \quad (30)$$

Δp and Δq being, respectively, the uncertainties associated with the momentum

and with the position of the particle. As usual, h denotes Planck's constant. Such an uncertainty derives from two causes:

- u.1) the perturbation on particle behavior exercised by the act of measure itself;
- u.2) the fact that particles are embedded, and interacting with, into an intrinsically noisy environment.

A first consequence of such an uncertainty is that a complete characterization of particle dynamical state with unlimited precision is impossible. One is, then, forced to introduce the concept of *representation* of the state of system being considered. In rough, nontechnical, terms the choice of a representation consists in selecting a subset of dynamical variables describing the state of the system, such that all variables belonging to the subset can be measured simultaneously with arbitrary precision. In a sense, owing to uncertainty principle and its generalizations, every representation can offer only a *partial* description of system's dynamics. However, an important theorem, proved by Von Neumann (cfr. Von Neumann, 1955), asserts that in QM all possible representations are reciprocally *equivalent*. This means that they give rise to the same values of probabilities of occurrence of results of all possible measures relative to the physical system under consideration, independently from the particular representation chosen.

A second consequence of uncertainty is that particles cannot be longer considered as localized objects (like in classical physics), but rather as diffuse entities, virtually spreaded over entire space. Thus, given a particle, we can only speak of its most probable location at a given time, by remembering, however, that in each space position (within allowable space volume) there will be a nonzero probability (even if in most cases very small) of finding the particle itself. Besides, given two particles whatsoever, independently from the distance between their most probable locations, and a generic spatial location, there will be always a nonzero probability of finding them simultaneously in this location. Often physicists speak of *waves* associated to the particles, and of nonzero *overlap* of the waves associated to two different particles in every spatial location. What is important, is that such overlap gives rise to a correlation (even if very small) between the behaviors of the two particles. Such a correlation is, from all points of view, entirely equivalent to an *interaction* (of quantum nature) between the two particles, independently from the fact that there exist other interactions between them mediated by suitable force fields.

A third consequence of uncertainty is that, when we describe, in QM, assemblies of identical particles (i.e. particles of the same type, as characterized by intrinsic features such as their mass, electric charge, spin, etc.), the particles themselves are *indistinguishable*. This means that, by interchanging the spatial positions of two particles associated to two different labels, the state of the system will remain unchanged (apart from a multiplicative phase factor). On the contrary, in classical mechanics all particles are *distinguishable* and the overall state of the system depends in a crucial way on the association between particle labels and their spatial locations. Such a circumstance lets us characterize usual neural network models of cognition as *classical*, because within them an assumption universally adopted is that the single microcognitive units deputed to detect particular microfeatures are distinguishable. Namely, all people supposes that the

outputs of these units can be known with unlimited precision; besides, the labeling of the units themselves, once chosen, cannot be changed during neural network operation without altering in a dramatic way the performance of the network itself.

The aforementioned consequences of uncertainty entail the occurrence of a typically quantum phenomenon, named *Bose-Einstein condensation* (BEC). The latter occurs in presence of suitable conditions (e.g. low temperatures) and consists in the fact that all particles belonging to a given system fall simultaneously in the same quantum state. This implies that their behaviors become all correlated, or synchronized, giving thus rise to a macroscopic state, which appears as globally *coherent*. BEC can be considered as the prototype of formation of macroscopic entities emerging, as collective effects, from the laws ruling the behaviors of microscopic particles. For this reason, in their paper of 1967, Ricciardi and Umezawa made appeal to BEC in order to explain how cognitive processes (such as memorization, and recall) emerge from individual behaviors of brain neurons.

Let's now turn to the other category of quantum models, the ones belonging to QFT. The latter, contrarily to what happens in QM, assumes that the main physical entities are *fields* (of force) and not particles. Such a standpoint has a long tradition in physics, lasting to Faraday and Maxwell, and gave rise to the most powerful theoretical architectures ever built in theoretical physics, such as Einstein's General Relativity, and unified gauge theories. Within this framework, the world is populated by fields (of force), and the concept of particle has to be considered nothing but as an auxiliary concept to denote space regions where a field has a particularly high intensity. The laws ruling physical phenomena coincide with *field equations*, driving space-time evolution of fields. When such equations are linear (as in the case of classical electromagnetic field) in field quantities, then a nontrivial evolution needs the introduction of suitable *external sources*, whereas, in the case of nonlinear field equations, a field can be self-generating, without the need for the sources themselves.

Following such an approach, QFT attempts to treat fields as defined by uncertain quantities. As fields, in principle, are not restricted to definite volumes, QFT deals typically with infinite volumes. In this way it becomes easy to introduce a sharp distinction between *macroscopic* phenomena (the ones surviving when the volume tends to infinity) and *microscopic* phenomena (which appear as fluctuations when the scale of observation is large enough). Of course, the approach followed by QFT is very difficult to implement in a concrete way, much more difficult than in the case of QM. For this reason, QFT can still be considered an incomplete theory, of which only particular realizations have been so far worked out, holding in particular domains, and at the expense of introduction of a very complex mathematical machinery. Besides, some general conceptual problems raised by this approach are still unsolved. This, after all is not so strange, as a full realization of QFT would be equivalent to the building of a *General Systems Theory*, in the sense of Von Bertalanffy (cfr. Von Bertalanffy, 1968), a sort of *Theory of Whole*, including the observer himself formulating the theory.

Despite these difficulties (which prevent from a wide diffusion of ideas and methods of QFT within domains different from physics of many-body systems), QFT, first proposed

in 1926 by P.A.M. Dirac, obtained in the last fifty years remarkable successes in describing and forecasting phenomena occurring within domain of particle physics and condensed matter physics. Quantum electrodynamics, the explanation of laser effect, the unified theory of weak and electromagnetic forces are between the most beautiful achievements of QFT, marking the whole history of physical research in the twentieth century (see, to this regard, Itzykson & Zuber, 1986; Umezawa, 1993). Here we will limit ourselves to mention an important feature of QFT, which is central for our future discussion about emergence: within QFT, differently from QM, there is the possibility of having different, *nonequivalent*, representations of the same physical system (cfr. Haag, 1961; Hepp, 1972). As each representation is associated to a particular class of macroscopic states of the system (via quantum statistical mechanics) and this class, in turn, can be identified with a particular thermodynamical *phase* of the system (for a proof of the correctness of such an identification see Sewell, 1986), we are forced to conclude that *only* QFT allows for the existence of different phases of the system itself. A further consequence is that only within QFT we can deal with *phase transitions*, i.e. with global structural changes of the system under study. Before closing our short discussion of QFT, we stress that the existence of nonequivalent representations is strictly connected to the fact that, if we interpret quantum fields as equivalent to sets of suitable particles lying in suitable dynamical states, then QFT describes, even in its simplest implementations, situations in which the total number of particles is *no longer conserved*. In other words, within QFT (and *only* within it) are allowed processes of *creation* and of *destruction* of particles. This gives to QFT a descriptive power enormously greater than the one of QM, where the number of particles was kept fixed.

In the following paragraph we will make use of these introductory notions in order to discuss the validity of a basic claim, according to which the modeling of emergent phenomena and collective effects of the cooperative dynamics of neural units can be done *only* by resorting to the methods and the framework of QFT. In other words, the only possible emergence would be *quantum* emergence.

7. Emergence and Spontaneous Symmetry Breaking

To prove this claim, let us start by stressing that in an ideal model, by definition, we can always introduce a function playing a role analogous to the one of *energy* in physical systems, so that stable and metastable equilibrium states are directly associated to local minima of such a function. The occurring of intrinsic emergence, then, can be identified with a transition, triggered by the change of value of a given parameter, in which (at least) a local energy minimum is splitted in a number (finite or infinite, but always greater than one) of different local energy minima, all *equivalent*, i.e. characterized by the same value of minimum energy. The intrinsic emergence is due to the fact that, if the system was, before the transition, in the state corresponding to the old energy minimum, surely the transition will provoke the settling of the system into one of the new energy minima, but we cannot forecast *which* of them will be chosen, on the basis of the model we have,

because all minima are equivalent one to another. Such a form of transition is usually called *spontaneous symmetry breaking* (SSB), and appears as the only way, so far known, to introduce intrinsic emergence in systems described by ideal models.

In order to deal with SSB by using a language similar to the one used in theoretical physics, we will restrict ourselves to sub-models describing system's behaviour not very far from the local energy minimum undergoing a splitting during the transition producing intrinsic emergence, in such a way that, within our sub-model, this energy minimum is an *absolute* minimum, and can be identified with system's *ground state*. Then, we must take into account that in such a situation, both the form of model evolution equations and the value of energy in the absolute minimum are invariant with respect to suitable *symmetry transformations* (cfr. Olver, 1995). Now, by using a terminology originally coined by L.D. Landau in his theory of second-order phase transitions, we will say that a *SSB occurs* when ground states are not invariant with respect to the symmetry transformations which leave unchanged both the form of evolution equations and energy values. As SSB takes place when the value of a suitable parameter crosses a given critical value, we will have that, before crossing (i.e. when there isn't SSB), the system will be characterized by only one ground state, and its behavior (of relaxation toward this state) will be predictable in advance, whereas, after crossing (i.e. when SSB occurs), the system will fall in one of its many possible ground states, without any possibility of forecasting, on the basis of evolution equations, which one of them will occur.

As regards SSB, some remarks are in order. The first one is that, in most cases, the multiplicity of ground states exists only if we go at the infinite volume limit. Namely in finite systems such an effect is often hidden by the existence of suitable boundary conditions. The most known case is that of the transition from paramagnetic to ferromagnetic state, in which the unique ground state of paramagnetic phase is splitted, in the ferromagnetic phase, into two equivalent ground states: one with all spins parallel to the inducing external field, and the other with all spins antiparallel. However, the equivalence between these two ground states holds only in the infinite volume limit, and disappears when the volume of magnets becomes finite (as it is the case in real world), so that the ground state with all spins parallel becomes the favored one.

A second remark is that, both in classical and QFT-based descriptions of SSB, the system will be anyway forced to choose one particular ground state. States corresponding to linear combinations of different ground states are not allowed, even in QFT, because it can be proved that any operator connecting two different ground states vanishes at the infinite volume limit (cfr. Huang, 1987).

A third remark is that, if we describe SSB within the context of QFT, the occurring of a SSB implies the appearance of collective excitations, which can be viewed as zero-mass particles carrying long-range interactions. They are generally called *Goldstone bosons* (cfr. Goldstone, Salam & Weinberg, 1962; for a more general approach see Umezawa, 1993). Such a circumstance endows systems, in which SSB takes place, with a sort of *generalized rigidity*, in the sense that, acting on one side of the system with an external perturbation, we can transmit such a perturbation to a very distant location essentially

unaltered. The reason for the appearance of Goldstone bosons is that they act as *order-preserving* messengers, preventing the system from a change of the particular ground state chosen at the moment of SSB transition. Besides, they are a direct manifestation of intrinsic emergence, as none of the forces acting between system's elementary constituents is able to produce generalized rigidity. We further remind that Goldstone bosons themselves can undergo the phenomenon of BEC, giving thus rise to new macroscopic objects and to new emergent symmetries (cfr. Wadati, Matsumoto & Umezawa, 1978a, 1978b; Matsumoto, Sodano & Umezawa, 1979; Umezawa, 1993).

Let's now reformulate the fundamental question under the form: can classical, but ideal, models of SSB, not based on framework of QFT, give rise to Goldstone bosons and to generalized rigidity? In order to answer such a question, we will take into consideration a classical system of individual particles described by laws of classical mechanics (which lets us classify such a system as ideal, being derivable from a single variational principle), whose global potential energy has a form such as to undergo a SSB in correspondence to a suitable value of a given critical parameter. Let's now focus on what happens when, after the occurrence of a SSB transition, the system relaxes into one of its many, equivalent, ground states. Such a relaxation process is described, from a microscopic point of view, by a part of theoretical physics usually called *Kinetics*. Between the main results of this latter we will quote the description of relaxation process when individual particles obey classical mechanics, and, as a consequence, their statistical behavior is ruled by the well known *Maxwell-Boltzmann* probability distribution. These results agree fairly well with experimental data relative to behavior of particular types of ideal gases, and let one derive the macroscopic form of the celebrated *Second Principle of Thermodynamics*, holding for these gases (such a derivation is today known as *H-theorem*). The forecastings based on classical Maxwell-Boltzmann probability distribution, however, fail to be valid in presence of other condensed states of matter. In modern times, a careful study of *quantum* probability distributions (both of *Fermi-Dirac* and of *Bose-Einstein* statistics) showed how the classical Maxwell-Boltzmann distribution is nothing but a particular approximation of quantum ones, holding in some special conditions, and helped us to understand the limits of validity of classical statistical mechanics.

Notwithstanding the partial successes achieved by kinetics based on classical statistical mechanics, for a long time many physicists lasted in a state of deep dissatisfaction, as these successes were based on equations (such as the well known *Boltzmann transport equation*) relying heavily on simplifying approximations introduced only on the basis of phenomenological or *bona fide* arguments. In other words, classical kinetics was not grounded on general principles. Such a situation changed after Second World War, when N. Bogoliubov was able to show that all classical kinetics, including the form itself of Maxwell-Boltzmann distribution, could be derived only on the basis of two general principles:

pb.1) the validity of the laws of classical mechanics;

pb.2) a principle ruling the space-time behavior of an aggregate of classical particles, named *correlation weakening principle*.

Without entering into technical details (cfr. Rumer & Ryvkin, 1980; Akhiezer & Péletminski, 1980; De Boer & Uhlenbeck, 1962) we will limit ourselves to say that correlation weakening principle asserts that, during a relaxation process, all long-range correlations between the individual motions of single particles tend to vanish when the volume tends to infinity (such a vanishing takes place, of course, when time tends to infinity). We can interpret such a principle as asserting the disappearance of Goldstone bosons (or of classical long-range collective excitations) in the relaxation process of a system of classical particles. As a consequence, classical systems *cannot* be endowed with generalized rigidity. This, in turn, implies that, when volume tends to infinity, the choice of a particular ground state, between the many possible after a SSB transition, can be destroyed even by very small perturbations. Namely, order-preserving messengers in classical case are absent. As a conclusion, SSB, even if possible in classical case, is *unstable* with respect to (thermal) perturbations. Such a result was already found by many researchers (cfr. Chafee, 1975; Fernandez, 1985; Nitzan & Ortoleva, 1980; Stein, 1980) in the early days of successes of theories of pattern formation based on dissipative structures.

Such a circumstance gives a sound basis for the claim advanced by the Nobel Prize P.W Anderson and his coworkers (cfr. Anderson, 1981; Anderson & Stein, 1985), according to which the *only* possible description of intrinsic emergence and of SSB is the one made within QFT. A similar point of view was held also by H Umezawa (cfr. Umezawa, 1993). As a conclusion of previous arguments we can assert that the only possible emergence is a *quantum* emergence. Therefore, if cognition is a collective phenomenon, emergent from the interaction of neural units, these latter must necessarily be interpreted as quanta of a suitable *quantum field*. In other words, if cognition has to be emergent, it must be a quantum phenomenon, as already proposed many years ago by Ricciardi and Umezawa in their celebrated paper (Ricciardi & Umezawa, 1967).

8. Quantum Neural Networks

From the previous arguments it follows that every neural-network based model of emergence of cognitive processing should be formulated in terms of a *quantum neural network*. But, what is the meaning of such an expression? To this regard, we shall distinguish between two different ways for building a quantum neural network model:

Qn.1) bottom-up ; in this case a quantum neural network is a set of interconnected units, named *quantum neurons*; the time evolution of output activity of a typical quantum neuron is driven by a quantum dynamical law, even if associated to neural-like features; in order to introduce a quantum field, however, one is forced to resort to a suitable continuum limit, performed on the quantum neural network, or to introduce a background neural field (modelling, for instance, the space-time distribution of chemical substances contained within extracellular liquid);

Qn.2) top-down ; in this case we start from the beginning from a quantum neural field (for instance the field of neurotransmitters), and the quantum neural network is nothing but a discrete approximation of the original field; what is difficult to obtain, however, is

that such an approximation be automatically endowed with neural-like features.

The bottom-up approach, even if directly inspired by recent advances in quantum computation and adopted by some authors (cfr. Samsonovich, 1994; Zak, 2000; Altaisky, 2001), meets a fundamental difficulty: the behavior of a typical neuron (described, for example, by a sigmoidal activation law) is *dissipative*, as it doesn't fulfill an energy conservation principle. Such a circumstance prevents from a direct application of traditional quantization procedures, based on this principle, to dynamical laws ruling neural units. This difficulty can be dealt with following two different approaches: either resorting to new methods of quantization, designed for dissipative systems, which from the starting come out of the usual Hamiltonian framework (cfr. Tarasov, 2001), or introducing the so-called *doubling mechanism* (Celeghini, Rasetti & Vitiello, 1992). The latter starts from the fact that a dissipative harmonic oscillator, that is endowed with a friction term directly proportional to the velocity, can be always associated to a *mirror* harmonic oscillator, characterized by a negative friction, but exactly of the same magnitude as the original dissipative oscillator. In other words, the mirror oscillator absorbs exactly the energy dissipated by the first oscillator. What is interesting is that the whole system constituted by the two oscillators, the original one and its mirror, can be described through a suitable Hamiltonian, a circumstance due to the obvious fact that the total energy of the system of the two oscillators is conserved. Thus, the introduction of mirror oscillators (that is, a *doubling* of the total number of oscillators) allows for the introduction of traditional quantization procedures even in the case of dissipative systems.

Without entering in further details on this topic, we will limit ourselves to remark that, when adopting a bottom-up approach, we are not necessarily forced to start from the introduction of quantum neurons. Another possibility is to start from usual equations, describing the time evolution of activations of traditional neural-like units, and to derive from them, through a suitable continuum limit, the evolution equations of suitable *classical* neural fields, trying to write them under a form derivable from some Hamiltonian. Then, it is easy to apply the standard field quantization methods so as to obtain a *neural quantum field theory*. In order to illustrate how such a procedure can be started, we will make resort to a simple example, based on a toy model of a 1-dimensional neural network, in which each neural unit can receive inputs only from its two immediately neighboring units. If we introduce, as customary, a discretized time scale $t, t + \Delta t, t + 2\Delta t, \dots$, and a discretized spatial scale $i, i + \Delta i, i + 2\Delta i, \dots$, the equation ruling the time evolution of output activation of the unit located in the i -th spatial location will have the form:

$$s(i, t + \Delta t) = F[w_s(i, i - \Delta i)s(i - \Delta i, t) + w_d(i, i + \Delta i)s(i + \Delta i, t)], \quad (31)$$

where $s(i, t)$ denotes the output activation of the i -th unit at time t , whereas $w_s(i, i - \Delta i)$ is the value of connection weight associated to the interconnection line coming to the i -th unit from its left-hand side neighboring unit located in the position $i - \Delta i$ and $w_d(i, i + \Delta i)$ is the analogous connection weight relative to the right-hand side unit located in the position $i + \Delta i$. Besides, F will denote the traditional sigmoidal function

defined by (6).

Let us now introduce the hypothesis of existence of a maximal neural signal propagation speed, hereafter denoted by c (to stress the analogy with the light propagation speed), which implies that time increments Δt and spatial increments Δi are connected by the relationship:

$$\Delta i = c\Delta t. \quad (32)$$

A further hypothesis is that, by taking the limit for $\Delta i \rightarrow 0$, the two-point functions $w_s(i, i - \Delta i)$ and $w_d(i, i + \Delta i)$ tend, respectively, to two functions $w_s(i)$ and $w_d(i)$, to be interpreted as a measure of the spontaneous ‘receptivity’ of the i -th unit with respect to stimulations coming, respectively, from its left-hand and from its right-hand side.

By taking the limit for $\Delta t \rightarrow 0$ in (31) and making use of the previous hypotheses, we will obtain immediately the relationship:

$$s(i, t) = F\{[w_s(i) + w_d(i)]s(i, t)\}. \quad (33)$$

We will now make use of the fact that, at the first order in Δi , we can write:

$$\begin{aligned} s(i - \Delta i, t) &= s(i, t) - (\partial s / \partial i)\Delta i, \quad s(i + \Delta i, t) = s(i, t) + (\partial s / \partial i)\Delta i \\ w_s(i, i - \Delta i) &= w_s(i) - (\partial w_s / \partial i)\Delta i, \quad w_d(i, i + \Delta i) = w_d(i) + (\partial w_d / \partial i)\Delta i. \end{aligned} \quad (34)$$

At this point, if we subtract from both members of (31) the quantity $s(i, t)$ and we divide both members by Δt , we will obtain, by taking the limit for $\Delta t \rightarrow 0$ and by taking into account the relationships (32), (33), (34), as well as the well known first-order formula holding for the sigmoidal function:

$$F(\xi + \Delta\xi) = F(\xi) + F(\xi)[1 - F(\xi)]\Delta\xi, \quad (35)$$

that the output activation field will satisfy the partial differential equation:

$$\partial s / \partial t = cF[(w_s + w_d)s][1 - F[(w_s + w_d)s]][s\partial(w_d - w_s) / \partial x + (w_d - w_s)\partial s / \partial x]. \quad (36)$$

By deriving (36) with respect to t and by taking into account that, owing to (32), we have:

$$\partial / \partial t = c\partial / \partial x, \quad (37)$$

we will finally obtain the following classical neural field equation:

$$\begin{aligned} \partial^2 s / \partial t^2 &= c^2 G(1 - G)(1 - 2G)(s\partial P / \partial x + P\partial s / \partial x)(s\partial Q / \partial x + Q\partial s / \partial x) + \\ &+ c^2 G(1 - G)[s\partial^2 Q / \partial x^2 + Q\partial^2 s / \partial x^2 + 2(\partial s / \partial x)(\partial Q / \partial x)], \end{aligned} \quad (38)$$

where:

$$G = F[(w_s + w_d)s], \quad P = w_s + w_d, \quad Q = w_d - w_s. \quad (39)$$

The equation (38) gives rise to a number of interesting limiting cases. For instance, if we assume that $P = 0$, $Q = \text{const}$, we will obtain immediately that the neural field equation will reduce to:

$$\partial^2 s / \partial t^2 = c^2 (Q/4) \partial^2 s / \partial x^2, \quad (40)$$

that is, to standard wave equation. If, instead, we suppose that the argument of F be so small as to have a value of G practically coincident with $1/2$, and we work in a very high frequency approximation, so as to neglect the first order derivatives of Q with respect to the second order ones, we can write the equation (38) under the form:

$$\partial^2 s / \partial t^2 = (c^2/4)(Q \partial^2 s / \partial x^2 + s \partial^2 Q / \partial x^2), \quad (41)$$

coincident with a generalization of the usual Klein-Gordon equation. It is to be remarked that both neural field equations (40) and (41) can be quantized through standard methods, so as to obtain a quantum neural field theory.

Before leaving such a topic, we underline that even a simple toy model, like the one presented before, allows for the existence of spontaneous symmetry breaking phenomena which, at the classical level, manifest themselves under the form of bifurcation points. To this regard, if we suppose that the argument of F be so small as to write, in a first approximation;

$$F(z) \cong (1/2) + (1/4)z, \quad (42)$$

then the homogeneous steady-state equilibrium solution of (38) will satisfy the equation:

$$-(1/4)P(\partial P / \partial x)(\partial Q / \partial x)s^3 + (\partial^2 Q / \partial x^2)s = 0. \quad (43)$$

For generic choices of the form of the functions P and Q the equation (43) will have one and only one solution corresponding to $s = 0$ (the ground state). However, if we assume that these functions fulfill the (compatible and easily solvable) constraints:

$$(\partial^2 Q / \partial x^2 = \alpha, \partial Q / \partial x = \alpha x + \beta, P \partial P / \partial x = \gamma / (\alpha x + \beta)), \quad (44)$$

where α , β and γ are suitable parameters, we will immediately see that (43) will assume the form:

$$-(\gamma/4)s^3 + \alpha s = 0, \quad (45)$$

which has three real solutions if α and γ have the same sign, whereas has only one real solution, corresponding to $s = 0$, in the opposite case. The critical situation in which we have a bifurcation point corresponds, whence, to the case in which one of the

two aforementioned parameters is vanishing (which one, depends on the choice of the bifurcation parameter).

Another very interesting topic, highly relevant to the design of quantum neural networks according to the top-down approach, is connected to the following question: do exist neural network models which, even if devoid of any quantum feature deliberately introduced *a priori*, exhibit a behavior typical of quantum models? In other words, it is possible to design neural-like systems which are not ideal models of emergence, in the sense specified in the fifth paragraph, but behave like suitable approximate versions of ideal models? Such a question arises from the well known fact that there exist models of physical systems, based on traditional classical physics and, therefore, devoid of any quantum property, which, however, exhibit a quantum-like behavior (see, for a review of this topic, Carati & Galgani, 2001). The answer is, of course, yes, provided we introduce suitable choices of interconnections, activation and learning laws, and, chiefly, we limit our considerations to small-world situations, avoiding infinite volume limits. This is due to the fact that the specification of a neural-like activity within a finite domain needs the introduction of suitable constraints (such as boundary conditions, connection topology, and so on) which, in a way or in another, could give rise to some form of *nonlocality* inside the modelled systems, manifesting itself through the occurrence of long-range correlations.

Such a circumstance, of course, is to be expected if neural networks models are built by starting directly from a quantum field model and following a reduction procedure of this sort:

- look at the explicit form of quantum field equations
- undertake a suitable spatial discretization of these equations
- substitute the original time evolution laws of field operators with approximate neural-like evolution laws
- replace the time evolution of neural-like operators with a time evolution of c-numbers, representing neural-like quantities, mimicking in some way the quantum evolutionary features
- introduce a finite spatial domain
- discretize the time scale.

A still unanswered question is whether *any other* neural network model, exhibiting quantum features, but not built in advance according the previous procedure, could or not be *always* viewed as deriving from a suitable quantum-field-theoretic ancestor model.

We will remark here that quantum neural networks don't coincide with *quantum computers*. These latter are devices whose operation is based on standard quantum-mechanical laws and, as such, have nothing to do with quantum field theory. For instance, they cannot allow for phase transitions. However, when a quantum-field-theoretical model has been discretized and constrained within a finite domain in a neural-like fashion, according to the procedure previously described, it could happen that its operation can mimic the operation of a suitable quantum computer. Thus, quantum computers appear as special cases of approximate forms of quantum neural networks. Within these latter,

however, we could observe even behaviors impossible for a quantum computer, such as self-assembly, structure change, variations in the number of components, self-repair, and like.

As we have seen, the study of quantum neural networks is a fascinating and difficult topic, which, however, cannot be explored within the short space of a single paper. We will limit ourselves to underline, from one hand, the need for this extension of traditional neural network models and, on the other hand, its feasibility.

Conclusions

The main conclusions of this paper, based on the arguments presented in the previous paragraphs, can be thus synthesized:

- neural networks are universal approximators of whatever kind of system; thus, every complex system can, in principle, be modelled through a suitable neural network;
- some traditional neural network models could be identified with not-ideal models of emergence and, as such, we cannot control the occurrence of this emergence nor know in advance whether it could or not disappear with increasing network size, evolution time, and so on;
- the understanding of complex systems behavior requires a firmly grounded theory of emergence;
- the only possible ideal models of emergence so far known can be formulated within the framework of quantum field theory;
- as a consequence, any ideal model of emergence in a complex system, allowing for a quantitative analysis, should coincide with a quantum neural network;
- the actual mathematical formulation of quantum field theory, however, precludes its direct application to neural network models, as they have been designed so far; in order to build quantum neural network models we still need to resort to approximations which, however, in many cases could be incompatible with modelling constraints;
- a suitable generalization, or reformulation, of quantum field theory is in order, if we want to extend its applicability to domains beyond the realm of atomic and particle physics;
- the actual tools of theoretical physics are the only possible starting point for a theory of emergence; they, however, should be suitably generalized in order to concretely produce such a theory.

These conclusions pose a formidable challenge for the future: the extraction, from the actual machinery of quantum theories, of the deep logical principles underlying them, in order to allow for a comparison with the logic underlying classical physics and for mathematical formulations totally independent from the particular features associated to the context of point particles. To face this challenge, we shall need a deep rearrangement of our cognitive schemata, which appears as the only possible way for a shift from the study of *unorganized complexity* to the one of organizations, be they biological, cognitive,

or socio-economical. We feel that only the latter framework will leave us some hope of managing in a successful way the formidable problems posed by the actual world.

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Entanglement Revisited

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Abstract: Quantum-classical hybrid that preserves the topology of the Schrödinger equation (in the Madelung form), but replaces the quantum potential with other, specially selected, function of probability density is introduced. Non-locality associated with a global geometrical constraint that leads to entanglement effect is demonstrated. Despite such a quantum-like characteristic, the hybrid can be of classical scale and all the measurements can be performed classically. This new emergence of entanglement shed light on the concept of non-locality in physics. Application of hybrid systems to instantaneous transmission of conditional information on remote distances is discussed.

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1. Introduction

Quantum entanglement is a phenomenon in which the quantum states of two or more objects have to be described with reference to each other, even though the individual objects may be spatially separated. This leads to correlations between observable physical properties of the systems. As a result, measurements performed on one system seem to be instantaneously influencing other systems entangled with it. Different views of what is actually occurring in the process of quantum entanglement give rise to different interpretations of quantum mechanics. In this paper we will demonstrate that entanglement is not a prerogative of quantum systems: it occurs in other non-classical systems such as quantum-classical hybrids [1], and that will shed light on the concept of entanglement as a special type of global constraint imposed upon a broad class of dynamical systems. In order to do that, we will turn to quantum mechanics. Representing the Schrödinger

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equation in the Madelung form, we observe the feedback from the Liouville equation to the Hamilton-Jacobi equation in the form of the quantum potential. Preserving the same topology, we will replace the quantum potential by other type of feedbacks [2] and investigate the property of these hybrid systems.

2. Destabilizing Effect of Liouville Feedback

We will start with derivation of an auxiliary result that illuminates departure from Newtonian dynamics. For mathematical clarity, we will consider here a one-dimensional motion of a unit mass under action of a force f depending upon the dimensionless *velocity* v and time t

$$\dot{v} = f(v, t), \quad (1)$$

If initial conditions are not deterministic, and their probability density is given in the form

$$\rho_0 = \rho_0(V), \quad \text{where } \rho \geq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \rho dV = 1 \quad (2)$$

while ρ is a *single-valued* function, then the evolution of this density is expressed by the corresponding Liouville equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial v}(\rho f) = 0 \quad (3)$$

The solution of this equation subject to initial conditions and normalization constraints (2) determines probability density as a function of V and t : $\rho = \rho(V, t)$.

In order to deal with the constraint (2), let us integrate Eq. (3) over the whole space assuming that $\rho \rightarrow 0$ at $|V| \rightarrow \infty$ and $|f| < \infty$. Then

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \rho dV = 0, \quad \int_{-\infty}^{\infty} \rho dV = \text{const}, \quad (4)$$

Hence, the constraint (3) is satisfied for $t > 0$ if it is satisfied for $t = 0$.

Let us now specify the force f as a feedback from the Liouville equation

$$f(v, t) = \varphi[\rho(v, t)] \quad (5)$$

and analyze the motion after substituting the force (5) into Eq.(1)

$$\dot{v} = \varphi[\rho(v, t)], \quad (6)$$

This is a fundamental step in our approach. Although the theory of ODE does not impose any restrictions upon the force as a function of space coordinates, the Newtonian physics does: equations of motion are never coupled with the corresponding Liouville equation. Moreover, it can be shown that such a coupling leads to non-Newtonian properties of the

underlying model. Indeed, substituting the force f from Eq. (5) into Eq. (4), one arrives at the *nonlinear* equation for evolution of the probability density

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial V} \{ \rho \varphi[\rho(V, t)] \} = 0 \quad (7)$$

Let us now demonstrate the destabilizing effect of the feedback (5). For that purpose, it should be noted that the derivative $\partial \rho / \partial v$ must change its sign, at least once, within the interval $-\infty < v < \infty$, in order to satisfy the normalization constraint (2).

But since

$$\text{Sign} \frac{\partial \dot{v}}{\partial v} = \text{Sign} \frac{d\varphi}{d\rho} \text{Sign} \frac{\partial \rho}{\partial v} \quad (8)$$

there will be regions of v where the motion is unstable, and this instability generates randomness with the probability distribution guided by the Liouville equation (8). It should be noticed that the condition (9) may lead to exponential or polynomial growth of v (in the last case the motion is called neutrally stable, however, as will be shown below, it causes the emergence of randomness as well if prior to the polynomial growth, the Lipchitz condition is violated).

3. Emergence of Randomness

In order to illustrate mathematical aspects of the concepts of Liouville feedback, as well as associated with it instability and randomness let us take the feedback (6) in the form

$$f = -\sigma^2 \frac{\partial}{\partial v} \ln \rho \quad (9)$$

to obtain the following equation of motion

$$\dot{v} = -\sigma^2 \frac{\partial}{\partial v} \ln \rho, \quad (10)$$

This equation should be complemented by the corresponding Liouville equation (in this particular case, the Liouville equation takes the form of the Fokker-Planck equation)

$$\frac{\partial \rho}{\partial t} = \sigma^2 \frac{\partial^2 \rho}{\partial V^2} \quad (11)$$

Here v stands for a particle velocity, and σ^2 is the constant diffusion coefficient.

The solution of Eq. (11) subject to the sharp initial condition is

$$\rho = \frac{1}{2\sigma\sqrt{\pi t}} \exp\left(-\frac{V^2}{4\sigma^2 t}\right) \quad (12)$$

Substituting this solution into Eq. (10) at $V = v$ one arrives at the differential equation with respect to $v(t)$

$$\dot{v} = \frac{v}{2t} \quad (13)$$

and therefore,

$$v = C\sqrt{t} \quad (14)$$

where C is an arbitrary constant. Since $v = 0$ at $t = 0$ for any value of C , the solution (14) is consistent with the sharp initial condition for the solution (12) of the corresponding Liouville equation (11). The solution (14) describes the simplest irreversible motion: it is characterized by the “beginning of time” where all the trajectories intersect (that results from the violation of Lipchitz condition at $t = 0$, Fig.1), while the backward motion obtained by replacement of t with $(-t)$ leads to imaginary values of velocities. One can notice that the probability density (13) possesses the same properties.

For a fixed C , the solution (14) is *unstable* since

$$\frac{d\dot{v}}{dv} = \frac{1}{2t} > 0 \quad (15)$$

and therefore, an initial error always grows generating *randomness*. Initially, at $t = 0$, this growth is of infinite rate since the Lipchitz condition at this point is violated

$$\frac{d\dot{v}}{dv} \rightarrow \infty \quad \text{at} \quad t \rightarrow 0 \quad (16)$$

This type of instability has been introduced and analyzed in [3].

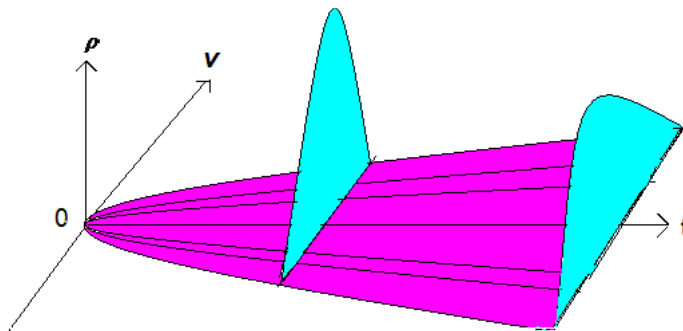


Fig. 1 Stochastic process and probability density

Considering first Eq. (14) at fixed C as a sample of the underlying stochastic process (12), and then varying $C(\omega)$ (where ω is a variable running over different samples of the stochastic process, and $\omega \subseteq V$), one arrives at the whole ensemble characterizing that process, (see Fig. 1). One can verify that, as follows from Eq. (12), [4], the expectation and the variance of this process are, respectively

$$MV = 0, \quad DV = 2\sigma^2 t \quad (17)$$

The same results follow from the ensemble (14) at $-\infty \leq C(\omega) \leq \infty$. Indeed, the first equality in (17) results from symmetry of the ensemble with respect to $v = 0$; the second one follows from the fact that

$$DV \propto v^2 \propto t \quad (18)$$

It is interesting to notice that the stochastic process (14) is an alternative to the following Langevin equation, [4]

$$\dot{v} = \Gamma(t), \quad M\Gamma = 0, \quad D\Gamma = \sigma \quad (19)$$

that corresponds to the *same* Fokker-Planck equation (11). Here $\Gamma(t)$ is the Langevin (random) force with zero mean and constant variance σ .

The results described in this sub-section can be generalized to n-dimensional case, [5].

4. Emergence of Global Constraint

In order to introduce and illuminate a fundamentally new non-Newtonian phenomenon similar to quantum entanglement, let us assume that the function $\varphi(\rho)$ in Eq. (5) is invertible, i.e. $\rho = \varphi^{-1}(f)$. Then as follows from Eq. (7) with reference to the normalization constraint (2)

$$\int_{-\infty}^{\infty} \varphi^{-1}[\dot{v}(\omega, t)] d\omega = 1 \quad (20)$$

Other non-Newtonian properties of solution to Eq. (7) such as shock waves in *probability space* have been studied in [2] and [5]. Similar shock wave effects have been investigated in [6].

Thus, the motions of the particles emerged from instability of Eq. (6) must satisfy the global kinematical constraint (20).

It should be emphasized that the concept of a global constraint is one of the main attribute of Newtonian mechanics. It includes such idealizations as a rigid body, an incompressible fluid, an inextensible string and a membrane, a non-slip rolling of a rigid ball over a rigid body, etc. All of those idealizations introduce geometrical or kinematical restrictions to positions or velocities of particles and provides “instantaneous” speed of propagation of disturbances. However, the global constraint

$$\int_{-\infty}^{\infty} \rho dV = 1 \quad (21)$$

is fundamentally different from those listed above for two reasons. Firstly, this constraint is not an idealization, and therefore, it cannot be removed by taking into account more subtle properties of matter such as elasticity, compressibility, etc. Secondly, it imposes restrictions not upon positions or velocities of particles, but upon the *probabilities* of their positions or velocities, and that is where the non-locality comes from.

Continuing this brief review of global constraints, let us discuss the role of the reactions to these constraints, and in particular, let us find the analog of reactions of global constraints in quantum mechanics. One should recall that in an incompressible fluid, the reaction to the global constraint $\nabla \cdot v \geq 0$ (expressing non-negative divergence of the velocity v) is a non-negative pressure $p \geq 0$; in inextensible flexible (one- or two-dimensional) bodies, the reaction of the global constraint $g_{ij} \leq g_{ij}^0, i, j = 1, 2$ (expressing

that the components of the metric tensor cannot exceed their initial values) is a non-negative stress tensor $\sigma_{ij} \geq 0$, $i, j=1,2$. Turning to quantum mechanics and considering the uncertainty inequality

$$\Delta x \Delta p \geq \hbar/2 \quad (22)$$

in which Δx and Δp are the standard deviation of coordinate and impulse, respectively as a global constraint, one arrives at the quantum potential $\frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}}$ as a “reaction” of this constraint in the Madelung equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\frac{\rho}{m} \nabla S \right) = 0 \quad (23)$$

$$\frac{\partial S}{\partial t} + (\nabla S)^2 + V - \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}} = 0 \quad (24)$$

Here ρ and S are the components of the wave function $\varphi = \sqrt{\rho} e^{iS/\hbar}$, and \hbar is the Planck constant divided by 2π . But since Eq. (23) is actually the Liouville equation, the quantum potential represents a Liouville feedback similar to those introduced above via Eqs. (5) and (9). Due to this topological similarity with quantum mechanics, the models that belong to the same class as those represented by the system (7), (8) are expected to display properties that are closer to quantum rather than to classical ones.

5. Emergence of Entanglement

Prior to introduction of the entanglement phenomenon, we will demonstrate additional non-classical effects displayed by the solutions to Eqs. (6) and (7) when

$$\varphi(\rho) = \zeta \rho, \quad \zeta = \text{const.} > 0$$

and therefore

$$\dot{v} = \zeta \rho \quad (25)$$

$$\frac{\partial \rho}{\partial t} + \zeta \frac{\partial}{\partial V} (\rho^2) = 0 \quad (26)$$

The solution of Eq. (26) subject to the initial conditions $\rho_0(V)$ and the normalization constraint (2) is given in the following implicit form [7]

$$\rho = \rho_0(\lambda), \quad V = \lambda + \rho_0(\lambda)t \quad (27)$$

This solution subject to the initial conditions and the normalization constraint, describes propagation of initial distribution of the density $\rho_0(V)$ with the speed V that is proportional to the values of this density, i.e. the higher values of ρ propagates faster than lower ones. As a result, any compressive part of the wave, where the propagation velocity is a decreasing function of V , ultimately “breaks” to give a *triple-valued* (but still continuous) solution for $\rho(V, t)$. Eventually, this process leads to the formation of strong discontinuities that are related to propagating jumps of the probability density. In the theory of nonlinear waves, this phenomenon is known as the formation of a shock wave. Thus,

as follows from the solution (27), a single-valued continuous probability density spontaneously transforms into a triple-valued, and then, into discontinuous distribution. In aerodynamical application of Eq. (26), when ρ stands for the gas density, these phenomena are eliminated through the model correction: at the small neighborhood of shocks, the gas viscosity ν cannot be ignored, and the model must include the term describing dissipation of mechanical energy. The corrected model is represented by the Burgers' equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial V}(\rho^2) = \nu \frac{\partial^2 \rho}{\partial V^2} \quad (28)$$

As shown in [7], this equation has continuous single-valued solution (no matter how small is the viscosity ν), and that provides a perfect explanation of abnormal behavior of the solution to Eq. (26). Similar correction can be applied to the case when ρ stands for the probability density if one includes Langevin forces $\Gamma(t)$ into Eq. (25)

$$\dot{v} = \rho + \sqrt{\nu}\Gamma(t), \quad \langle \Gamma(t) \rangle = 0, \quad \langle \Gamma(t)\Gamma(t') \rangle = 2\delta(t-t') \quad (29)$$

Then the corresponding Fokker-Planck equation takes the form (28). It is reasonable to assume that small random forces of strength $\sqrt{\nu} \ll 1$ are always present, and that protects the mathematical model (25), (26) from singularities and multi-valuedness in the same way as it does in the case of aerodynamics.

It is interesting to notice that Eq. (28) can be obtained from Eq. (25) in which random force is replaced by additional Liouville feedback

$$\dot{v} = \zeta\rho - \nu \frac{\partial}{\partial V} \ln \rho, \quad \zeta > 0, \quad \nu > 0, \quad (30)$$

An interesting non-classical property of a solution of this equation is decrease of entropy. Indeed,

$$\begin{aligned} \frac{\partial H}{\partial t} &= -\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \rho \ln \rho dV = -\int_{-\infty}^{\infty} \frac{1}{\zeta} \dot{\rho} (\ln \rho + 1) dV = \int_{-\infty}^{\infty} \frac{1}{\zeta} \frac{\partial}{\partial V} (\rho^2) \ln(\rho + 1) dV \\ &= \frac{1}{\zeta} \left[\int_{-\infty}^{\infty} \rho^2 (\ln \rho + 1) - \int_{-\infty}^{\infty} \rho dV \right] = -\frac{1}{\zeta} < 0 \end{aligned} \quad (31)$$

Obviously, presence of small diffusion, when $\nu \ll 1$, does not change the inequality (31) during certain period of time. (However, eventually, for large times, diffusion takes over, and the inequality (31) is reversed). It is easily verifiable that the solution to Eq. (28) satisfies the constraint (2) if the corresponding initial condition does, [7].

Let us concentrate now on the solution of the system (30) and (28) remembering that it is a particular case of the system (6), (7)

$$\dot{v} = \zeta\rho - \nu \frac{\partial}{\partial V} \ln \rho, \quad \zeta > 0, \quad [\zeta] = \frac{1}{\text{sec}}, \quad [\nu] = \frac{1}{\text{sec}}, \quad (32)$$

$$\frac{\partial \rho}{\partial t} + \zeta \frac{\partial}{\partial V} (\rho^2) = \nu \frac{\partial^2 \rho}{\partial V^2}, \quad (33)$$

subject to a single-hump initial condition

$$\rho_0(V) = A\delta(V) \quad \text{at } t = 0, \quad A = \text{const.} \quad (34)$$

where A is the initial area of the hump, and

$$v(t = 0) = v_0 \quad (35)$$

The variable v in Eq. (32) is a dimensionless velocity $v \rightarrow v/v_0$, and the “Reynolds” number

$$R = \zeta \frac{A}{2\nu} \quad (36)$$

We will be interested in the solution of the system (32), (33) for the case of large Reynolds number

$$R \rightarrow \infty, \quad \text{and } \zeta \gg \nu \quad (37)$$

In this case, Eq. (32) can be simplified by omitting the “viscose” term

$$\dot{v} = \zeta\rho, \quad \zeta > 0 \quad (38)$$

However, omitting the last term in Eq. (33) would lead to qualitative changes outlined above, and in particular, it would prevent us to start with the sharp initial conditions (34).

We will start with the solution to Eq. (33). It is different from the standard Burger’s equation only by a physical interpretation of the variable ρ that is now a probability density (instead of density of a gas), but as shown in [7], the constraint (2) is satisfied automatically if it is satisfied for the initial condition (34).

Thus, the solution to Eq.(33) subject to the conditions (2), (34) and (37) reads

$$\rho = \frac{V}{\zeta t} \quad \text{in } 0 < V < \sqrt{2A\zeta t} \quad \text{and } \rho = 0 \quad \text{outside} \quad (39)$$

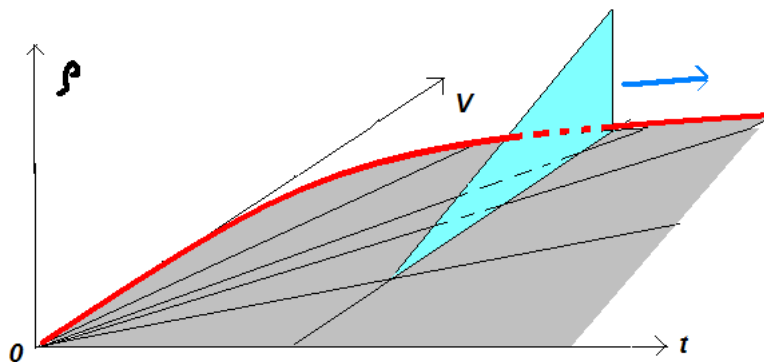


Fig. 2 The triangular shock wave of probability density and samples of associated stochastic process.

The solution has a shock of density

$$[\rho] = \sqrt{\frac{2A}{\zeta t}} \quad \text{at } V = \sqrt{2A\zeta t} \quad (40)$$

Substituting the solution (39) into equation (38) one obtains

$$\dot{v} = \frac{v}{t} \quad \text{in} \quad 0 < v < \sqrt{2A\zeta t} \quad (41)$$

and

$$\dot{v} = 0 \quad \text{in} \quad v > \sqrt{2A\zeta t} \quad (42)$$

whence

$$v = Ct \quad \text{in} \quad 0 < v < \sqrt{2A\zeta t} \quad (43)$$

$$v = C \quad \text{in} \quad v > \sqrt{2A\zeta t} \quad (44)$$

We will be interested here only in the region of Eqs. (41) and (43). Here C is an arbitrary constant. Since $v = 0$ at $t = 0$ for any value of C , the solution (43) is consistent with the sharp initial condition (34). For a fixed C , the solution (43) is *unstable* since

$$\frac{d\dot{v}}{dv} = \frac{1}{2t} > 0 \quad (45)$$

and therefore, an initial error always grows generating *randomness* whose probability is controlled by Eq.(33). Initially, at $t = 0$, this growth is of infinite rate since the Lipschitz condition at this point is violated

$$\frac{d\dot{v}}{dv} \rightarrow \infty \quad \text{at} \quad t \rightarrow 0 \quad (46)$$

Considering first Eq. (41) at fixed C as a sample of the underlying stochastic process (39), and then varying $C(\omega)$ (where ω is a variable running over different samples of the stochastic process, and $\omega \subseteq V$), one arrives at the whole ensemble characterizing that process, (see Fig. 2).

The same phenomenon has been observed in the solution to Eq. (13). However, here the stochastic process converges to the attractor represented by the curve (40) on the $V - t$ plane where the shock of the probability density occurs (see the red line in Fig. 2). The displacement x corresponding to the velocity v , strictly speaking, includes a diffusion term. However, because of vanishing viscosity (see Eq. (37), this term can be ignored, and therefore, samples with the same initial positions have the same velocities.

Let us turn to physical interpretation of the solution. In terms of Eq.(38), the curve (40) is a superposition of different samples of the stochastic process that have the same position and velocity at the same instance of time. However, the accelerations of the superimposed samples, as well as the probability of their occurrence, are different. If one introduces a continuous variable μ to distinguish the accelerations (assuming that this variable changes from zero to one, while $\mu \subset V$), than, as follows from Eq. (38) with reference to Eq. (40)

$$\int_0^1 \dot{v}(\mu) d\mu = \zeta[\rho] = \sqrt{\frac{2A\zeta}{t}} \quad (47)$$

This is a global kinematical constraint that bounds the accelerations of those samples of the stochastic process that are superimposed at the same point of the $V - t$ plane, (see the red curve in Fig. 2).

In order to investigate the properties of the constrained samples (47), let us slightly modify the original system (32), (33) as

$$\dot{v} = \zeta\rho - \nu\frac{\partial}{\partial V}\ln\rho, \quad \zeta > 0, \quad \text{if } t \leq T, \quad \text{and } \zeta < 0 \quad \text{if } t > T \quad (48)$$

$$\frac{\partial\rho}{\partial t} + \zeta\frac{\partial}{\partial V}(\rho^2) = \nu\frac{\partial^2\rho}{\partial V^2} \quad (49)$$

Now we will be interested in the case $\zeta < 0$. Starting from $t > T$, the triangular shock wave (see Fig. 2) will move backwards as a wave of expansion, and the shock will start dissipating. In order to avoid unnecessary mathematical details, we will concentrate attention only on the area around the shock itself disregarding behavior of the rest area of the triangle. Therefore we can drop viscose terms in both equations since formation of new shock waves will start up-stream away from the old shock, (see the grey area in Fig. 3). Hence, now we will deal with the system

$$\dot{v} = \zeta\rho\zeta < 0 \quad (50)$$

$$\frac{\partial\rho}{\partial t} + \zeta\frac{\partial}{\partial V}(\rho^2) = 0 \quad (51)$$

The solution to these equations is,

$$\begin{aligned} \rho &= \sqrt{\frac{2A}{\zeta T}}, \quad \text{and } v = \sqrt{\frac{2A}{\zeta T}}t \quad \text{for } \rho < \frac{V}{\zeta t}, \\ \rho &= \frac{V}{\zeta t}, \quad \text{and } v = C(\omega)t \quad \text{for } 0 < \frac{V}{\zeta t} < \sqrt{\frac{2A}{\zeta t}}, \\ \rho &= 0, \quad \text{and } v = 0 \quad \text{for } \frac{V}{\zeta t} < 0 \end{aligned} \quad (52)$$

The first and the last parts of the solution (52) describe the motion in front and in rear of the dispersing shock, (see the grey and the yellow areas, respectively, in Fig. 3). The middle part of the solution (52) describes the process of shock dispersion (see the red fan in Fig.3). The most important property of the motion in this area is the preservation of the global constraint (47): although samples of the stochastic process that occur inside of the fan are not superimposed any more, i.e. they have different locations and different velocities, nevertheless their accelerations are still bounded by the same constraint as those as they had during their superposition, and such a “memory” expresses qualitative effect of entanglement similar to those in quantum mechanics. However, there is a difference: in quantum mechanics, entanglement is referred to *different particles*, while in the system introduced above it is referred to *different samples of the same particle*. To illuminate this effect, assume that we observed some portion of entangled samples of a stochastic process. Then, based upon the global constraint (47), we can predict

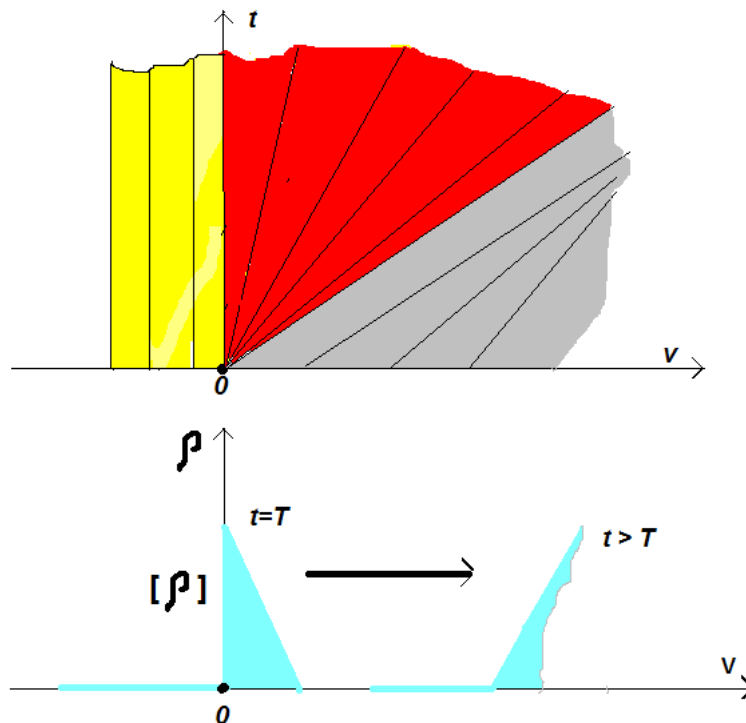


Fig. 3 Dispersion of the shock, and associated entangled fan.

properties of the rest, never observed, samples of the same stochastic process. It could be expected that extension of the model (48), (49) to multi-dimensional case

$$\dot{v}_i = \zeta_i \rho - \nu \sum \frac{\partial}{\partial V_i} \ln \rho, \quad (53)$$

$$\frac{\partial \rho}{\partial t} + \sum \zeta_i \frac{\partial}{\partial V_i} (\rho^2) = \nu \frac{\partial^2 \rho}{\partial V^2}, \quad i=1, 2, \dots, n \quad (54)$$

would create entanglement of samples of *different* particles.

6. Discussion and Conclusion

The formal mathematical difference between quantum and classical mechanics is better pronounced in the Madelung (rather than the Schrödinger) equation. Two factors contribute to this difference: the scale of the system introduced through the Planck constant and the topology of the Madelung equations (23), (24) that includes the feedback (in the form of the quantum potential) from the Liouville equation to the Hamilton-Jacobi equation. Ignoring the scale factor as well as the concrete form of the feedback, we concentrated upon preserving the topology while varying the types of the feedbacks. As a result, we arrived at a new class of dynamical systems: quantum-classical hybrids. A general approach to the choice of the feedback was introduced and discussed in [2]. More specific feedbacks linked to the behavioral models of Livings were presented in [5] via replacement of quantum potential with information potential. That replacement leads to

the capability of the model to evolve from disorder to order (compare to Eq. (31). The computational capabilities of the quantum-classical hybrids have been investigated in [1] where a special feedback – computational potential- was selected

The most effective way of implementation of quantum-classical hybrid is by means of analog devices such as VLSI chips used for neural net’s analog simulations, [10]. Of special importance there is the square root circuit that was extended to the circuit for terminal repeller by Cetin,B, [11].

Thus, the objective of this paper is to demonstrate another fundamental property of quantum-classical hybrids: entanglement. So far this mysterious property that caused many discussions on the highest level of scientific community was considered as a prerogative of quantum mechanics. In this paper we demonstrate that a special form of Liouville feedback, Fig. 4, provides quantitatively similar entanglement effects: different samples of a stochastic process after being superimposed for a certain period of time attain a “memory”: their accelerations satisfy a global kinematical constraint (47).

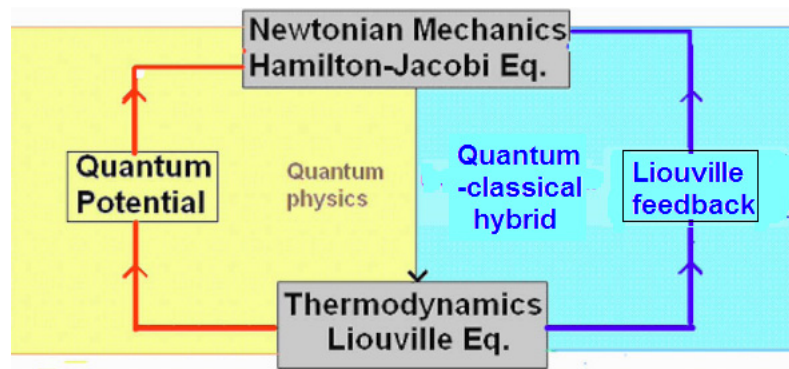


Fig. 4 Classical physics, Quantum physics, and quantum-classical hybrid.

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Innerness of ρ -derivations on Hyperfinite Von Neumann Algebras

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Abstract: Suppose that $\mathfrak{M}, \mathfrak{N}$ are von Neumann algebras acting on a Hilbert space and \mathfrak{M} is hyperfinite. Let $\rho : \mathfrak{M} \rightarrow \mathfrak{N}$ be an ultraweakly continuous $*$ -homomorphism and let $\delta : \mathfrak{M} \rightarrow \mathfrak{N}$ be a $*$ - ρ -derivation such that $\delta(I)$ commutes with $\rho(I)$. We prove that there is an element U in \mathfrak{N} with $\|U\| \leq \|\delta\|$ such that $\delta(A) = U\rho(A) - \rho(A)U$ for all $A \in \mathfrak{M}$.

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Keywords: $*$ - ρ -Derivation; Inner ρ -Derivation; $*$ -Homomorphism; Hyperfinite Von Neumann Algebra; C^* -Algebra

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1. Introduction

The theory of algebras of operators on Hilbert spaces started around 1930 with papers of von Neumann and Murray. The principal motivations of these authors were the theory of unitary group representations and certain aspects of the quantum mechanical formalism. The von Neumann algebras are significant for mathematical physics since the most fruitful algebraic reformulation of quantum statistical mechanics and quantum field theory was in terms of these algebras, cf. [1, 2]. The study of theory of derivations in operator algebras is motivated by questions in quantum physics and statistical mechanics. One of important questions in the theory of derivations is that “When are all bounded derivations inner?” Forty years ago, Kadison [8] and Sakai [16] independently proved that every derivation

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of a von Neumann algebra \mathfrak{M} into itself is inner, see also papers of [7, 9]. This was the starting point for the study of the so-called bounded cohomology groups (see [14]). This nice result can be restated as saying that the first cohomology group $H^1(\mathfrak{M}; \mathfrak{M})$ (i.e. the vector space of derivations modulo the inner derivations) vanishes.

Let \mathfrak{A} and \mathfrak{B} be two algebras, \mathfrak{X} be a \mathfrak{B} -bimodule and $\rho : \mathfrak{A} \rightarrow \mathfrak{B}$ be a homomorphism. A linear mapping $\delta : \mathfrak{A} \rightarrow \mathfrak{X}$ is called a ρ -derivation if $\delta(ab) = \delta(a)\rho(b) + \rho(a)\delta(b)$ for all $a, b \in \mathfrak{A}$. These maps have been extensively investigated in pure algebra. Recently, they have been treated in the Banach algebra theory (see [3, 6, 10, 11, 18] and references therein).

A wide range of examples are as follows:

i. Every ordinary derivation of an algebra \mathfrak{A} into an \mathfrak{A} -bimodule \mathfrak{X} is an $\iota_{\mathfrak{A}}$ -derivation (throughout the paper, $\iota_{\mathfrak{A}}$ denotes the identity map on the algebra \mathfrak{A});

ii. For a given homomorphism ρ on \mathfrak{A} and a fixed arbitrary element X in an \mathfrak{A} -bimodule \mathfrak{X} , the linear mapping $\delta(A) = X\rho(A) - \rho(A)X$ is a ρ -derivation of \mathfrak{A} into \mathfrak{X} which is said to be an inner ρ -derivation.

iii. Every point derivation $\delta : \mathfrak{A} \rightarrow \mathbb{C}$ at the character θ on \mathfrak{A} is a θ -derivation.

A von Neumann algebra \mathfrak{M} is said to be hyperfinite if there exists an increasing net of finite dimensional subalgebras $\{\mathfrak{M}_\lambda\}_{\lambda \in \Lambda}$ such that $\cup_\lambda \mathfrak{M}_\lambda$ is ultraweakly dense in \mathfrak{M} . There is a known characterization of hyperfiniteness that says \mathfrak{M} is hyperfinite if and only if $H^n(M; X) = 0$ for all dual normal \mathfrak{M} -bimodule X ; cf. [4]. Recall that by the weak (operator) topology on $B(\mathfrak{H})$ we mean the topology generated by the semi-norms $T \mapsto |\langle T\xi, \eta \rangle|$ ($\xi, \eta \in \mathfrak{H}$). We also use the terminology ultraweak (operator) topology for the weak*-topology on $B(\mathfrak{H})$ considered as the dual space of the nuclear operators on \mathfrak{H} . We refer the reader to [17] for undefined phrases and notations.

Using some strategies of [15], we prove that every ρ -derivation is inner under certain conditions in the setting of hyperfinite von Neumann algebras. In [12], we prove a similar result for von Neumann algebras under some other conditions (see also [13]).

Throughout the paper, \mathfrak{A} and \mathfrak{B} denote unital C^* -algebras and $\mathfrak{M}, \mathfrak{N}$ denote von Neumann algebras acting on a Hilbert space \mathfrak{H} .

2. The Main Results

We now establish our first result for ρ -derivations on hyperfinite von Neumann algebras when ρ is an ultraweakly continuous $*$ -homomorphism. By a $*$ -mapping we mean a mapping preserving $*$. We start with the following theorem.

Theorem 2.1. Let \mathfrak{A} and \mathfrak{B} be two unital C^* -algebras, $\rho : \mathfrak{A} \rightarrow \mathfrak{B}$ be a strongly continuous and ultraweakly continuous $*$ -homomorphism with the extension $\bar{\rho}$ of the ultraweak closure $\bar{\mathfrak{A}}$ of \mathfrak{A} into the ultraweak closure $\bar{\mathfrak{B}}$ of \mathfrak{B} , and $\delta : \mathfrak{A} \rightarrow \mathfrak{B}$ be a ρ -derivation. Then δ is ultraweakly continuous and can be extended to a $\bar{\rho}$ -derivation $\bar{\delta} : \bar{\mathfrak{A}} \rightarrow \bar{\mathfrak{B}}$ in such a way that $\|\bar{\delta}\| = \|\delta\|$.

Proof. Let \mathfrak{A}_1 and \mathfrak{A}_1^+ denote the closed unit balls of \mathfrak{A} and the positive cone \mathfrak{A}^+ , respectively. Fix the vectors $\xi, \eta \in \mathfrak{H}$ and let $A \in \mathfrak{A}^+$ with the positive square root B . We have

$$\begin{aligned} |\langle \delta(A)\xi, \eta \rangle| &= |\langle \delta(B^2)\xi, \eta \rangle| \\ &= |\langle \delta(B)\rho(B)\xi, \eta \rangle| + |\langle \rho(B)\delta(B)\xi, \eta \rangle| \\ &= |\langle \delta(B)\rho(B)\xi, \eta \rangle| + |\langle \delta(B)\xi, \rho(B)\eta \rangle| \\ &\leq \|\delta\| \|B\| \|\rho(B)\xi\| \|\eta\| + \|\delta\| \|B\| \|\xi\| \|\rho(B)\eta\|. \end{aligned}$$

Hence

$$\|\rho(B)\xi\|^2 = \langle \rho(B)\rho(B)\xi, \xi \rangle = \langle \rho(A)\xi, \xi \rangle = \|\rho(A)\xi\| \|\xi\|$$

Thus

$$\begin{aligned} |\langle \delta(A)\xi, \eta \rangle| &\leq \|\delta\| \|A\|^{1/2} (\|\xi\|^{1/2} \|\eta\| \|\rho(A)\xi\|^{1/2} \\ &\quad + \|\eta\| \|\xi\|^{1/2} \|\rho(A)\eta\|^{1/2}). \end{aligned}$$

Hence $A \mapsto |\langle \delta(A)\xi, \eta \rangle|$ is continuous at 0 on \mathfrak{A}_1^+ in the relative strong topology. Using the standard reasoning as in the proof of Theorem 2.2.2 of [15] we deduce that δ is ultraweakly continuous on \mathfrak{A} . Hence a limit argument let us extend δ and ρ on $\overline{\mathfrak{A}}$ to ultraweakly continuous linear mappings $\bar{\delta}$ and $\bar{\rho}$, respectively. Since the multiplication is separately ultraweakly continuous we conclude that $\bar{\delta}$ is a $\bar{\rho}$ -derivation. ■

The next statement is a known fact about finite dimensional C^* -algebras; cf. Lemma 2.4.1 of [15].

Lemma 2.2. Let \mathfrak{M} be a finite dimensional von Neumann algebra. Then there exists a finite group \mathcal{G} of unitary elements whose span is \mathfrak{M} .

We are ready to show that every $*$ - ρ -derivation on a finite dimensional von Neumann algebra is inner.

Proposition 2.3. Let \mathfrak{M} be a finite dimensional von Neumann algebra with the unit I , \mathfrak{N} be a von Neumann algebra, $\rho : \mathfrak{M} \rightarrow \mathfrak{N}$ is a $*$ -homomorphism, $\delta : \mathfrak{M} \rightarrow \mathfrak{N}$ be a $*$ - ρ -derivation and $\delta(I)$ commutes with $\rho(I)$. Then there exists an element U in \mathfrak{N} with $\|U\| \leq \|\delta\|$ such that $\delta(A) = U\rho(A) - \rho(A)U$ for each $A \in \mathfrak{M}$.

Proof. First, we have

$$\begin{aligned} \delta(A)\rho(I) &= \delta(AI)\rho(I) \\ &= \delta(A)\rho(I)\rho(I) + \rho(A)\delta(I)\rho(I) \\ &= \delta(A)\rho(I) + \rho(A)\rho(I)\delta(I) \\ &= \delta(A)\rho(I) + \rho(A)\delta(I) \\ &= \delta(A), \end{aligned}$$

for all $A \in \mathfrak{M}$. In addition, $\delta(A) = \delta(A^*)^* = (\delta(A^*)\rho(I))^* = \rho(I)\delta(A)$, for all $A \in \mathfrak{M}$.

Next, let $\mathcal{G} = \{U_1, \dots, U_n\}$ be a group of unitaries spanning \mathfrak{M} . Set

$$U = -\frac{1}{n} \sum_{i=1}^n \rho(U_i^*) \delta(U_i).$$

Since $*$ -homomorphisms on C^* -algebras are norm decreasing we have $\|U\| \leq \|\delta\|$ and

$$\delta(U_i U_j) = \delta(U_i) \rho(U_j) + \rho(U_i) \delta(U_j).$$

Thus

$$\begin{aligned} \rho(U_j) (\rho(U_i U_j))^* \delta(U_i U_j) &= \rho(U_i^*) \rho(I) \delta(U_i U_j) \\ &= \rho(U_i^*) \delta(U_i U_j) \\ &= \rho(U_i^*) \delta(U_i) \rho(U_j) + \rho(U_i^*) \rho(U_i) \delta(U_j) \\ &= \rho(U_i^*) \delta(U_i) \rho(U_j) + \delta(U_j). \end{aligned}$$

Therefore

$$\rho(U_j) \sum_{i=1}^n (\rho(U_i U_j))^* \delta(U_i U_j) = \sum_{i=1}^n (\rho(U_i^*) \delta(U_i)) \rho(U_j) + n \delta(U_j).$$

Since $\{U_1 U_j, \dots, U_n U_j\} = \{U_1, \dots, U_n\}$ we have

$$\rho(U_j) (-nU) = (-nU) \rho(U_j) + n \delta(U_j).$$

This shows that $\delta(U_j) = U \rho(U_j) - \rho(U_j) U$ for all $j = 1, \dots, n$. Thus $\delta(A) = U \rho(A) - \rho(A) U$ for all $A \in \mathfrak{M}$. ■

The above proposition enables us to address the problem whether $*$ - ρ -derivations on hyperfinite von Neumann algebras are inner.

Theorem 2.4. Let \mathfrak{M} be a hyperfinite von Neumann algebra with the unit I , \mathfrak{N} be a von Neumann algebra, $\rho : \mathfrak{M} \rightarrow \mathfrak{N}$ be an ultraweakly continuous $*$ -homomorphism and $\delta : \mathfrak{M} \rightarrow \mathfrak{N}$ be a $*$ - ρ -derivation such that $\delta(I)$ commutes with $\rho(I)$. Then there is an element U in \mathfrak{N} with $\|U\| \leq \|\delta\|$ such that $\delta(A) = U \rho(A) - \rho(A) U$ for all $A \in \mathfrak{M}$.

Proof. Suppose $\{\mathfrak{M}_\lambda\}_{\lambda \in \Lambda}$ is an increasing net of finite dimensional subalgebras of \mathfrak{M} (containing I) such that $\cup_\lambda \mathfrak{M}_\lambda$ is ultraweakly dense in \mathfrak{M} . The ρ -derivation $\delta : \mathfrak{M} \rightarrow \mathfrak{N}$ induces a family of ρ -derivations $\delta_\lambda : \mathfrak{M}_\lambda \rightarrow \mathfrak{N}$ by restriction. By Proposition 2. there are elements $U_\lambda \in \mathfrak{N}$, with $\|U_\lambda\| \leq \|\delta\|$ such that

$$\delta_\lambda(A) = U_\lambda \rho(A) - \rho(A) U_\lambda, \quad A \in \mathfrak{M}_\lambda$$

Since the ball in \mathfrak{N} of radius $\|\delta\|$ is ultraweakly compact $\{U_\lambda\}_{\lambda \in \Lambda}$ has a co-final ultraweakly convergent subnet $\{U_\gamma\}_{\gamma \in \Gamma}$ with limit $U \in \mathfrak{N}$. Hence $\|U\| \leq \|\delta\|$. If $A \in \cup_\lambda \mathfrak{M}_\lambda$ then $\delta(A) = U \rho(A) - \rho(A) U$. It follows from Theorem (2.1) that δ is ultraweakly continuous. Therefore $\delta(A) = U \rho(A) - \rho(A) U$ for all $A \in \mathfrak{M}$. ■

Remark 2.5. For two arbitrary von Neumann algebras \mathfrak{M} and \mathfrak{N} , a positive linear mapping $\psi : \mathfrak{M} \rightarrow \mathfrak{N}$ is said to be normal if for every increasing filtering set $\mathcal{F} \subseteq \mathfrak{M}^+$ with

supremum $T \in \mathfrak{M}^+$, $\psi(\mathcal{F})$ has supremum $\psi(T)$. Theorem 2 of page 59 of [5] implies that each normal $*$ -homomorphism $\psi : \mathfrak{M} \rightarrow \mathfrak{N}$ is ultraweakly and ultrastrongly continuous and hence the restriction of ψ to bounded subsets of \mathfrak{M} is weakly and strongly continuous.

Corollary 2.6. Suppose that $\mathfrak{A} \subseteq \mathfrak{M} \subseteq \mathfrak{N}$ are von Neumann algebras, where \mathfrak{A} is hyperfinite, $\rho : \mathfrak{M} \rightarrow \mathfrak{N}$ is a ultraweakly continuous $*$ -homomorphism and $\delta : \mathfrak{M} \rightarrow \mathfrak{N}$ is a $*$ - ρ -derivation such that $\delta(I)$ commutes with $\rho(I)$. Then there is an inner $*$ - ρ -derivation δ_1 such that $\delta - \delta_1$ annihilates \mathfrak{A} .

Proof. By Theorem 2., for the restriction of δ to \mathfrak{A} there is an element $N \in \mathfrak{N}$ such that $\delta(A) = N\rho(A) - \rho(A)N$, $A \in \mathfrak{A}$. Defining $\delta_1 : \mathfrak{M} \rightarrow \mathfrak{N}$ by $\delta_1(A) = N\rho(A) - \rho(A)N$, $A \in \mathfrak{M}$, we conclude that $\delta - \delta_1$ annihilates \mathfrak{A} . ■

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A Novel Pseudo Random Bit Generator Based on Chaotic Standard Map and its Testing

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Abstract: During last one and half decade an interesting relationship between chaos and cryptography has been developed, according to which many properties of chaotic systems such as: ergodicity, sensitivity to initial conditions/system parameters, mixing property, deterministic dynamics and structural complexity can be considered analogous to the confusion, diffusion with small change in plaintext/secret key, diffusion with a small change within one block of the plaintext, deterministic pseudo randomness and algorithmic complexity properties of traditional cryptosystems. As a result of this close relationship, several chaos based cryptosystems have been put forward since 1990. In one of the stages of the development of chaotic stream ciphers, the application of discrete chaotic dynamical systems in pseudo random bit generation has been widely studied recently. In this communication, we propose a novel pseudo random bit generator (PRBG) based on two chaotic standard maps running side-by-side and starting from random independent initial conditions. The pseudo random bit sequence is generated by comparing the outputs of both the chaotic standard maps. We also present the detailed results of the statistical testing on generated bit sequences, done by using two statistical test suites: the NIST suite and DIEHARD suite, which are developed independently and considered the most stringent statistical test suites to detect the specific characteristics expected of truly random sequences.

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1. Introduction

New rapid developments in the telecommunication technologies especially the Internet and mobile networks have extended the domain of information transmission, which in turn

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present new challenges for protecting the information from unauthorized eavesdropping. It has intensified the research activities in the field of cryptography to fulfill the strong demand of new secure cryptographic techniques [1, 2].

Recently researchers from the nonlinear dynamics community have noticed an interesting relationship between chaos and cryptography. According to that many properties of chaotic systems such as: ergodicity, sensitivity to initial conditions/system parameters, mixing property, deterministic dynamics and structural complexity can be considered analogous to the confusion, diffusion with small change in plaintext/secret key, diffusion with a small change within one block of the plaintext, deterministic pseudo randomness and algorithmic complexity properties of traditional cryptosystems [3]. As a result of this close relationship several chaos based cryptosystems have been put forward since 1990 [4]. These chaos based cryptosystems can be broadly classified into two categories: analog and digital. Analog chaos based cryptosystems are based on the techniques of control [5, 6] and synchronization [5, 7, 8] of chaos. There are several ways through which analog chaos based cryptosystems can be realized such as: chaotic masking [9], chaotic modulation [10], chaotic switching [11], inverse system approach [12] etc. On the other hand in digital chaos based cryptosystems, chaotic discrete dynamical systems are implemented in finite computing precision. Again there are number of ways through which digital chaos based cryptosystems be realized: block ciphers based on forward and/or reverse iterations of chaotic maps [4, 13,14,15], block ciphers based on chaotic round functions [16], stream ciphers implementing chaos based pseudo random bit generators (PRBG) [17] etc.

The subject of the present manuscript is the generation of cryptographically secure pseudo random bit sequences, which can be further used in the development of fool-proof stream ciphers and its statistical testing. The very first, relatively unnoticed, idea of designing a pseudo-random number generator by making use of chaotic first order non-linear difference equation was proposed by Oishi and Inoue [18] in 1982 where they could construct a uniform random number generator with an arbitrary Kolmogorov entropy. After a long gap, in 1993 Lin and Chua [19] designed a pseudo random number generator by using a second-order digital filter and realized it on digital hardware. After 1993, a significant progress has been made in this direction. We refer the readers to our recent contribution [20] for the detail review of the existing work on random bit generation using chaotic systems.

In this paper, we propose a pseudo random bit generator (PRBG) based on two chaotic standard maps. Most of the existing pseudo random bit generators [18, 19] are based on a single chaotic system and there are known techniques in chaos theory to extract information about the chaotic systems from its trajectory, which makes such chaos based pseudo random bit generators insecure [21]. However the proposed pseudo random bit generator is based on two chaotic systems running side-by-side, which of course increases the complexity in the random bit generation and hence becomes difficult for an intruder to extract information about the chaotic system. In the next section, we briefly introduce the standard mapping, which is a basic building block of the proposed pseudo random

bit generator and its properties, which make it a suitable choice for the generation of random bit sequences.

2. The standard map

The origin of the well known and widely used standard map lies in the field of particle physics. The problem examined by Fermi [22], as an analogue to a possible cosmic ray acceleration mechanism in which charged particles are accelerated by collision with moving magnetic field structures, is that of a ball bouncing between a fixed and an oscillating wall. For every impact of the ball on the wall, the phase of the oscillation is chosen at random, the ball will get accelerated. The question was now that, if the ball would be also accelerated, when the wall oscillation is a periodic function of time. This problem was investigated by Ulam [23] who found that the particle motion appeared to be stochastic, but did not increase its energy. Other people [24-26] demonstrated that in case of smooth forcing functions, the phase plane shows three distinct regions with increasing ball velocity: (i) a low-velocity region where all fixed points of period -1 are unstable and thus leading to stochastic motion, (ii) an intermediate velocity region in which islands of stability around elliptic fixed points are embedded in a stochastic sea and (iii) a high velocity region in which bands of stochastic motion are separated from each other by regular orbits. As this problem of particle acceleration can be approximated by the simple mapping, it became a well-suited case to study the parameter regions of phase space and the corresponding KAM surfaces. The exact Ulam mapping for the motion of the ball bouncing between a fixed and an oscillating wall, where the velocity is defined by a saw tooth function, is given by a set of four exact difference equations. Under the area-preserving condition and if we allow the wall to add momentum to the ball according to its velocity (without a change in the position of the wall), this set of four difference equation be converted into a set of two difference equations:

$$u_{n+1} = |u_n + \sin \psi_n|, \quad (1)$$

$$\psi_{n+1} = \psi_n + \frac{2\pi M}{u_{n+1}}. \quad (2)$$

where u_n and ψ_n respectively, are directly related to the ball velocity and phase just before the n th impact.

The standard mapping is obtained by the linearization of the above set of difference equations in action space. The set of difference equations thus obtained are

$$X_{n+1} = X_n + K \sin Y_n \quad \text{mod } 2\pi, \quad (3)$$

$$Y_{n+1} = Y_n + X_{n+1} \quad \text{mod } 2\pi, \quad (4)$$

where X_n is the new action parameter, which has explicit dependence on u_n , Y_n is the new angle obtained by shifting the phase ψ_n by π and K is stochasticity parameter. The period-1 fixed point of the standard mapping can be easily obtained by requiring that

the phase (mod 2π) and the action are stationary i.e., $X_1 = 2\pi m$ ($m \in Z$) and $Y_1 = 0, \pi$. The fixed point $Y = 0$ is always stable whereas the fixed point $Y = \pi$ is stable for $K < 4$ and changes from an elliptic fixed point to a hyperbolic fixed point with the increase in K . For the detailed derivation of the standard mapping from the original Fermi problem and its stability analysis, we refer the readers to [27].

In Figure 1, we have shown how the chaotic region in the phase space increases with K . For $K = 0.5$ one can see the primary period-1 and period -2 orbits very clearly, only local stochasticity near the separatrix occurs. For $K = 1.0$ the KAM curve between the period-1 and period-2 islands has been destroyed and the chaos is global now, only the islands of stability remain. As the value of K increases the size of the islands of stability decreases. In a recent numerical experimentation [28], it has been shown that at very

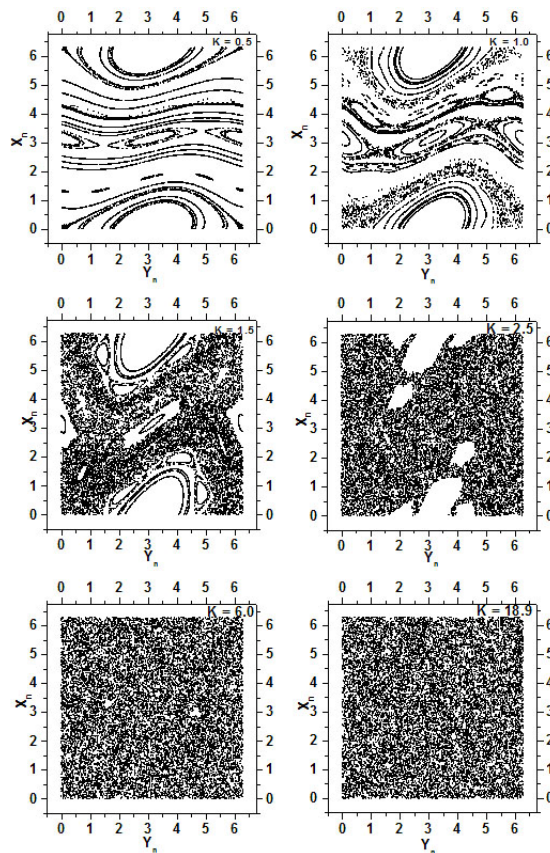


Fig. 1 Phase space for the standard map for $K = 0.5, 1.0, 1.5, 2.5, 6.0$ and 18.9 .

large values of the stochasticity parameter, there exist areas of regular motion- in fact there is a ‘creation and decay’ of the periodic orbits, which happens to repeat itself with a period of 2π .

3. The proposed PRBG

A *random bit generator (RBG)* is a device or algorithm, which outputs a sequence of statistically independent and unbiased binary digits. Such generator requires a naturally

occurring source of randomness (non-deterministic). In most practical environments designing a hardware device or software programme to exploit the natural source of randomness and produce a bit sequence free from biases and correlation is a difficult task. In such situations, the problem can be ameliorated by replacing a random bit generator with a pseudo random bit generator (PRBG).

A *pseudo random bit generator (PRBG)* is a deterministic algorithm, which uses a truly random binary sequence of length k as input called seed and produces a binary sequence of length $l \gg k$, called pseudo random sequence, which appears to be random. The output of a PRBG is not truly random; in fact the number of possible output sequences is at most a small fraction ($2^k/2^l$) of all possible binary sequences of length l . The basic intent is to take a small truly random sequence of length k and expand it to a sequence of much larger length l in such a way that an adversary can not efficiently distinguish between output sequence of PRBG and truly random sequence of length l [2].

In this paper, we are proposing a PRBG, which is based on two standard maps, starting from random independent initial conditions ($X_{1,0}, Y_{1,0}, X_{2,0}, Y_{2,0} \in [0, 2\pi]$)

$$X_{1,n+1} = X_{1,n} + K \sin Y_{1,n} \pmod{2\pi}, \quad (5)$$

$$Y_{1,n+1} = Y_{1,n} + X_{1,n+1} \pmod{2\pi}, \quad (6)$$

$$X_{2,n+1} = X_{2,n} + K \sin Y_{2,n} \pmod{2\pi}, \quad (7)$$

$$Y_{2,n+1} = Y_{2,n} + X_{2,n+1} \pmod{2\pi}, \quad (8)$$

The bit sequence is generated by comparing the outputs of both the standard maps in the following way:

$$h(X_{2,n+1}, Y_{1,n+1}) = \begin{cases} 1 & \text{if } X_{2,n+1} > Y_{1,n+1} \\ 0 & \text{if } X_{2,n+1} \leq Y_{1,n+1} \end{cases}, \quad (9)$$

The set of initial conditions ($X_{1,0}, Y_{1,0}, X_{2,0}, Y_{2,0} \in [0, 2\pi]$) serves as the seed for the PRBG, if we supply the exactly same seed to the PRBG, it will produce the same bit sequence due to the above deterministic procedure. The schematic block diagram of the proposed PRBG is shown in Figure 2.

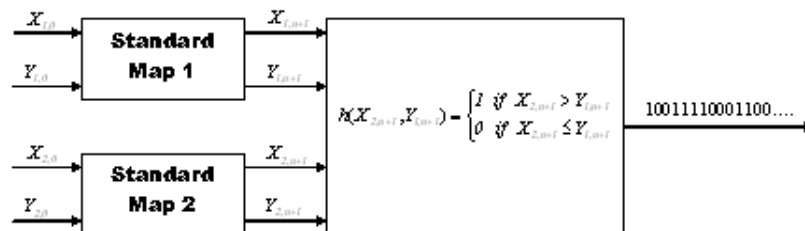


Fig. 2 Schematic block diagram of the proposed pseudo random bit generator (PRBG)

In the next section, we discuss the results of the statistical testing of the proposed PRBG using the NIST (National Institute of Standards and Technology, Gaithersberg, MD, USA) tests suite and DIEHARD tests suite, which are considered the most stringent statistical tests suites to test randomness.

4. Statistical Testing

In order to gain the confidence that newly developed pseudo random bit generators are cryptographically secure, they should be subjected to a variety of statistical tests designed to detect the specific characteristics expected of truly random sequences. There are several options available for analyzing the randomness of the newly developed pseudo random bit generators. The four most popular options are: NIST suite of statistical tests [29], the DIEHARD suite of statistical tests [30], The Crypt-XS suite of statistical tests and the Donald Knuth's statistical tests set. There are different number of statistical tests in each of the above mentioned test suites to detect distinct types of non-randomness in the binary sequences. Various efforts based on the principal component analysis show that not all the above mentioned suites are needed to implement at a time as there are redundancy in the statistical tests (i.e., all the tests are not independent).

In this communication, we use the NIST suite and DIEHARD suite to test the randomness of the bit sequences generated by the proposed pseudo random bit generator. In the following paragraph, we discuss the results of our analysis of the proposed pseudo random bit generator with the NIST and DIEHARD tests suites.

Testing strategy

The NIST tests and DIEHARD tests, like many statistical tests, are based on hypothesis testing. A hypothesis test is a procedure for determining if an assertion about a characteristic of a population is reasonable. In the present case, the test involves determining whether or not a specific sequence of zeroes and ones is random (it is called null hypothesis H_0).

For each test, a relevant randomness statistic be chosen and used to determine the acceptance or rejection of the null hypothesis. Under an assumption of randomness, such a statistic has a distribution of possible values. A theoretical reference distribution of this statistic under the null hypothesis is determined by mathematical methods and corresponding probability value (*P-value*) is computed, which summarizes the strength of the evidence against the null hypothesis. For each test, the *P-value* is the probability that a perfect random number generator would have produced a sequence less random than the sequence that was tested, given the kind of non-randomness assessed by the test. If a *P-value* for a test is determined to be equal to 1, then the sequence appears to have perfect randomness. A *P-value* equal to zero indicates that the sequence appears to be completely non-random. A significance level (α) be chosen for the tests and if *P-value* $\geq \alpha$, then the null hypothesis is accepted i.e., the sequence appears to be random. If *P-value* $< \alpha$, then the null hypothesis is rejected; i.e., the sequence appears to be non-random. Typically, the significance level (α) is chosen in the interval [0.001, 0.01]. The $\alpha = 0.01$ indicates

that one would expect 1 sequence out of 100 sequences to be rejected. A P -value ≥ 0.01 would mean that the sequence would be considered to be random with a confidence of 99%.

4.1 The NIST Suite:

The NIST tests suite is a statistical package comprising of 16 tests that are developed to test the randomness of (arbitrary long) binary sequences produced by either hardware or software based cryptographic random or pseudo random bit generators. These tests focus on variety of different types of non-randomness that could exist in a binary sequence. Broadly, we may classify these sixteen tests into two categories: (i) *Non-parameterized tests*: Frequency (monobit) test, Runs test, Test for longest run of ones in a block, Lempel-Ziv compression test, Binary matrix rank test, Cumulative sums test, Discrete Fourier transform (spectral) test, Random excursions test and Random excursions variant test and (ii) *Parameterized tests*: Frequency test within a block, Approximate entropy test, Linear complexity test, Maurer's universal statistical test, Serial test, Overlapping template matching test and Non-overlapping template matching test. For the detailed description of all 16 tests of NIST suite, we refer the readers to the NIST document [29].

For the numerical experimentations on the proposed pseudo random bit generator, we have generated 100 (sample size $m = 100$) different binary sequences (each sequence has been generated from a randomly chosen seed $(X_{1,0}, Y_{1,0}, X_{2,0}, Y_{2,0} \in [0, 2\pi])$, $K_1 = 150.72$, $K_2 = 210.37$) each of length 10^6 bits and computed the P -value corresponding to each sequence for all the 16 tests of NIST Suite (in all we have computed total $50 \times 100 = 5000$ P -values). All the computations have been performed in the double precision floating point representation. We refer the readers to Rukhin et al [29] for the detailed mathematical procedure for calculating the P -value for each individual test of NIST suite. For the analysis of P -values obtained from various statistical tests, we have fixed the significance level at $\alpha = 0.01$. In Tables 1 and 2 respectively, we have summarized the results obtained after implementing non-parameterized and parameterized tests of NIST suite on the binary sequences produced by the proposed pseudo random bit generator.

Interpretation of results

(i) *Uniform distribution of P-values*: For each test, the distribution of P -values for a large number of binary sequences ($m = 100$) has been examined. The uniformity of the P -values has been examined quantitatively via an application of χ^2 test and the determination of a P -value corresponding to the Goodness-of-Fit distributional test on the P -values obtained for each statistical test (i.e., a P -value of the P -values, which is denoted by $P - value_T$). The computation is as follows:

$$\chi^2 = \sum_{i=1}^{10} \frac{\left(f_i - \frac{m}{10}\right)^2}{\frac{m}{10}}, \quad (10)$$

where f_i is the number of P -values in the sub-interval i and m is the size of the sample, which is $m = 100$ for the present analysis. The P -value of the P -values (i.e., $P - value_T$)

is obtained from the χ^2 by using

$$P - value_T = igamc\left(\frac{9}{2}, \frac{\chi^2}{2}\right), \quad (11)$$

where $igamc(\)$ is the incomplete Gamma function.

If $P - value_T \geq 0.0001$ then the P-values are considered to be uniformly distributed. The computed $P - value_T$ corresponding to each statistical test has been given in Tables 1 and 2. In Figures 3(a) and (b) respectively, we have graphically depicted the computed $P - value_T$ for each non-parameterized and parameterized test along with the threshold value (0.0001).

(ii) *Proportions of the sequences passing the tests:* We have calculated the proportion of the sequences passing a particular statistical test and compared it with the range of acceptable proportion. The range of acceptable proportion is determined by using the confidence interval given by

$$\hat{p} \pm 3\sqrt{\frac{\hat{p}(1 - \hat{p})}{m}}, \quad (12)$$

where m is the sample size and $\hat{p} = 1 - \alpha$, which are $m = 100$ and $\hat{p} = 1 - 0.01 = 0.99$ for the present analysis. So the range of acceptable proportion is [1.01985, 0.96015].

The quantitative results of proportions are given the Tables 1 and 2 respectively for various non-parameterized and parameterized statistical tests of NIST suite. In Figures 4(a) and (b) respectively, we have graphically depicted the computed proportions for each non-parameterized and parameterized test along with the confidence interval i.e., [1.01985, 0.96015].

It is clear that the computed proportion for each test lies inside the confidence interval; hence the tested binary sequences generated by the proposed PRBG are random with respect to all the 16 tests of NIST suite.

4.2 The DIEHARD Suite

The DIEHARD suite is a battery of statistical tests for measuring the quality of a set of random numbers. They are developed by George Marsaglia and first published in 1995. The tests are: Birthday spacings, Overlapping permutations, Ranks of matrices, Monkey tests, Count the 1s, Parking lot test, Minimum distance test, Random spheres test, The squeeze test, Overlapping sums test, Runs test, and The craps test. For the detailed description of all tests in the DIEHARD suite, we refer the readers to [30].

The tests available in the DIEHARD suite return a p-values, which should be uniform on [0,1] if the input file contains truly independent random bits. These p-values are obtained by $p = F(X)$, where F is the assumed distribution of the sample random variable X , often normal. But the assumed F is just an asymptotic approximation, for which the fit will be worst in the tails. Thus we should not be surprised with occasional p-values near 0 or 1, such as .0011 or .9981. When a bit stream really *FAILS BIG*, we will get p's of 0 or 1 to six or more places. By all means, do not, as a statistician might, think

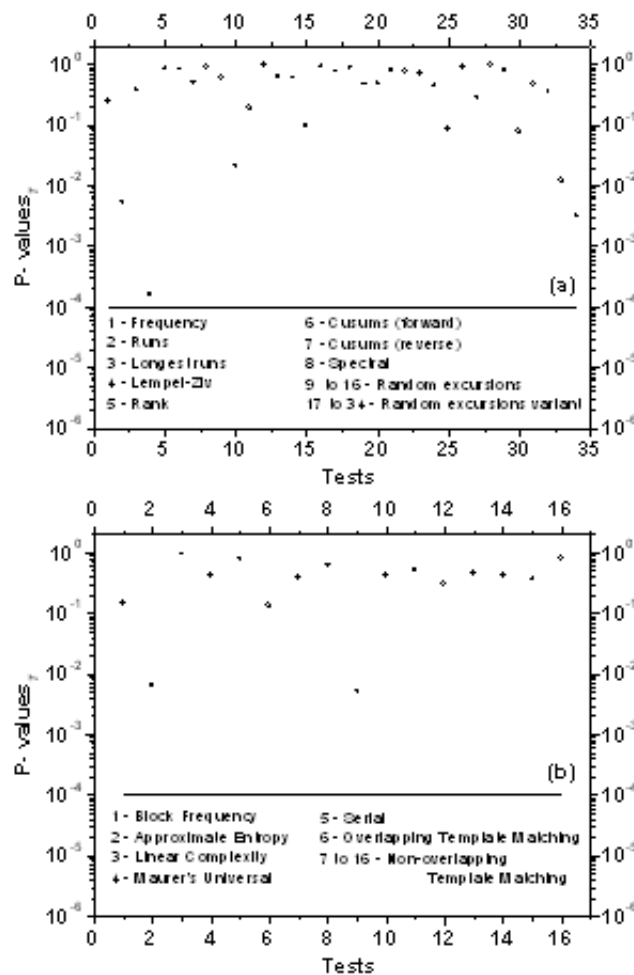


Fig. 3 $P - value_T$ (i.e., P -value of the P -values) for (a) non-parameterized tests and (b) parameterized tests of NIST suite. The horizontal line represents the threshold value of $P - value_T$.

that a $p < .025$ or $p > .975$ means that the RNG has "failed the test at the significance level $\alpha = .05$ ". Such p 's happen among the hundreds that DIEHARD produces, even with good RNG's [30].

For the numerical experimentations on the proposed pseudo random bit generator, we have generated an input file for the DIEHARD testing consists of 30,00,000 32-bit integers produced using the proposed random bit generator with randomly chosen seed ($X_{1,0}, Y_{1,0}, X_{2,0}, Y_{2,0} \in [0, 2\pi]$, $K_1 = 150.72$, $K_2 = 210.37$) (an ASCII file in hex form, 8 hex 'digits' per integer, 10 integers per line with no intervening spaces). Then the P -values corresponding to the various statistical tests of DIEHARD suite are computed (in all it computes 234 P -values). The details of the computed P -values are given in Table 3.

It is clear from the results of DIEHARD testing given in Table 3 that the random bit sequences produced using the proposed pseudo random bit generator have the characteristics expected of truly random sequences.

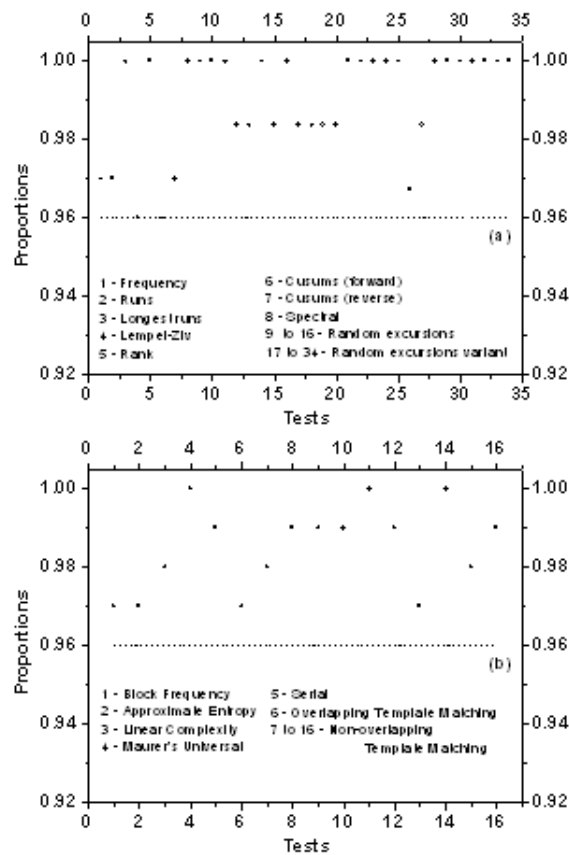


Fig. 4 Proportions of the sequences passing the tests for (a) non-parameterized and (b) parameterized tests of NIST suite. The region between two horizontal dashed lines is the acceptable range of proportion.

Conclusion

We have proposed a design of a pseudo random bit generator (PRBG) based on two chaotic standard maps iterated independently starting from independent initial conditions. The pseudo random bit sequence is obtained by comparing the outputs of both the chaotic maps. We have also tested rigorously the generated sequences using the NIST suite and DIEHARD suite, which are considered the most stringent statistical tests suites to detect the specific characteristics expected of truly random sequences. The results of statistical testing are encouraging and show that the proposed PRBG has perfect cryptographic properties and hence can be used in the design of new stream ciphers.

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Table 1: Non-parameterized tests results

<ul style="list-style-type: none"> • Number of binary sequences tested (m): 100 • Length of each binary sequence: 1,000,000 bits • Significance level (α) = 0.01 • The range of acceptable proportion is 0.99 ± 0.02985 • Null hypothesis (H_0): The binary sequence is random • If $P - value \geq \alpha$ (0.01) then the null hypothesis (H_0) is accepted. • If $P - value < \alpha$ (0.01) then the null hypothesis (H_0) is rejected. • If $P - value_T$ ($P - value$ corresponding to the Goodness-of-Fit distributional test on the $P - values$ obtained for a particular test i.e., a $P - value$ of the $P - values$) ≥ 0.0001 then $P - values$ can be considered uniformly distributed. 			
S. No.	Statistical Test	$P - value$ corresponding to the goodness of fit ($P - value_T$)	Proportion of sequences passing the test
1.	Frequency (monobit) test	0.249284	0.9700
2.	Runs test	0.005428	0.9700
3.	Test for longest run of ones in a block	0.383827	1.0000
4.	Lempel-Ziv compression test	0.000157	0.9600
5.	Binary matrix rank test	0.867692	1.0000
6.	Cumulative sums test		
	1) Forward sums test	0.851383	0.9600
	2) Reverse sums test	0.514124	0.9700
7.	Discrete Fourier transform (spectral) test	0.897763	1.0000
8.	Random excursions test		
	1) $x = -4$	0.585209	1.0000
	2) $x = -3$	0.021262	1.0000
	3) $x = -2$	0.186566	1.0000
	4) $x = -1$	0.997147	0.9836
	5) $x = 1$	0.619772	0.9836
	6) $x = 2$	0.619772	1.0000
	7) $x = 3$	0.095617	0.9836
	8) $x = 4$	0.941144	1.0000

Table 1: continued

Random excursions variant test			
9.	1) $x = -9$	0.788728	0.9836
	2) $x = -8$	0.900104	0.9836
	2) $x = -7$	0.484646	0.9836
	3) $x = -6$	0.484646	0.9836
	4) $x = -5$	0.788728	1.0000
	5) $x = -4$	0.756476	1.0000
	7) $x = -3$	0.723129	1.0000
	8) $x = -2$	0.452799	1.0000
	9) $x = -1$	0.086458	1.0000
	10) $x = 1$	0.900104	0.9672
	11) $x = 2$	0.287306	0.9836
	12) $x = 3$	0.957319	1.0000
	13) $x = 4$	0.788728	1.0000
	14) $x = 5$	0.078086	1.0000
	15) $x = 6$	0.452799	1.0000
	16) $x = 7$	0.364146	1.0000
	17) $x = 8$	0.011931	1.0000
	18) $x = 9$	0.003161	1.0000

Table 2: Parameterized tests results

<ul style="list-style-type: none"> • Number of binary sequences tested (m): 100 • Length of each binary sequence: 1,000,000 bits • Significance level (α) = 0.01 • The range of acceptable proportion is 0.99 ± 0.02985 • Null hypothesis (H_0): The binary sequence is random • If $P - value \geq \alpha$ (0.01) then the null hypothesis (H_0) is accepted. • If $P - value < \alpha$ (0.01) then the null hypothesis (H_0) is rejected. • If $P - value_T$ ($P - value$ corresponding to the Goodness-of-Fit distributional test on the $P - values$ obtained for a particular test i.e., a $P - value$ of the $P - values$) ≥ 0.0001 then $P - values$ can be considered uniformly distributed. 			
S. No.	Statistical Test	$P - value$ corresponding to the goodness of fit ($P - value_T$)	Proportion of sequences passing the test
1.	Frequency test within a block (Block length = 128)	0.145326	0.9700
2.	Approximate entropy test (Block length = 10)	0.006265	0.9700
3.	Linear complexity test (Block length = 500)	0.971699	0.9800
4.	Maurer's universal statistical test (No. of blocks = 7, Block length = 1280)	0.419021	1.0000
5.	Serial test (Block length = 16)	0.798139	0.9900
6.	Overlapping template matching test (Template = 111111111)	0.129620	0.9700

Table 2: Continued

7.	Non-overlapping template matching test (Template length=9)		
	9) Template = 000000001	0.383827	0.9800
	10) Template = 000100111	0.637119	0.9900
	11) Template = 001010011	0.005358	0.9900
	12) Template = 010001011	0.419021	0.9900
	13) Template = 011101111	0.514124	1.0000
	14) Template = 101101000	0.304126	0.9900
	15) Template = 110101000	0.455937	0.9700
	16) Template = 111000010	0.437274	1.0000
	17) Template = 111100000	0.366918	0.9800
	18) Template = 111111110	0.779188	0.9900

Table 3: DIEHARD tests results

Test name & corresponding p-values		No. of p-values
1	Birthday spacing test	9
	0.988763 0.219153 0.144182 0.760248 0.187449 0.512483 0.910415 0.923849 0.094988	
2	The overlapping permutations test	2
	0.973546 0.980946	
3	Binary rank test	27
	(i) 31x31 0.901002	
	(ii) 32x32 0.855994	
	(iii) 6x8 0.064125 0.302897 0.941658 0.426312 0.766173 0.715484 0.838059 0.080130 0.195773 0.974842 0.249858 0.984198 0.971825 0.275149 0.517412 0.885706 0.052722 0.026866 0.022423 0.628286 0.244628 0.990259 0.437097 0.100355 0.918104	
4	The monkey tests	102
	(i) The bitstream test 0.82420 0.87322 0.77173 0.76389 0.99775 0.82420 0.27074 0.66439 0.95206 0.87563 0.82958 0.99766 0.91765 0.76604 0.94620 0.80225 0.73796 0.23206 0.82055 0.99361	
	(ii) Overlapping-Pairs-Sparse-Occupancy (OPSO) 0.9996 0.9629 0.8813 0.9645 0.9572 0.9701 0.9950 0.3366 0.9742 0.8976 0.9723 0.9961 0.8411 0.9892 0.9877 0.9999 0.9985 0.9990 0.5599 0.9918 0.9849 0.6664 0.3518	
	(iii) Overlapping-Quadruples-Sparse-Occupancy (OQSO) 0.9548 0.4820 0.9170 0.8539 0.3184 0.9484 0.9724 0.4111 0.9211 0.9812 0.9371 0.9392 0.9175 0.9621 0.6748 0.7697 0.9236 0.4833 0.8615 0.9264 0.9924 0.9747 0.9977 0.9817 0.8097 0.9862 0.6893 0.6649	
	(iv) DNA Test 0.1918 0.2893 0.2186 0.4491 0.6801 0.3066 0.9841 0.2773 0.6620 0.4363 0.9588 0.3915 0.9646 0.4890 0.9600 0.7328 0.4843 0.8404 0.3769 0.6588 0.0778 0.3139 0.8361 0.5055 0.9282 0.9290 0.2588 0.9566 0.2743 0.1815 0.8883	

Table 3: Continued

5	Count the 1's test		27
	(i) in stream of bytes	0.821983 0.888099	
	(ii) for specific bytes	0.295819 0.663301 0.931640 0.030505 0.718611 0.237751 0.309827 0.297175 0.455678 0.956925 0.140010 0.120595 0.383329 0.339572 0.356567 0.592706 0.885688 0.445925 0.724707 0.770896 0.271070 0.781559 0.888849 0.234706 0.502525	
6	Parking lot test	0.692266 0.481790 0.037471 0.554479 0.781201 0.041356 0.108811 0.117571 0.291865 0.572463	10
7	The minimum distance test	0.849829 0.932613 0.815004 0.036007 0.323248 0.376892 0.899944 0.614205 0.553456 0.132279 0.258550 0.035251 0.804337 0.992632 0.654338 0.332773 0.668850 0.043199 0.808551 0.661693	20
8	The 3D spheres test	0.77778 0.47420 0.25062 0.76921 0.48096 0.37490 0.82961 0.92404 0.62866 0.11081 0.36001 0.30289 0.01981 0.00226 0.92195 0.40129 0.78319 0.81086 0.08832 0.64228	20
9	The squeeze test	0.224066	1
10	The overlapping sums test	0.602870 0.979228 0.159121 0.588719 0.942497 0.290170 0.687853 0.083611 0.234358 0.446009	10
11	The runs test		4
	(i) Up	0.529492 0.258671	
	(ii) Down	0.281869 0.089151	
12	The craps test	0.251346 0.339478	2

Heisenberg Hamiltonian with Second Order Perturbation for Spinel Ferrite Ultra-thin Films

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Abstract: The solution of Heisenberg Hamiltonian with second order perturbation will be described for non-oriented spinel cubic ferrimagnetic materials. The perturbation related to the change of angle at the interface of two cells will be considered. The energy peaks become sharper and peak position varies in energy-angle curve as N is increased from 2 to 3. But the separation between two consecutive major maximums remains same. The 3-D plot of total energy versus angle and stress becomes smoother as N is increased from 2 to 3. The energy decreases with number of layers indicating that the behavior of oriented and non-oriented films is different. In N=2 case, minor maximums next to major maximum can be observed. When second order anisotropy constant does not vary within the film with N=2, film behaves as an oriented film.

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Keywords: Heisenberg Hamiltonian; Ferrimagnetic Materials; Spinel Ferrites; Spin Models; Magnetic Anisotropy

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1. Introduction

In some early reports, the structure of spinel ferrites with the position of octahedral and tetrahedral sites is given in detail [1-5]. Only the occupied octahedral and tetrahedral sites were used for the calculation in this report, although there are many filled and vacant octahedral and tetrahedral sites in cubic spinel cell [1]. Only few previous reports could be found on the theoretical works of ferrites [6-9]. The solution of Heisenberg ferrites only with spin exchange interaction term has been found earlier by means of the retarded Green function equations [6].

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All the relevant energy terms such as spin exchange energy, dipole energy, second and fourth order anisotropy terms, interaction with magnetic field and stress induced anisotropy in Heisenberg Hamiltonian were taken into consideration. These equations derived here can be applied for spinel ferrites with unit cell AFe_2O_4 such as Fe_3O_4 , $NiFe_2O_4$ and $ZnFe_2O_4$ only. The spin exchange interaction energy and dipole interaction have been calculated only between two nearest spin layers and within same spin plane. Also the azimuthal angle of spins within one cubic cell is assumed to be constant. The change of angle at the interface of cubic cell will be considered. According to some of our early experimental reports, the anisotropy energy of Nickel ferrite and Lithium mixed ferrite depends on the stress of the film induced during cooling or heating process of the film [15, 16].

2. The Model

Classical Heisenberg Hamiltonian of a thin film can be written as following.

$$H = -J \sum_{m,n} \vec{S}_m \cdot \vec{S}_n + \omega \sum_{m \neq n} \left(\frac{\vec{S}_m \cdot \vec{S}_n}{r_{mn}^3} - \frac{3(\vec{S}_m \cdot \vec{r}_{mn})(\vec{r}_{mn} \cdot \vec{S}_n)}{r_{mn}^5} \right) - \sum_m D_{\lambda m}^{(2)} (S_m^z)^2 - \sum_m D_{\lambda m}^{(4)} (S_m^z)^4 - \sum_m \vec{H} \cdot \vec{S}_m - \sum_m K_s \sin 2\theta_m \quad (1)$$

Here J , ω , θ , $D_m^{(2)}$, $D_m^{(4)}$, H_{in} , H_{out} , K_s , m , n and N are spin exchange interaction, strength of long range dipole interaction, azimuthal angle of spin, second and fourth order anisotropy constants, in plane and out of plane applied magnetic fields, stress induced anisotropy constant, spin plane indices and total number of layers in film, respectively. When the stress applies normal to the film plane, the angle between m^{th} spin and the stress is θ_m .

The cubic cell was divided into 8 spin layers with alternative A and Fe spins layers [1]. The spins of A and Fe will be taken as 1 and p , respectively. While the spins in one layer point in one direction, spins in adjacent layers point in opposite directions. A thin film with (001) spinel cubic cell orientation will be considered. The length of one side of unit cell will be taken as “a”. Within the cell the spins orient in one direction due to the super exchange interaction between spins (or magnetic moments). Therefore the results proven for oriented case in one of our early report [11] will be used for following equations. But the angle θ will vary from θ_m to θ_{m+1} at the interface between two cells.

For a thin film with thickness Na ,

Spin exchange interaction energy=

$$E_{exchange} = N(-10J + 72Jp - 22Jp^2) + 8Jp \sum_{m=1}^{N-1} \cos(\theta_{m+1} - \theta_m)$$

Dipole interaction energy= E_{dipole}

$$E_{dipole} = -48.415\omega \sum_{m=1}^N (1 + 3 \cos 2\theta_m) + 20.41\omega p \sum_{m=1}^{N-1} [\cos(\theta_{m+1} - \theta_m) + 3 \cos(\theta_{m+1} + \theta_m)]$$

Here the first and second term in each above equation represent the variation of energy within the cell [11] and the interface of the cell, respectively.

Total energy

$$\begin{aligned}
 E = & N(-10J + 72Jp - 22Jp^2) + 8Jp \sum_{m=1}^{N-1} \cos(\theta_{m+1} - \theta_m) \\
 & - 48.415\omega \sum_{m=1}^N (1 + 3 \cos 2\theta_m) + 20.41\omega p \sum_{m=1}^{N-1} [\cos(\theta_{m+1} - \theta_m) + 3 \cos(\theta_{m+1} + \theta_m)] \\
 & - \sum_{m=1}^N [D_m^{(2)} \cos^2 \theta_m + D_m^{(4)} \cos^4 \theta_m] - 4(1-p) \sum_{m=1}^N [H_{in} \sin \theta_m + H_{out} \cos \theta_m + K_s \sin 2\theta_m]
 \end{aligned} \tag{2}$$

Here the anisotropy energy term and the last term have been explained in our previous report for oriented spinel ferrite [11]. If the angle is given by $\theta_m = \theta + \varepsilon_m$ with perturbation ε_m , after taking the terms up to second order perturbation of ε only,

The total energy can be given as

$$E(\theta) = E_0 + E(\varepsilon) + E(\varepsilon^2)$$

Here

$$\begin{aligned}
 E_0 = & -10JN + 72pNJ - 22Jp^2N + 8Jp(N-1) - 48.415\omega N - 145.245\omega N \cos(2\theta) \\
 & + 20.41\omega p [(N-1) + 3(N-1)\cos(2\theta)] \\
 & - \cos^2 \theta \sum_{m=1}^N D_m^{(2)} - \cos^4 \theta \sum_{m=1}^N D_m^{(4)} - 4(1-p)N(H_{in} \sin \theta + H_{out} \cos \theta + K_s \sin 2\theta)
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 E(\varepsilon) = & 290.5\omega \sin(2\theta) \sum_{m=1}^N \varepsilon_m - 61.23\omega p \sin(2\theta) \sum_{m=1}^{N-1} (\varepsilon_m + \varepsilon_{m+1}) \\
 & + \sin 2\theta \sum_{m=1}^N D_m^{(2)} \varepsilon_m + 2 \cos^2 \theta \sin 2\theta \sum_{m=1}^N D_m^{(4)} \varepsilon_m \\
 & + 4(1-p)[-H_{in} \cos \theta \sum_{m=1}^N \varepsilon_m + H_{out} \sin \theta \sum_{m=1}^N \varepsilon_m - 2K_s \cos 2\theta \sum_{m=1}^N \varepsilon_m]
 \end{aligned} \tag{4}$$

$$\begin{aligned}
E(\varepsilon^2) = & -4Jp \sum_{m=1}^{N-1} (\varepsilon_n - \varepsilon_m)^2 + 290.5\omega \cos(2\theta) \sum_{m=1}^N \varepsilon_m^2 - 10.2\omega p \sum_{m=1}^{N-1} (\varepsilon_n - \varepsilon_m)^2 \\
& - 30.6\omega p \cos(2\theta) \sum_{m=1}^{N-1} (\varepsilon_n + \varepsilon_m)^2 \\
& - (\sin^2 \theta - \cos^2 \theta) \sum_{m=1}^N D_m^{(2)} \varepsilon_m^2 + 2 \cos^2 \theta (\cos^2 \theta - 3 \sin^2 \theta) \sum_{m=1}^N D_m^{(4)} \varepsilon_m^2 \\
& + 4(1-p) \left[\frac{H_{in}}{2} \sin \theta \sum_{m=1}^N \varepsilon_m^2 + \frac{H_{out}}{2} \cos \theta \sum_{m=1}^N \varepsilon_m^2 + 2K_s \sin 2\theta \sum_{m=1}^N \varepsilon_m^2 \right] \quad (5)
\end{aligned}$$

The sin and cosine terms in equation number 2 have been expanded to obtain above equations. Here $n=m+1$.

Under the constraint $\sum_{m=1}^N \varepsilon_m = 0$, first and last three terms of equation 4 are zero.

Therefore, $E(\varepsilon) = \vec{\alpha} \cdot \vec{\varepsilon}$

Here $\vec{\alpha}(\varepsilon) = \vec{B}(\theta) \sin 2\theta$ are the terms of matrices with

$$B_\lambda(\theta) = -122.46\omega p + D_\lambda^{(2)} + 2D_\lambda^{(4)} \cos^2 \theta \quad (6)$$

Also $E(\varepsilon^2) = \frac{1}{2} \vec{\varepsilon} \cdot C \cdot \vec{\varepsilon}$, and matrix C is assumed to be symmetric ($C_{mn} = C_{nm}$).

Here the elements of matrix C can be given as following,

$$C_{m,m+1} = 8Jp + 20.4\omega p - 61.2p\omega \cos(2\theta)$$

For $m=1$ and N ,

$$\begin{aligned}
C_{mm} = & -8Jp - 20.4\omega p - 61.2p\omega \cos(2\theta) + 581\omega \cos(2\theta) - 2(\sin^2 \theta - \cos^2 \theta) D_m^{(2)} \\
& + 4 \cos^2 \theta (\cos^2 \theta - 3 \sin^2 \theta) D_m^{(4)} + 4(1-p) [H_{in} \sin \theta + H_{out} \cos \theta + 4K_s \sin(2\theta)] \quad (7)
\end{aligned}$$

For $m=2, 3, \dots, N-1$

$$\begin{aligned}
C_{mm} = & -16Jp - 40.8\omega p - 122.4p\omega \cos(2\theta) + 581\omega \cos(2\theta) - 2(\sin^2 \theta - \cos^2 \theta) D_m^{(2)} \\
& + 4 \cos^2 \theta (\cos^2 \theta - 3 \sin^2 \theta) D_m^{(4)} + 4(1-p) [H_{in} \sin \theta + H_{out} \cos \theta + 4K_s \sin(2\theta)]
\end{aligned}$$

Otherwise, $C_{mn}=0$

Therefore, the total energy can be given as

$$E(\theta) = E_0 + \vec{\alpha} \cdot \vec{\varepsilon} + \frac{1}{2} \vec{\varepsilon} \cdot C \cdot \vec{\varepsilon} = E_0 - \frac{1}{2} \vec{\alpha} \cdot C^+ \cdot \vec{\alpha} \quad (8)$$

Here C^+ is the pseudo-inverse given by

$$C \cdot C^+ = 1 - \frac{E}{N} \quad (9)$$

Here E is the matrix with all elements $E_{mn}=1$.

3. Results and Discussion

First a film with $N=2$ will be considered. When anisotropy constants $D_m^{(2)}$ and $D_m^{(4)}$ do not vary within the film $\alpha_1 = \alpha_2$ from equation 6.

Then from equation 7, $C_{11}=C_{22}$ and $C_{12}=C_{21}$.

From equation 9, $C_{12}^+ = C_{21}^+ = \frac{1}{2(C_{21}-C_{22})} = -C_{11}^+ = -C_{22}^+$

Therefore, $\vec{\alpha} \cdot C^+ \cdot \vec{\alpha} = (\alpha_1 - \alpha_2)^2 C_{11}^+ = 0$

Finally the energy $E(\theta)=E_0$ and film behaves as an oriented film. The exactly same result was obtained for thin ferromagnetic films earlier [12].

If the anisotropy constants vary within the film, then $C_{12}=C_{21}$ and $C_{11} \neq C_{22}$.

Then $C_{11}^+ = -C_{12}^+ = \frac{C_{22}+C_{21}}{2(C_{11}C_{22}-C_{21}^2)}$ and $C_{21}^+ = -C_{22}^+ = \frac{C_{21}+C_{11}}{2(C_{21}^2-C_{11}C_{22})}$.

Hence, $\vec{\alpha} \cdot C^+ \cdot \vec{\alpha} = (\alpha_1 - \alpha_2)(C_{21}^+ \alpha_2 - C_{12}^+ \alpha_1)$

When all the terms in equation 7 are taken into account, the C_{mn}^+ is contained more than 80 terms. Therefore only the spin exchange interaction, spin dipole interaction, second order anisotropy and stress induced anisotropy have been considered to avoid tedious calculations.

Then

$$C_{11} = -8Jp - 20.4\omega p - 61.2p\omega \cos(2\theta) + 581\omega \cos(2\theta) + 2(\cos 2\theta)D_1^{(2)} + 16(1-p)K_s \sin(2\theta)$$

$$C_{22} = -8Jp - 20.4\omega p - 61.2p\omega \cos(2\theta) + 581\omega \cos(2\theta) + 2(\cos 2\theta)D_2^{(2)} + 16(1-p)K_s \sin(2\theta)$$

$$C_{12} = 8Jp + 20.4\omega p - 61.2p\omega \cos(2\theta)$$

$$\alpha_1 = [-122.46\omega p + D_1^{(2)}] \sin(2\theta)$$

$$\alpha_2 = [-122.46\omega p + D_2^{(2)}] \sin(2\theta)$$

$$E(\theta) = E_0 - \frac{(\alpha_1 - \alpha_2)(C_{21}^+ \alpha_2 - C_{12}^+ \alpha_1)}{2}$$

$$E_0 = -20J + 144pJ - 44Jp^2 + 8Jp - 96.83\omega - 290.5\omega \cos(2\theta) + 20.41\omega p[1 + 3\cos(2\theta)]$$

$$- \cos^2 \theta [D_1^{(2)} + D_2^{(2)}] - 4(1-p)NK_s \sin(2\theta)$$

This simulation will be performed for Nickel ferrite with $p=2.5$.

Then

$$C_{11} = -20J - 51\omega + 428\omega \cos(2\theta) + 2(\cos 2\theta)D_1^{(2)} - 24K_s \sin(2\theta)$$

$$C_{22} = -20J - 51\omega + 428\omega \cos(2\theta) + 2(\cos 2\theta)D_2^{(2)} - 24K_s \sin(2\theta)$$

$$C_{12} = 20J + 51\omega - 153\omega \cos(2\theta)$$

$$E_0 = 85J - 96.83\omega - 290.5\omega \cos(2\theta) + 51.03\omega[1 + 3\cos(2\theta)] - \cos^2 \theta [D_1^{(2)} + D_2^{(2)}]$$

$$+ 12K_s \sin(2\theta)$$

Then $E(\theta)$ can be found using equation 8. When $\theta = 0^\circ$ and $\theta = 90^\circ$, the second order perturbation energy term is zero and film behaves as an oriented film. The graph between $\frac{E(\theta)}{\omega}$ and θ is given in figure 1, for $\frac{J}{\omega} = \frac{D_1^{(2)}}{\omega} = \frac{K_s}{\omega} = 10$, $\frac{D_2^{(2)}}{\omega} = 5$. Nearest maximum and minimum can be observed at 63° and 86° , respectively. Therefore the angle between easy and hard directions is not 90° in this case. Two consecutive maximums can be observed at 63° and 240.7° .

When $\frac{K_s}{\omega}$ is a variable, the 3-D plot of $\frac{E(\theta)}{\omega}$ versus θ and $\frac{K_s}{\omega}$ is given in figure 2. This graph shows a variation similar to oriented spinel ferrite films [11]. The graph indicates several minimums indicating that film can be easily oriented in some particular directions by applying a stress.

When $N=3$, the each C_{nm}^+ element found using equation 9 is contained more than 20 terms. To avoid this problem, matrix elements were found using $C \cdot C^+ = 1$. Then C_{mn}^+ is given by $C_{mn}^+ = \frac{\text{cofactor}C_{nm}}{\det C}$. Under this condition, $\vec{E} \cdot \vec{\alpha} = 0$, and the average value of first order perturbation is zero [12]. The second order anisotropy constant is assumed to be an invariant for the convenience.

Then $C_{11}=C_{33}$, $C_{12}=C_{21}=C_{23}=C_{32}$, $C_{13}=C_{31}=0$, $\alpha_1 = \alpha_2 = \alpha_3$.

$$C_{11}=C_{33} = -20J-51\omega+428\omega\cos(2\theta) + 2(\cos 2\theta)D_m^{(2)}-24K_s\sin(2\theta)$$

$$C_{22} = -40J-102\omega + 275\omega\cos(2\theta) + 2(\cos 2\theta)D_m^{(2)}-24K_s\sin(2\theta)$$

$$\text{Therefore}^{12}, C_{11}^+ = \frac{C_{11}C_{22}-C_{32}^2}{C_{11}^2C_{22}-2C_{32}^2C_{11}} = C_{33}^+, C_{13}^+ = \frac{C_{32}^2}{C_{11}^2C_{22}-2C_{32}^2C_{11}} = C_{31}^+$$

$$C_{12}^+ = \frac{-C_{32}C_{11}}{C_{11}^2C_{22} - 2C_{32}^2C_{11}} = C_{21}^+ = C_{23}^+ = C_{32}^+, C_{22}^+ = \frac{C_{11}^2}{C_{11}^2C_{22} - 2C_{32}^2C_{11}}$$

The total energy can be found using following equation.

$$E(\theta) = E_0 - 0.5[C_{11}^+(2\alpha_1^2) + C_{32}^+(4\alpha_1^2) + C_{31}^+(2\alpha_1^2) + \alpha_1^2C_{22}^+]$$

$$\text{Here } E_0 = 137.5J - 145.245\omega - 435.735\omega\cos(2\theta) + 102.05\omega[1 + 3\cos(2\theta)] \\ - \cos^2\theta[D_1^{(2)} + D_2^{(2)} + D_3^{(2)}] + 18K_s\sin(2\theta)$$

The graph between $\frac{E(\theta)}{\omega}$ and θ is given in figure 3, for $\frac{J}{\omega} = \frac{D_m^{(2)}}{\omega} = \frac{K_s}{\omega} = 10$.

These peaks become sharper compared with those given figure 1 for spinel ferrite films with $N=2$ and ferromagnetic thin films of $N=3$ with 2^{nd} order perturbation described in one of our early report [12]. Two consecutive maximums can be observed at 23° and 97° . But the peak positions are different from those given in figure 1. Energy is smaller compared with $N=2$ spinel ferrite films given in figure 1. But energy is higher compared with ferromagnetic thin films [12] with $N=3$. Although it is difficult to find any experimental reports related to the thickness or angle dependence of magnetic energy of Nickel ferrite thin films, this kind of phenomena has been observed for Co-ferrite and Mn-Zn ferrite thin films. The energy of these Co-ferrite films depends on the thickness [13]. Also the easy direction (or preferred orientation direction) of Mn-Zn ferrite thin films varies with the thickness of the film [14]. According to figure 1 and 3, the angle corresponding to energy minimum varies with the thickness or number of layers (N). But the numerical values of Nickel ferrite films obtained here can not be compared with the numerical values obtained for Co-ferrite or Mn-Zn ferrite films.

When $\frac{K_s}{\omega}$ is a variable, the 3-D plot of $\frac{E(\theta)}{\omega}$ versus θ and $\frac{K_s}{\omega}$ is given in figure 4. This graph shows a higher variation of energy and higher maximum energy compared with ferromagnetic thin films with $N=3$. Compared with figure 2, energy is less in this case. Some experimental evidences confirm that the energy of Co-ferrite thin films depend on the residual stress [13]. Also the anisotropy energy of Nickel ferrite and Lithium mixed ferrite films depends on the induced stress [15, 16].

Conclusion

The energy peaks become sharper and peak position varies in energy-angle curve as N is increased from 2 to 3. But the separation between two consecutive major maximums

remains same. The angle between easy and hard directions is not 90° . The 3-D plot of $\frac{E(\theta)}{\omega}$ versus θ and $\frac{K_s}{\omega}$ becomes smoother as N is increased from 2 to 3. Unlike the oriented spinel ferrite films, the energy decreases with number of layers in this case according to figure 1, 2, 3 and 4. Some experimental results also indicate that the anisotropy energy depends on the thickness and stress inside the film, and the easy direction depends on the thickness of the ferrite films. In case N=2, minor maximums next to major maximum can be observed. These simulations can be performed for any other ferrite and some other values of $\frac{J}{\omega}$, $\frac{D_m^{(2)}}{\omega}$ and $\frac{K_s}{\omega}$.

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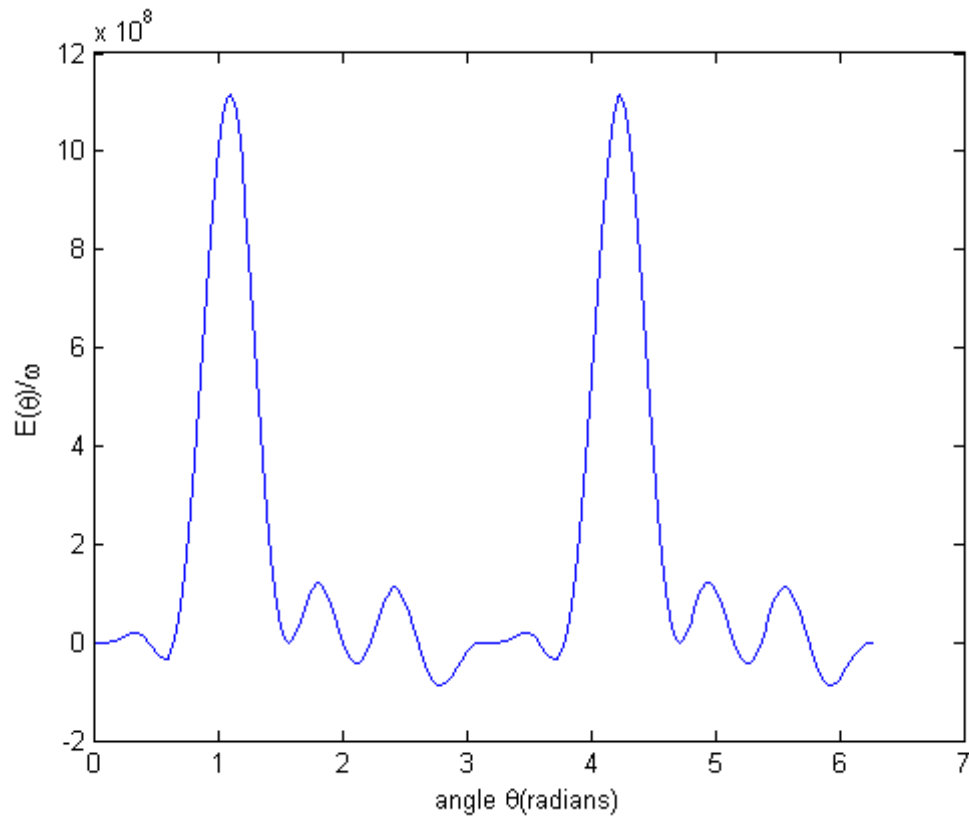


Fig. 1 Graph between $\frac{E(\theta)}{\omega}$ and θ for $\frac{K_s}{\omega}=10$ with the effect of variable second order anisotropy for $N=2$.

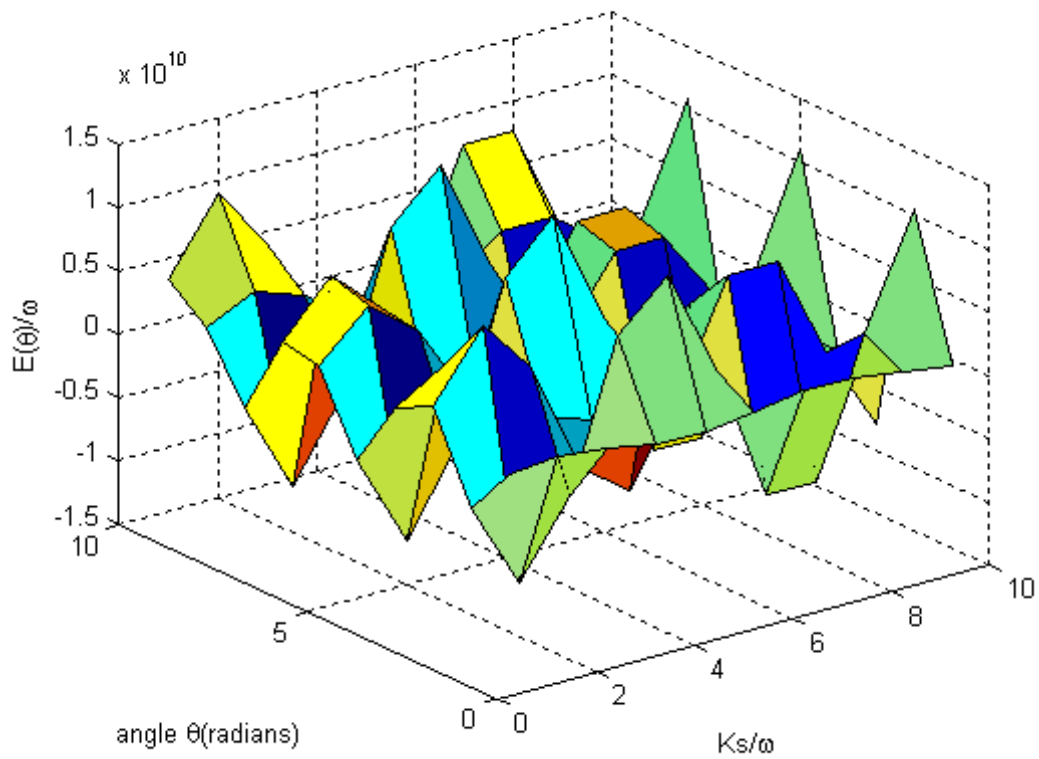


Fig. 2 3-D plot of $\frac{E(\theta)}{\omega}$ versus $\frac{Ks}{\omega}$ and θ with the effect of variable second order anisotropy for $N=2$.

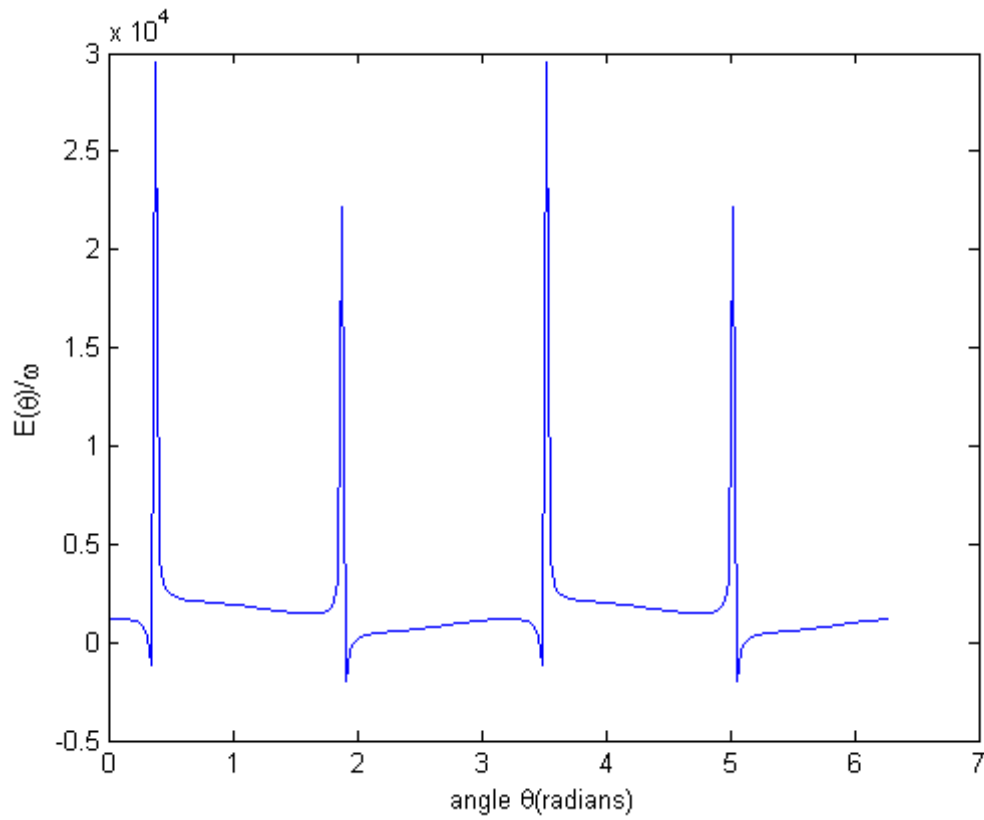


Fig. 3 Graph between $\frac{E(\theta)}{\omega}$ and θ for $\frac{K_s}{\omega}=10$ with the effect of invariant second order anisotropy for $N=3$.

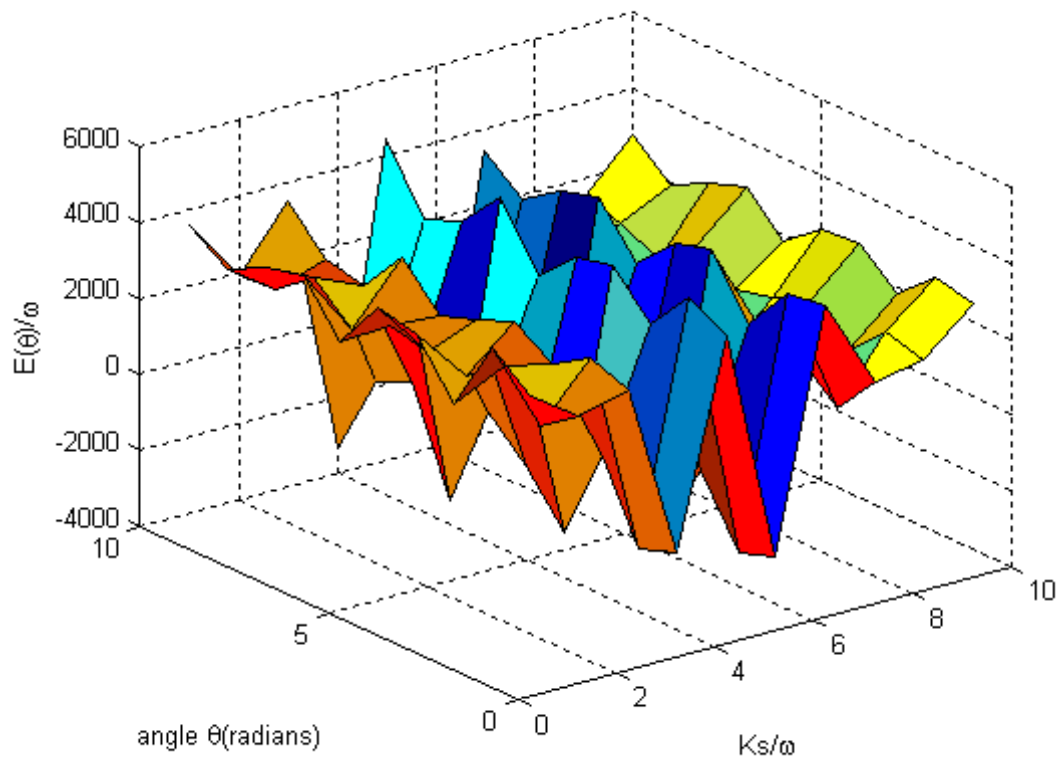


Fig. 4 3-D plot of $\frac{E(\theta)}{\omega}$ versus $\frac{Ks}{\omega}$ and θ with the effect of invariant second order anisotropy for $N=3$.

Study of Superconducting State Parameters of Alloy Superconductors

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Abstract: The theoretical study of the superconducting state parameters (SSP) viz. electron-phonon coupling strength λ , Coulomb pseudopotential μ^* , transition temperature T_C , isotope effect exponent α and effective interaction strength N_0V of Pb-Tl-Bi alloys viz. $\text{Tl}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.40}\text{Tl}_{0.60}$, $\text{Pb}_{0.60}\text{Tl}_{0.40}$, $\text{Pb}_{0.80}\text{Tl}_{0.20}$, $\text{Pb}_{0.60}\text{Tl}_{0.20}\text{Bi}_{0.20}$, $\text{Pb}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.80}\text{Bi}_{0.20}$, $\text{Pb}_{0.70}\text{Bi}_{0.30}$, $\text{Pb}_{0.65}\text{Bi}_{0.35}$ and $\text{Pb}_{0.45}\text{Bi}_{0.55}$ have been made extensively in the present work using a model potential formalism for the first time. A considerable influence of various exchange and correlation functions on λ and μ^* is found from the present study. The present results of the SSP are found in qualitative agreement with the available experimental data wherever exist.

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1. Introduction

During last several years, the superconductivity remains a dynamic area of research in condensed matter physics with continual discoveries of novel materials and with an increasing demand for novel devices for sophisticated technological applications. A large number of metals and amorphous alloys are superconductors, with critical temperature T_C ranging from 1-18K. Even some heavily doped semiconductors have also been found to be superconductors. Basically, all the metal superconductors are type-I superconductors at room temperature [1-13]. The pseudopotential theory has been used successfully in explaining the superconducting state parameters (SSP) for metallic complexes by many workers [1-13]. Many of them have used well known model pseudopotential in the calcu-

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lation of the SSP for the metallic complexes. Recently, Vora *et al.* [3-11] have studied the SSP of some metals, In-based binary alloys, alkali-alkali binary alloys and large number of metallic glasses using single parametric model potential formalism. The study of the SSP of the binary alloy based superconductors may be of great help in deciding their applications; the study of the dependence of the transition temperature T_C on the composition of metallic elements is helpful in finding new superconductors with high T_C . The application of pseudopotential to binary alloys involves the assumption of pseudoions with average properties, which are assumed to replace three types of ions in the binary systems, and a gas of free electrons is assumed to permeate through them. The electron-pseudoion is accounted for by the pseudopotential and the electron-electron interaction is involved through a dielectric screening function. For successful prediction of the superconducting properties of the alloying systems, the proper selection of the pseudopotential and screening function is very essential [3-11].

A well known empty core (EMC) model potential of Ashcroft [14] is applied here in the study of the SSP viz. electron-phonon coupling strength λ , Coulomb pseudopotential μ^* , transition temperature T_C , isotope effect exponent α and effective interaction strength N_0V of Pb-Tl-Bi alloys viz. Tl_{0.90}Bi_{0.10}, Pb_{0.40}Tl_{0.60}, Pb_{0.60}Tl_{0.40}, Pb_{0.80}Tl_{0.20}, Pb_{0.60}Tl_{0.20}Bi_{0.20}, Pb_{0.90}Bi_{0.10}, Pb_{0.80}Bi_{0.20}, Pb_{0.70}Bi_{0.30}, Pb_{0.65}Bi_{0.35} and Pb_{0.45}Bi_{0.55}. To see the impact of various exchange and correlation functions on the aforesaid properties, we have used five different types of local field correction functions proposed by Hartree (H) [15], Taylor (T) [16], Ichimaru-Utsumi (IU) [17], Farid *et al.* (F) [18] and Sarkar *et al.* (S) [19]. We have incorporated for the first time the more advanced and newly developed local field correction functions i.e. IU [17], F [18] and S [19] in the investigation of the SSP of Pb-Tl-Bi alloys.

To describe electron-ion interactions in the binary systems, the Ashcroft's empty core (EMC) single parametric local model potential [14] is employed in the present investigation. The form factor $W(q)$ of the EMC model potential in wave number space is (in au) [14]

$$W(q) = \frac{-8\pi Z}{\Omega_O q^2 \varepsilon(q)} \cos(qr_C). \quad (1)$$

here, Z , $\Omega_O \varepsilon(q)$ and r_C are the valence, atomic volume, Hartree dielectric function and parameter of the model potential of Pb-Tl-Bi alloys, respectively.

2. Method of Computation

In the present investigation for binary mixtures, the electron-phonon coupling strength λ is computed using the relation [3-11]

$$\lambda = \frac{m_b \Omega_o}{4\pi^2 k_F M \langle \omega^2 \rangle} \int_0^{2k_F} q^3 |W(q)|^2 dq \quad (2)$$

Here m_b is the band mass, M the ionic mass, Ω_O the atomic volume, k_F the Fermi wave vector and $W(q)$ the screened pseudopotential. The effective averaged square phonon

frequency $\langle\omega^2\rangle$ is calculated using the relation given by Butler [20], $\langle\omega^2\rangle^{1/2} = 0.69\theta_D$, where θ_D is the Debye temperature of the Pb-Tl-Bi alloys.

Using $X = q/2k_F$ and $\Omega_O = 3\pi^2 Z/(k_F)^3$, we get Eq. (2) in the following form,

$$\lambda = \frac{12m_b Z}{M\langle\omega^2\rangle} \int_0^1 X^3 |W(X)|^2 dX \quad (3)$$

where Z and $W(X)$ are the valence and the screened EMC pseudopotential [13] of the Pb-Tl-Bi alloys, respectively.

The Coulomb pseudopotential μ^* is given by [3-11]

$$\mu^* = \frac{\frac{m_b}{\pi k_F} \int_0^1 \frac{dX}{\varepsilon(X)}}{1 + \frac{m_b}{\pi k_F} \ln\left(\frac{E_F}{10\theta_D}\right) \int_0^1 \frac{dX}{\varepsilon(X)}} \quad (4)$$

Where E_F is the Fermi energy, m_b the band mass of the electron and $\varepsilon(X)$ the modified Hartree dielectric function, which is written as [15]

$$\varepsilon(X) = 1 + (\varepsilon_H(X) - 1) (1 - f(X)). \quad (5)$$

$\varepsilon_H(X)$ is the static Hartree dielectric function [15] and $f(X)$ the local field correction function. In the present investigation, the local field correction functions due to H [15], T [16], IU [17], F [18] and S [19] are incorporated to see the impact of exchange and correlation effects.

After evaluating λ and μ^* , the transition temperature T_C and isotope effect exponent α are investigated from the McMillan's formula [3-11]

$$T_C = \frac{\theta_D}{1.45} \exp\left[\frac{-1.04(1+\lambda)}{\lambda - \mu^*(1+0.62\lambda)}\right], \quad (6)$$

$$\alpha = \frac{1}{2} \left[1 - \left(\mu^* \ln \frac{\theta_D}{1.45 T_C} \right)^2 \frac{1+0.62\lambda}{1.04(1+\lambda)} \right]. \quad (7)$$

The expression for the effective interaction strength $N_O V$ is studied using [3-11]

$$N_O V = \frac{\lambda - \mu^*}{1 + \frac{10}{11}\lambda}. \quad (8)$$

3. Results and Discussion

The input parameters and constants used in the present calculations are given in Table 1. Table 2 shows the presently calculated values of the SSP viz. electron-phonon coupling strength λ , Coulomb pseudopotential μ^* , transition temperature T_C , isotope effect exponent α and effective interaction strength $N_O V$ at various concentrations for Pb-Tl-Bi alloys with available experimental findings [21].

The calculated values of the electron-phonon coupling strength λ for Pb-Tl-Bi alloys, using five different types of the local field correction functions with EMC model potential, are shown in Table 2 with the experimental data [21]. It is noticed from the present study that, the percentile influence of the various local field correction functions with respect to the static H-screening function on the electron-phonon coupling strength λ is 26.52%-49.52%, 26.42%-51.81%, 26.08%-52.21%, 25.67%-51.91%, 25.72%-60.06%, 25.20%-51.46%, 25.14%-51.69%, 25.11%-52.15%, 25.11%-52.67% and 25.28%-51.24% for $\text{Tl}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.40}\text{Tl}_{0.60}$, $\text{Pb}_{0.60}\text{Tl}_{0.40}$, $\text{Pb}_{0.80}\text{Tl}_{0.20}$, $\text{Pb}_{0.60}\text{Tl}_{0.20}\text{Bi}_{0.20}$, $\text{Pb}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.80}\text{Bi}_{0.20}$, $\text{Pb}_{0.70}\text{Bi}_{0.30}$, $\text{Pb}_{0.65}\text{Bi}_{0.35}$ and $\text{Pb}_{0.45}\text{Bi}_{0.55}$ alloys, respectively. Also, the H-screening yields lowest values of λ , whereas the values obtained from the F-function are the highest. It is also observed from the Table 2 that, λ goes on increasing from the values of 0.9788→1.8834 as the concentration 'x' of 'Tl' is decreased from 0.60→0.20, while for concentration 'x' of 'Bi' increases except $\alpha\text{Pb}_{0.45}\text{Bi}_{0.55}$ alloys, λ goes on increasing. The increase or decrease in λ with concentration 'x' of 'Tl' and 'Bi' shows a gradual transition from weak coupling behaviour to intermediate coupling behaviour of electrons and phonons, which may be attributed to an increase of the hybridization of sp-d electrons of 'Tl' and 'Bi' with increasing or decreasing concentration (x). This may also be attributed to the increase role of ionic vibrations in the Tl or Bi-rich region. The present results are found in qualitative agreement with the available experimental data [21]. The calculated results of the electron-phonon coupling strength λ for $\text{Tl}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.40}\text{Tl}_{0.60}$, $\text{Pb}_{0.60}\text{Tl}_{0.40}$, $\text{Pb}_{0.80}\text{Tl}_{0.20}$, $\text{Pb}_{0.60}\text{Tl}_{0.20}\text{Bi}_{0.20}$, $\text{Pb}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.80}\text{Bi}_{0.20}$, $\text{Pb}_{0.70}\text{Bi}_{0.30}$, $\text{Pb}_{0.65}\text{Bi}_{0.35}$ and $\text{Pb}_{0.45}\text{Bi}_{0.55}$ deviate in the range of 8.68%-36.54%, 7.60%-29.21%, 3.40%-24.83%, 1.84%-23.10%, 23.27%-52.06%, 0.81%-20.78%, 1.36%-29.07%, 1.58%-31.30%, 2.70%-32.31% and 33.28%-55.88% alloys from the available experimental findings [21], respectively.

The computed values of the Coulomb pseudopotential μ^* , which accounts for the Coulomb interaction between the conduction electrons, obtained from the various forms of the local field correction functions are tabulated in Table 2. It is observed from the Table 2 that for all binary alloys, the μ^* lies between 0.11 and 0.14, which is in accordance with McMillan [22], who suggested $\mu^* \approx 0.13$ for simple and non-simple metals. The weak screening influence shows on the computed values of the μ^* . The percentile influence of the various local field correction functions with respect to the static H-screening function on μ^* for the Pb-Tl-Bi alloys is observed in the range of 4.77%-9.04%, 4.73%-8.86%, 4.66%-8.82%, 4.60%-8.77%, 5.02%-9.56%, 4.52%-8.70%, 4.60%-8.69%, 4.59%-8.76%, 4.59%-8.75% and 4.63%-8.84% for $\text{Tl}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.40}\text{Tl}_{0.60}$, $\text{Pb}_{0.60}\text{Tl}_{0.40}$, $\text{Pb}_{0.80}\text{Tl}_{0.20}$, $\text{Pb}_{0.60}\text{Tl}_{0.20}\text{Bi}_{0.20}$, $\text{Pb}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.80}\text{Bi}_{0.20}$, $\text{Pb}_{0.70}\text{Bi}_{0.30}$, $\text{Pb}_{0.65}\text{Bi}_{0.35}$ and $\text{Pb}_{0.45}\text{Bi}_{0.55}$ alloys, respectively. Again the H-screening function yields lowest values of the μ^* , while the values obtained from the F-function are the highest. The theoretical or experimental data of the μ^* is not available for the further comparisons.

Table 2 contains calculated values of the transition temperature T_C for Pb-Tl-Bi alloys computed from the various forms of the local field correction functions along with the experimental findings [21]. From the Table 2 it can be noted that, the static H-screening

function yields lowest T_C whereas the F-function yields highest values of the T_C . The present results obtained from the H-local field correction functions are found in good agreement with available experimental data [21]. The calculated results of the transition temperature T_C for Pb-Tl-Bi alloys viz. $\text{Tl}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.40}\text{Tl}_{0.60}$, $\text{Pb}_{0.60}\text{Tl}_{0.40}$, $\text{Pb}_{0.80}\text{Tl}_{0.20}$, $\text{Pb}_{0.60}\text{Tl}_{0.20}\text{Bi}_{0.20}$, $\text{Pb}_{0.90}\text{Bi}_{0.10}$, $\text{Pb}_{0.80}\text{Bi}_{0.20}$, $\text{Pb}_{0.70}\text{Bi}_{0.30}$, $\text{Pb}_{0.65}\text{Bi}_{0.35}$ and $\text{Pb}_{0.45}\text{Bi}_{0.55}$ deviate in the range of 0.05%-115.39%, 0.01%-68.53%, 0.04%-55.03%, 0.02%-47.79%, 0.01%-99.72%, 0.03%-43.59%, 0.00%-42.90%, 0.02%-41.19%, 0.01%-41.10% and 0.06%-53.58% from the experimental findings [21], respectively.

The values of the isotope effect exponent α for Pb-Tl-Bi alloys are tabulated in Table 2. The computed values of the α show a weak dependence on the dielectric screening, its value is being lowest for the H- screening function and highest for the F-function. Since the experimental value of α has not been reported in the literature so far, the present data of α may be used for the study of ionic vibrations in the superconductivity of alloying substances. Since H-local field correction function yields the best results for λ and T_C , it may be observed that α values obtained from this screening provide the best account for the role of the ionic vibrations in superconducting behaviour of this system. The theoretical or experimental data of the α is not available for the further comparisons.

The values of the effective interaction strength $N_O V$ are listed in Table 2 for different local field correction functions. It is observed that the magnitude of $N_O V$ shows that the Pb-Tl-Bi alloys under investigation lie in the range of weak coupling superconductors. The values of the $N_O V$ also show a feeble dependence on dielectric screening, its value being lowest for the H-screening function and highest for the F-screening function. The variation of present values of the $N_O V$ show that, the Pb-Tl-Bi alloys under consideration falls in the range of weak coupling superconductors. The theoretical or experimental data of the $N_O V$ is not available for the further comparisons.

From the study of the Table 2, one can see that among the five screening functions the screening function due to H (only static–without exchange and correlation) gives the minimum value of the SSP while the screening function due to F gives the maximum value. The present findings due to T, IU and S-local field correction functions are lying between these two screening functions. The local field correction functions due to IU, F and S are able to generate consistent results regarding the SSP of Pb-Tl-Bi alloys as those obtained for more commonly employed H and T functions. The effect of local field correction functions plays an important role in the computation of λ and μ^* , which makes drastic variation on T_C , α and $N_O V$. Thus, the use of these more promising local field correction functions is established successfully. The computed results of α and $N_O V$ are not showing any abnormal values for Pb-Tl-Bi alloys.

The values of the electron-phonon coupling strength λ and the transition temperature T_C show an appreciable dependence on the local field correction function, whereas for the Coulomb pseudopotential μ^* , isotope effect exponent α and effective interaction strength $N_O V$ a weak dependence is observed. The magnitude of the λ , α and $N_O V$ values shows that Pb-Tl-Bi alloys are weak to intermediate superconductors. In the absence of experimental data for α and $N_O V$, the presently computed values of these parameters may

be considered to form reliable data for Pb-Tl-Bi alloys, as they lie within the theoretical limits of the Eliashberg-McMillan formulation.

Lastly, we would like to emphasize the importance of involving a precise form for the pseudopotential. It must be confessed that although the effect of pseudopotential in strong coupling superconductor is large, yet it plays a decisive role in weak coupling superconductors i.e. those substances which are at the boundary dividing the superconducting and nonsuperconducting region. In other words, a small variation in the value of electron-ion interaction may lead to an abrupt change in the superconducting properties of the material under consideration. In this connection we may realize the importance of an accurate form for the pseudopotential.

Conclusions

The comparison of presently computed results with available experimental findings is highly encouraging in the case of Pb-Tl-Bi alloys, which confirms the applicability of the model potential. The theoretically observed values of SSP are not available for most of the Pb-Tl-Bi alloys therefore it is difficult to draw any special remarks. However, the comparison with other such theoretical data supports the present computations of the SSP. Such study on SSP of other binary and multi component alloys as well as metallic glasses is in progress.

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Table 1. Input parameters and other constants.

Alloys	Z	r_C (au)	Ω_O (au) ³	k_F (au)	M (amu)	θ_D (K)	$\langle \omega^2 \rangle^2$ (au) ² x 10 ⁻⁶
Tl _{0.90} Bi _{0.10}	3.20	0.9910	196.47	0.7842	204.83	90.20	0.15644
Pb _{0.40} Tl _{0.60}	3.40	0.8935	196.38	0.8003	205.5	90.72	0.15825
Pb _{0.60} Tl _{0.40}	3.60	0.8482	198.72	0.8125	206.06	92.58	0.16481
Pb _{0.80} Tl _{0.20}	3.80	0.8237	201.06	0.8241	206.63	94.44	0.17150
Pb _{0.60} Tl _{0.20} Bi _{0.20}	4.00	0.2349	208.26	0.8285	206.98	190.00	0.69415
Pb _{0.90} Bi _{0.10}	4.10	0.8004	207.00	0.8370	207.37	98.57	0.18682
Pb _{0.80} Bi _{0.20}	4.20	0.7874	210.60	0.8390	207.55	100.84	0.19553
Pb _{0.70} Bi _{0.30}	4.30	0.7662	214.20	0.8408	207.73	103.11	0.20443
Pb _{0.65} Bi _{0.35}	4.35	0.7452	216.00	0.8417	207.82	104.25	0.20898
Pb _{0.45} Bi _{0.55}	4.55	0.8157	223.20	0.8451	208.17	108.79	0.22757

Table 2. Superconducting state parameters of the Pb-Tl-Bi alloys.

Alloys	SSP	Present results					Expt. [21]
		H	T	IU	F	S	
Tl _{0.90} Bi _{0.10}	λ	0.7123	1.0067	1.0621	1.0650	0.9012	0.78
	μ^*	0.1195	0.1288	0.1300	0.1303	0.1252	-
	T_C	2.3011	4.5437	4.9372	4.9540	3.7799	2.30
	α	0.4372	0.4558	0.4580	0.4580	0.4515	-
	N_0V	0.3598	0.4584	0.4742	0.4749	0.4265	-
Pb _{0.40} Tl _{0.60}	λ	0.9788	1.3999	1.4795	1.4859	1.2374	1.15
	μ^*	0.1185	0.1276	0.1288	0.1290	0.1241	-
	T_C	4.6004	7.2982	7.7222	7.7522	6.3648	4.60
	α	0.4627	0.4719	0.4730	0.4730	0.4695	-
	N_0V	0.4553	0.5599	0.5760	0.5772	0.5239	-
Pb _{0.60} Tl _{0.40}	λ	1.1317	1.6220	1.7141	1.7226	1.4269	1.38
	μ^*	0.1179	0.1269	0.1281	0.1283	0.1234	-
	T_C	5.9021	8.6916	9.1119	9.1465	7.7143	5.90
	α	0.4697	0.4764	0.4773	0.4773	0.4746	-
	N_0V	0.4997	0.6042	0.6199	0.6213	0.5674	-
Pb _{0.80} Tl _{0.20}	λ	1.2398	1.7739	1.8738	1.8834	1.5581	1.53
	μ^*	0.1174	0.1263	0.1275	0.1277	0.1228	-
	T_C	6.8014	9.6015	10.0139	10.0497	8.6140	6.80
	α	0.4733	0.4788	0.4794	0.4795	0.4772	-
	N_0V	0.5277	0.6306	0.6460	0.6473	0.5940	-
Pb _{0.60} Tl _{0.20} Bi _{0.20}	λ	0.8677	1.2976	1.3726	1.3888	1.0909	1.81
	μ^*	0.1276	0.1382	0.1396	0.1398	0.1340	-
	T_C	7.2606	13.4096	14.3079	14.4995	10.6459	7.26
	α	0.4460	0.4625	0.4641	0.4645	0.4564	-
	N_0V	0.4137	0.5319	0.5485	0.5520	0.4804	-
Pb _{0.90} Bi _{0.10}	λ	1.3151	1.8770	1.9815	1.9919	1.6465	1.66
	μ^*	0.1172	0.1260	0.1272	0.1274	0.1225	-
	T_C	7.6521	10.5294	10.9469	10.9849	9.5032	7.65
	α	0.4753	0.4800	0.4806	0.4806	0.4787	-
	N_0V	0.5456	0.6470	0.6619	0.6633	0.6104	-

Pb _{0.80} Bi _{0.20}	λ	1.3335	1.9055	2.0117	2.0228	1.6688	1.88
	μ^*	0.1174	0.1262	0.1274	0.1276	0.1228	-
	T_C	7.9501	10.8947	11.3199	11.3602	9.8361	7.95
	α	0.4756	0.4802	0.4808	0.4808	0.4789	-
	N_0V	0.5497	0.6512	0.6661	0.6676	0.6142	-
Pb _{0.70} Bi _{0.30}	λ	1.3809	1.9782	2.0888	2.1011	1.7277	2.01
	μ^*	0.1176	0.1265	0.1277	0.1279	0.1230	-
	T_C	8.4518	11.4578	11.8872	11.9309	10.3650	8.45
	α	0.4765	0.4808	0.4814	0.4814	0.4795	-
	N_0V	0.5601	0.6617	0.6765	0.6781	0.6243	-
Pb _{0.65} Bi _{0.35}	λ	1.4418	2.0711	2.1875	2.2012	1.8039	2.13
	μ^*	0.1177	0.1266	0.1278	0.1280	0.1231	-
	T_C	8.9511	11.9763	12.4032	12.4496	10.8663	8.95
	α	0.4776	0.4816	0.4821	0.4821	0.4804	-
	N_0V	0.5730	0.6745	0.6892	0.6908	0.6367	-
α Pb _{0.45} Bi _{0.55}	λ	1.1426	1.6289	1.7195	1.7281	1.4315	2.59
	μ^*	0.1188	0.1278	0.1291	0.1293	0.1243	-
	T_C	7.0042	10.2267	10.7090	10.7508	9.0714	7.0
	α	0.4696	0.4761	0.4769	0.4770	0.4743	-
	N_0V	0.5022	0.6051	0.6205	0.6219	0.5680	-

Riemann Zeta Function Zeros Spectrum

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Abstract: This paper shows that quantum chaotic oscillator Hamiltonian $H = px$ generates Riemann zeta function zeros as energy eigenvalues assuming validity of the Riemann hypothesis. We further put this on a firmer ground proving rigorously the Riemann hypothesis. We next introduce reformulation of special theory of relativity by which chaotic oscillator motion described via Hamiltonian $H = px$ is generated by gravitational potential, thus linking chaotic motion and Riemann zeta function to gravity.

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1. Introduction to Chaotic Quantum Oscillator

Chaos is all around us. Consider a flame from the cigarette lighter, for instance. Or fractal patterns in a leaf on a tree. Nature is chaotic.

One of the simplest chaotic systems is chaotic oscillator [1]. One is simply to complexify the harmonic oscillator Hamiltonian $H = p^2 + x^2$ by substituting $x \rightarrow ix$, turning it thus into chaotic Hamiltonian

$$H = p^2 - x^2 \quad (1)$$

After performing canonic rotation

$$\begin{aligned} p &\rightarrow p - x \\ x &\rightarrow p + x \end{aligned} \quad (2)$$

upon (1), we reach chaotic oscillator Hamiltonian in simpler form

$$H = px \quad (3)$$

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Classically, Hamilton equations for this Hamiltonian are

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p} = x \\ \dot{p} &= -\frac{\partial H}{\partial x} = -p\end{aligned}\quad (4)$$

These equations integrate to

$$\begin{aligned}x(t) &= x_0 e^t \\ p(t) &= p_0 e^{-t}\end{aligned}\quad (5)$$

These equations represent unstable trajectories reflecting the fact that system described by Hamiltonian $H = px$ is chaotic.

To rewrite Hamiltonian (3) for quantum system, we are simply to symmetrize it as follows,

$$H = \frac{1}{2}(px + xp) \quad (6)$$

Hamiltonian (6) is obviously hermitean. Its differential form is

$$H = -i \left(x \frac{d}{dx} + \frac{1}{2} \right) \quad (7)$$

Coordinate representation eigenfunctions $\psi(x)$ satisfying Schroedinger eigenequation

$$H\psi(x) = E\psi(x) \quad (8)$$

with energy E being constant of motion for given orbit are

$$\psi(x) = \frac{A}{x^{1/2-iE}} \quad (9)$$

Impulse representation eigenfunction $\phi(p)$ is simply Fourier transform of coordinate eigenfunction $\psi(x)$,

$$\phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx} dx = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ipx}}{x^{1/2-iE}} dx \quad (10)$$

To evaluate this integral, we have to handle singularity of integrand at $x = 0$. To do this, it is appropriate to require $x \rightarrow |x|$. This enables us to calculate integral (10) and we find

$$\phi(p) = \frac{A 2^{iE} \Gamma\left(\frac{1}{4} + \frac{iE}{2}\right)}{|p|^{1/2+iE} \Gamma\left(\frac{1}{4} - \frac{iE}{2}\right)} = \frac{A}{\sqrt{2\pi} \left|\frac{p}{2\pi}\right|^{1/2+iE}} e^{2i\theta(E)} \quad (11)$$

Here function $\theta(E) = \arg \Gamma\left(\frac{1}{4} + \frac{iE}{2}\right) - \frac{t}{2} \log \pi$ is actually the argument of the Riemann zeta function on the critical line $\Re s = 1/2$.

We have encountered the Riemann zeta function while investigating the chaotic oscillator. Riemann zeta function zeros density function has mean value [1]

$$\langle \mathcal{N}(E) \rangle = \frac{\theta(E)}{\pi} + 1 = \frac{E}{2\pi} \log \left(\frac{E}{2\pi} \right) + \frac{7}{8} + O\left(\frac{1}{E}\right) \quad (12)$$

Next we count quantum states of chaotic oscillator with Hamiltonian (6). Mean number of states of energy lesser than E is simply area under the contour $H = E$ in phase space (x, p) . For Hamiltonian (6) this area is unbound because classical motion (5) generated by H is not bound. To overcome this obstacle, we have to regularize position x and impulse p by truncating them, $x > x_{\min}$, $p > p_{\min}$. We can always truncate any physical system [2]. When thus calculated, mean density of states of H is exactly (12). Hence, there are asymptotically as many states of H as Riemann zeta function nontrivial zeros in the mean. Because of this fact we deliberately chose $x \rightarrow |x|$ to be the appropriate continuation over integrand singularity when evaluating integral (11).

Further connection of chaotic quantum oscillator with Riemann zeta function comes from considering an appropriate boundary condition [1] for wave-functions (9) and (11). Intriguing relation coming from this boundary condition is [1]

$$x^{1/2}\zeta(1/2 - iE)\psi(x) - p^{1/2}\zeta(1/2 + iE)\phi(p) = 0 \quad (13)$$

Here x and p are ordinary real numbers. This condition does not seem to generate Riemann zeta function nontrivial zeros because of the difference in signs of energy E in zeta functions in (13), nor can be easily explained as a boundary condition geometrically, since it mixes both x and p .

This is as far as one gets when considering chaotic quantum oscillator given by Hamiltonian $H = px$ so far.

2. Riemann Zeta Function Zeros Spectrum

We next consider equation (13). Let us rewrite it as

$$x^{1/2}\zeta(1/2 - iE)\psi(x) = p^{1/2}\zeta(1/2 + iE)\phi(p) \quad (14)$$

Since energy E being constant for given eigen-state $\psi(x)$, we notice that left hand side of Eq. (14) is function of x only. Similarly, right hand side of Eq. (14) is function of p only. These two sides being equal, we conclude that they are both equal to some constant $C(E)$, ie.

$$\begin{aligned} x^{1/2}\zeta(1/2 - iE)\psi(x) &= C(E) \\ p^{1/2}\zeta(1/2 + iE)\phi(p) &= C(E) \end{aligned} \quad (15)$$

These equations are now easily turned into a boundary condition.

With wave-function $\psi(x)$ defined in (9), first of equations (15) becomes

$$Ax^{iE}\zeta(1/2 - iE) = C(E) \quad (16)$$

Since the system we are considering is chaotic, positions x are not constants of motion, but change with time. So there is only one way to fulfill requirement (16), namely to have $C(E) = 0$. Then the value of x does not matter. Condition

$$C(E) = 0 \quad (17)$$

is fulfilled as soon as

$$\zeta(1/2 - iE) = 0 \quad (18)$$

This condition, being part of the boundary condition for wave-functions of chaotic quantum oscillator, obviously generates Riemann zeta function nontrivial zeros.

Hence, spectrum of chaotic quantum oscillator given by Hamiltonian (6) consists solely of imaginary parts of not necessarily all of the Riemann zeta function nontrivial zeros on the critical line $\Re s = 1/2$.

There is still one interesting point about this result. Hilbert and Polya showed that if one manages to find Hermitean Hamiltonian such that eigenenergies of that Hamiltonian be imaginary parts of all of the Riemann zeta function nontrivial zeros, then this way one actually proves the Riemann hypothesis, saying that all the Riemann zeta function nontrivial zeros have real part $\Re s = 1/2$. Similar to condition (18).

We actually did not prove the Riemann hypothesis, because condition (18) involves only the Riemann zeta function nontrivial zeros on the critical line $\Re s = 1/2$ and does not concern rest of them from entire critical strip $0 < \Re s < 1$. However, if one is able to prove the Riemann hypothesis independently of results exposed so far, ie. without concerning Hamiltonian $H = px$, say by purely analytic methods, then Hamiltonian $H = px$ would prove to be the candidate for Hamiltonian Hilbert and Polya wrote about.

Actually, we are able to prove the Riemann hypothesis rigorously by purely analytic and almost elementary methods, and this is what we do next.

3. Introduction to Riemann Zeta Function

Euler was the first to consider the Riemann zeta function. He was also the first to consider the prime number counting function $\pi(r)$ counting primes $p \leq r$.

Riemann gave the complete analytic treatment of zeta function [3,4,5,6] defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (19)$$

in the complex half plane $\Re s > 1$. He showed that zeta function continues analytically over entire complex plane as a meromorphic function with a pole at $s = 1$. Riemann showed that zeta function nontrivial zeros $\rho = \sigma + it$ are situated symmetrically with respect to point $s = 1/2$ in the strip $0 \leq \Re s \leq 1$.

I find necessary to point to a fact that there are no zeta function zeros[7] on the line $\Re s = 1$. Therefore, there are no zeta function zeros on line $\Re s = 0$ being centrally symmetric to the line $\Re s = 1$ with respect to point $s = 1/2$.

Hence, we may consider region $0 < \Re s < 1$ to be the critical strip where all the nontrivial zeta function zeros are located. Let us therefore henceforth refer to the strip $0 < \Re s < 1$ as to the critical strip.

Primes counting function behaves asymptotically as

$$\pi(r) = Li(r) + \mathcal{O}(r^\theta \log r) \quad (20)$$

with $1/2 \leq \theta < \sigma$. Hence, if $\sigma = 1/2$ then the uncertainty in θ is minimal and $\theta = 1/2$. Hence the importance of the Riemann hypothesis,

The Riemann hypothesis: *Zeta function nontrivial zeros have real part equal to 1/2.*

One of integral representations of zeta function valid for any s from the critical strip $0 < \Re s < 1$ is [8]

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\frac{1}{e^p - 1} - \frac{1}{p} \right) p^{s-1} dp \quad (21)$$

4. Definition and Lemma

4.1 Definition

Define family \mathcal{F} as family of functions $F(s)$ analytic and single-valued in complex variable s in critical strip $0 < \Re s < 1$, satisfying following three conditions:

Condition A Zeros $\rho = \sigma + it$ of $F(s)$ are located symmetrically with respect to point $s = 1/2$ in the critical strip $0 < \Re s < 1$.

Condition B There exists real function $f(p)$, continuous in real variable $p \in R_+$, such that Mellin transform

$$F(s) = \int_0^{\infty} f(p) p^{s-1} dp \quad (22)$$

exists for s in the critical strip.

Condition C $\lim_{p \rightarrow 0^+} p^\sigma (1 + \log^2 p) \frac{df(p)}{dp} = 0$

Hence, by the theory of Mellin transformation, we conclude from Eq. (22) that $f(p) = \mathcal{O}(1)$ as $p \rightarrow 0^+$ for $0 < \Re s < 1$.

We notice that condition **C** follows from $f(p) = \mathcal{O}(1)$ as $p \rightarrow 0^+$ for $0 < \Re s < 1$. We keep condition **C** as a distinct condition simply for having it at our disposal explicitly.

We also notice by inspecting Mellin transform (22) that function $F(s)$ is real on real axis. Hence by the Schwarz principle of reflection we conclude that $F(\bar{s}) = \bar{F}(s)$. Therefore nontrivial zeros ρ are distributed symmetrically with respect to real axis.

4.2 Lemma

In order to prove the Riemann hypothesis we should start by proving the following lemma.

Lemma (Riemann hypothesis for Mellin transforms analytic in critical strip with zeros symmetric with respect to $s = 1/2$): *Let $F(s) \in \mathcal{F}$. Then all zeros $\rho = \sigma + it$ of $F(s)$ in critical strip have real part σ equal to 1/2.*

To prove this lemma, let us introduce continuous increasing parametrization $p(q)$, $0 \leq q \leq 1$, such that $p(0) = 0$, $\lim_{q \rightarrow 1^-} p(q) = +\infty$.

With such parametrization we reparametrize integral (22) yielding

$$F(s) = \int_0^1 f(p(q))p(q)^{s-1}p'(q)dq \quad (23)$$

Let us decompose this integral into its real and imaginary parts,

$$\begin{aligned} \Re F(s) &= \int_0^1 f(p(q))p(q)^{x-1} \cos(y \log p(q))p'(q)dq \\ \Im F(s) &= \int_0^1 f(p(q))p(q)^{x-1} \sin(y \log p(q))p'(q)dq \end{aligned} \quad (24)$$

with $x = \Re s$, $y = \Im s$, $s = x + iy$ and using the real logarithm.

Hence, by the mean value theorem, we conclude that for any s from the critical strip there exist at least one a and b , $0 \leq a, b \leq 1$, such that Eq. (24) becomes

$$\begin{aligned} \Re F(s) &= f(p(a))p(a)^{x-1} \cos(y \log p(a))p'(a) \\ \Im F(s) &= f(p(b))p(b)^{x-1} \sin(y \log p(b))p'(b) \end{aligned} \quad (25)$$

Equation (25) is numerical, meaning we cannot differentiate (25) and expect to yield any result since equation (25) simply states that for any s there exist some a and b such that number on the right in (25) equals number on the left in (25). Left and right hand side of (25) represent two numbers, not functions.

When we change $s = x + iy$ we consequently change a and b . We are therefore naturally inclined to introduce new functions $a(x, y)$ and $b(x, y)$ such that Eq. (25) becomes

$$\begin{aligned} \Re F(s) &= f(p(a(x, y)))p(a(x, y))^{x-1} \cos(y \log p(a(x, y)))p'(a(x, y)) \\ \Im F(s) &= f(p(b(x, y)))p(b(x, y))^{x-1} \sin(y \log p(b(x, y)))p'(b(x, y)) \end{aligned} \quad (26)$$

Equation (26) defines functions $a(x, y)$ and $b(x, y)$ implicitly.

Let us next introduce parametrization $p(q) = e^{\tan(\pi(q-1/2))}$. It satisfies all requirements parametrization $p(q)$ is required to satisfy. Namely, it is continuous, increasing, $p(0) = 0$, $\lim_{q \rightarrow 1^-} p(q) = +\infty$ and it maps interval $0 \leq q \leq 1$ onto R_+ . Parametrization $p(q) = e^{\tan(\pi(q-1/2))}$ is bijective as soon as we restrict values of q to $0 \leq q \leq 1$.

Let us further inspect properties of parametrization $p(q) = e^{\tan(\pi(q-1/2))}$. We notice that $p'(q) = \pi \frac{e^{\tan(\pi(q-1/2))}}{\cos^2(\pi(q-1/2))} = \pi p(q) (1 + \log^2(p(q)))$. We further notice that $\lim_{q \rightarrow 0^+} p(q)^{x-1}p'(q) = \pi \lim_{q \rightarrow 0^+} \frac{e^{x \tan(\pi(q-1/2))}}{\cos^2(\pi(q-1/2))} = 0$ for any $0 < x < 1$.

Hence whenever $F(s)$ is in \mathcal{F} , by inspecting (26) having $\lim_{q \rightarrow 0^+} p(q)^{x-1}p'(q) = 0$, $p'(q) = \pi p(q) (1 + \log^2(p(q)))$ and $f(0) = \mathcal{O}(1)$, we conclude that

$$\begin{aligned} \Re F(\rho) &= \pi f(p(0))p(0)^\sigma \cos(t \log p(0)) (1 + \log^2(p(0))) = 0 \\ \Im F(\rho) &= \pi f(p(0))p(0)^\sigma \sin(t \log p(0)) (1 + \log^2(p(0))) = 0 \end{aligned} \quad (27)$$

in zeros $\rho = \sigma + it$ of function $F(s)$, whenever we set $a(\sigma, t), b(\sigma, t) = 0$. Whenever we set $a(x, y), b(x, y) = 0$ in (26) we reproduce result $F(s) = 0$ as demonstrated in (27).

Therefore we can require $a(\sigma, t), b(\sigma, t) = 0$ for all zeros $\rho = \sigma + it$ from the critical strip, as we will.

I find useful to pay some attention to our choice $a(\sigma, t), b(\sigma, t) = 0$. I'd like to stress the fact that by the mean value theorem points a and b are completely arbitrary as long as Eq. (26) is satisfied for given s . Hence any $a, b \in [0, 1]$ that reproduce result $F(s) = 0$ as in Eq. (27) are admissible at any zero ρ . We use this arbitrariness to set $a(\sigma, t), b(\sigma, t) = 0$ for all zeros $\rho = \sigma + it$. Once we made our choice for $a(\sigma, t), b(\sigma, t)$, we explicitly do not allow any other value for $a(\sigma, t), b(\sigma, t)$ for simple fact that one value for $a(\sigma, t), b(\sigma, t)$ is enough to completely describe function $F(s)$ at any point $s = \sigma + it$.

Next we demonstrate that derivative $\frac{\partial \Re F(\rho)}{\partial y}$ exists at zeros $\rho = \sigma + it$. To show this, we suppose that derivative $\frac{\partial p(a(\sigma, t))}{\partial y}$ indeed exists at zeros $\rho = \sigma + it$. We differentiate (27) formally with respect to y as a product of functions and yield

$$\begin{aligned} \frac{\partial \Re F(\rho)}{\partial y} &= \\ &= \pi [f'(p)p^\sigma \cos(t \log p)(1 + \log^2 p) + \\ &+ \sigma f(p)p^{\sigma-1} \cos(t \log p)(1 + \log^2 p) - \\ &- t f(p)p^{\sigma-1} \sin(t \log p)(1 + \log^2 p) + \\ &+ 2f(p)p^{\sigma-1} \cos(t \log p) \log p] \frac{\partial p}{\partial y} - \\ &- \pi f(p)p^\sigma \sin(t \log p)(1 + \log^2 p) \log p \end{aligned} \quad (28)$$

as $p \rightarrow 0^+$. Variable p in (28) stands for function $p(a(x, y))$ at $a(\sigma, t) = 0$ as a shorthand abbreviation.

We notice that $\lim_{p \rightarrow 0^+} p^x (1 + \log^2 p) \log p = 0$ for any $0 < x < 1$. We further notice that by condition **C** the first summand in square brackets in (28) vanishes.

Thus we are left with

$$\begin{aligned} \frac{\partial \Re F(\rho)}{\partial y} &= \\ &= \pi f(p)p^{\sigma-1} \cos(t \log p) [(\sigma - t \tan(t \log p)) (1 + \log^2 p) + 2 \log p] \frac{\partial p}{\partial y} \end{aligned} \quad (29)$$

Let us next suppose that

$$\sigma - t \tan(t \log p) \neq 0 \quad (30)$$

as $p \rightarrow 0^+$.

We notice that $\lim_{p \rightarrow 0^+} \frac{1 + \log^2 p}{\log p} = 2 \lim_{p \rightarrow 0^+} \log p$ by virtue of the L'Hospital theorem. Hence, since $\sigma - t \tan(t \log p) \neq 0$ we find terms 1 and $2 \log p$ neglectable compared to $\log^2 p$ as $p \rightarrow 0^+$. Hence Eq. (29) becomes

$$\begin{aligned} \frac{\partial \Re F(\rho)}{\partial y} &= \\ &= 2\pi f(p)p^{\sigma-1} \cos(t \log p)(\sigma - t \tan(t \log p)) \log^2 p \frac{\partial p}{\partial y} \end{aligned} \quad (31)$$

At this point we notice that $\frac{\partial \Re F(\rho)}{\partial y}$ exists. We further notice that product $p^{\sigma-1} \log^2 p$ is unbounded as $p \rightarrow 0^+$. Further, $f(p) = \mathcal{O}(1)$ as $p \rightarrow 0^+$. We also supposed that $\sigma - t \tan(t \log p) \neq 0$. Thus Eq. (31) suggests

$$\cos(t \log p) \frac{\partial p}{\partial y} = 0 \quad (32)$$

This is true if either $\frac{\partial p}{\partial y} = 0$ or if $\cos(t \log p) = 0$.

If $\frac{\partial p}{\partial y} = 0$ we have proven that partial derivative $\frac{\partial p(a(\sigma, t))}{\partial y}$ exists as $p \rightarrow 0^+$.

If $\cos(t \log p) = 0$, then $\sin(t \log p) = \pm 1$ and Eq. (31) becomes

$$\frac{\partial \Re F(\rho)}{\partial y} = \mp 2t\pi f(p)p^{\sigma-1} \log^2 p \frac{\partial p}{\partial y} \quad (33)$$

Since $\frac{\partial \Re F(\rho)}{\partial y}$ exists, since product $p^{\sigma-1} \log^2 p$ is unbounded as $p \rightarrow 0^+$ and since $f(p) = \mathcal{O}(1)$ as $p \rightarrow 0^+$, Eq. (33) suggests again that $\frac{\partial p}{\partial y} = 0$.

Therefore if $\sigma - t \tan(t \log p) \neq 0$ we conclude that partial derivative $\frac{\partial p(a(\sigma, t))}{\partial y}$ exists as $p \rightarrow 0^+$.

We next consider assumption

$$\sigma - t \tan(t \log p) = 0 \quad (34)$$

From (34) we conclude that $\tan(t \log p) = \sigma/t$. Having finite σ and nonvanishing t we notice that $\cos(t \log p) \neq 0$.

Consider (34). We know that there is some $r > 0$, $r \in \mathbb{R}$, such that $\sigma - t \tan(t \log p)$ tends to zero as p^r as $p \rightarrow 0^+$. Hence, factor $(\sigma - t \tan(t \log p)) \log^2 p$ in (29) behaves as $p^r \log^2 p$ as $p \rightarrow 0^+$ under assumption $\sigma - t \tan(t \log p) = 0$.

Therefore we next consider limit $\lim_{p \rightarrow 0^+} p^r \log^2 p$ and find by the use of the L'Hospital rule that $\lim_{p \rightarrow 0^+} p^r \log^2 p = \frac{2}{r^2} \lim_{p \rightarrow 0^+} p^r = 0$ for any $r > 0$.

We should also consider the case with $\sigma - t \tan(t \log p)$ tending to zero identically. Then we find $(\sigma - t \tan(t \log p)) \log^2 p = 0$ again, identically.

Hence, by putting $(\sigma - t \tan(t \log p)) \log^2 p = 0$ back to Eq. (29) we conclude

$$\frac{\partial \Re F(\rho)}{\partial y} = 2\pi f(p)p^{\sigma-1} \cos(t \log p) \log p \frac{\partial p}{\partial y} \quad (35)$$

At this point we again notice that $\frac{\partial \Re F(\rho)}{\partial y}$ exists. We further notice that product $p^{\sigma-1} \log p$ is unbounded as $p \rightarrow 0^+$. Further, $f(p) = \mathcal{O}(1)$ as $p \rightarrow 0^+$ and $\cos(t \log p) \neq 0$. Thus Eq. (35) suggests $\frac{\partial p}{\partial y} = 0$ as $p \rightarrow 0^+$.

Therefore we conclude that partial derivative $\frac{\partial p(a(\sigma, t))}{\partial y}$ exists for every zero $\rho = \sigma + it$.

Next we show that partial derivative $\frac{\partial p(a(\sigma,t))}{\partial x}$ exists at zeros $\rho = \sigma + it$. To show this, we suppose that derivative $\frac{\partial p(a(\sigma,t))}{\partial x}$ indeed exists at zeros $\rho = \sigma + it$. We differentiate (27) formally with respect to x as a product of functions and yield

$$\begin{aligned} \frac{\partial \Re F(\rho)}{\partial x} &= \\ &= \pi[f'(p)p^\sigma \cos(t \log p)(1 + \log^2 p) + \\ &+ \sigma f(p)p^{\sigma-1} \cos(t \log p)(1 + \log^2 p) - \\ &- t f(p)p^{\sigma-1} \sin(t \log p)(1 + \log^2 p) + \\ &+ 2f(p)p^{\sigma-1} \cos(t \log p) \log p] \frac{\partial p}{\partial x} - \\ &- \pi f(p)p^\sigma \cos(t \log p)(1 + \log^2 p) \log p \end{aligned} \quad (36)$$

as $p \rightarrow 0^+$. Variable p in (36) again stands for function $p(a(x,y))$ at $a(\sigma,t) = 0$ as a shorthand abbreviation.

We notice that $\lim_{p \rightarrow 0^+} p^x (1 + \log^2 p) \log p = 0$ for any $0 < x < 1$. We further notice that by condition **C** the first summand in square brackets in (36) vanishes.

Thus we are left with

$$\begin{aligned} \frac{\partial \Re F(\rho)}{\partial x} &= \\ &= \pi f(p)p^{\sigma-1} \cos(t \log p) [(\sigma - t \tan(t \log p)) (1 + \log^2 p) + 2 \log p] \frac{\partial p}{\partial x} \end{aligned} \quad (37)$$

We notice that Eqs. (29) and (37) are identical as soon as we substitute $\frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial x}$ in (29). Hence, by following steps (29) through (35) with $\frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial x}$, we find that partial derivative $\frac{\partial p(a(\sigma,t))}{\partial x}$ exists for every zero $\rho = \sigma + it$.

Hence we conclude that partial derivatives $\frac{\partial p(a(\sigma,t))}{\partial x}$ and $\frac{\partial p(a(\sigma,t))}{\partial y}$ exist at any zero $\rho = \sigma + it$.

We would like now to examine the behavior of ratio $F(s)/F(1-s)$ at zeros $s = \rho$ and $1-s = 1-\rho$. This ratio involves ratio $p(a(\sigma,y))/p(a(1-\sigma,y))$ at zeros $s = \rho$ and $1-s = 1-\rho$. So let us continue proving lemma by inspecting the following limit,

$$\lim_{y \rightarrow t} \frac{p(a(\sigma,y))}{p(a(1-\sigma,y))} \quad (38)$$

We first notice that point $1-\sigma+it$ is a zero as soon as $\sigma+it$ is a zero. This is so because zeros are located symmetrically with respect to point $s = 1/2$ as well as with respect to real axis – and therefore symmetrically with respect to critical line $\Re s = 1/2$ as well – for any function $F(s) \in \mathcal{F}$. Therefore both $p(a(\sigma,t))$ and $p(a(1-\sigma,t))$ equal zero following discussion of Eq. (27).

Limit (38) may or may not exist. If it does not exist, it does not exist because it is unbounded, since $p(a(\sigma,t))$ being continuous at any zero ρ and since zeros ρ and $1-\rho$ being of the same order, and hence we conclude that if limit (38) does not exist, then reciprocal limit, namely $\lim_{y \rightarrow t} \frac{p(a(1-\sigma,y))}{p(a(\sigma,y))}$, vanishes, therefore exists.

Let us first assume that $\lim_{y \rightarrow t} \frac{p(a(\sigma, y))}{p(a(1-\sigma, y))}$ exists. Since $p(a(\sigma, t)) = 0$ for any zero $\rho = \sigma + it$ and since $p(a(\sigma, t))$ being continuous at every zero $\rho = \sigma + it$, we conclude that we can employ L'Hospital rule upon limit (38) and yield

$$\begin{aligned} \lim_{y \rightarrow t} \frac{p(a(\sigma, y))}{p(a(1-\sigma, y))} &= \\ &= \lim_{y \rightarrow t} \frac{\frac{\partial}{\partial y} p(a(\sigma, y))}{\frac{\partial}{\partial y} p(a(1-\sigma, y))} \end{aligned} \tag{39}$$

choosing complex number s to approach zero ρ keeping real part $x = \sigma$ constant.

Limit involving partial derivatives in (39) may be ill defined as well – for instance, partial derivatives in (39) vanish. However, we are not interested in the exact value of this limit. All that we require is the formal identity (39) as it is.

We notice that

$$\lim_{y \rightarrow t} \frac{p(a(\sigma, y))^\sigma}{p(a(1-\sigma, y))^{1-\sigma}} \tag{40}$$

exists as soon as limit $\lim_{y \rightarrow t} \frac{p(a(\sigma, y))}{p(a(1-\sigma, y))}$ exists, for

$$\lim_{y \rightarrow t} \frac{p(a(\sigma, y))^\sigma}{p(a(1-\sigma, y))^{1-\sigma}} = \lim_{y \rightarrow t} \left[\frac{p(a(\sigma, y))}{p(a(1-\sigma, y))} \right]^\sigma \frac{1}{p(a(1-\sigma, y))^{1-2\sigma}}$$

exists as soon as $\sigma > 1/2$.

Since condition **A** required zeros to be located symmetrically with respect to point $1/2$, and since zeros are distributed symmetrically with respect to real axis, therefore zeros appear symmetrically with respect to the critical line $\Re s = x = 1/2$, therefore we conclude that we may restrict our analysis to zeros with $\sigma > 1/2$ without loss of generality.

There is of course one more possibility left, namely $\sigma = 1/2$, but then the lemma is proved automatically.

Hence, let us assume $\sigma > 1/2$ without loss of generality. We notice that $\lim_{q \rightarrow 0^+} p(q)^\sigma = 0$ for any $0 < \sigma < 1$. Hence we may employ the L'Hospital rule upon limit $\lim_{y \rightarrow t} \frac{p(a(\sigma, y))^\sigma}{p(a(1-\sigma, y))^{1-\sigma}}$ still choosing complex number s to approach zero ρ keeping real part $x = \sigma$ constant. This way,

$$\begin{aligned} \lim_{y \rightarrow t} \frac{p(a(\sigma, y))^\sigma}{p(a(1-\sigma, y))^{1-\sigma}} &= \\ &= \frac{\sigma}{1-\sigma} \lim_{y \rightarrow t} \frac{p(a(\sigma, y))^\sigma}{p(a(1-\sigma, y))^{1-\sigma}} \frac{p(a(1-\sigma, y))}{p(a(\sigma, y))} \frac{\frac{\partial}{\partial y} p(a(\sigma, y))}{\frac{\partial}{\partial y} p(a(1-\sigma, y))} \end{aligned} \tag{41}$$

Result (39) when employed upon (41) implies

$$\begin{aligned} \lim_{y \rightarrow t} \frac{p(a(\sigma, y))^\sigma}{p(a(1-\sigma, y))^{1-\sigma}} &= \\ &= \frac{\sigma}{1-\sigma} \lim_{y \rightarrow t} \frac{p(a(\sigma, y))^\sigma}{p(a(1-\sigma, y))^{1-\sigma}} \end{aligned} \tag{42}$$

Since limit on the left hand side of (42) is identical to the one on the right hand side of (42), equation (42) demands $\sigma = 1/2$, since (42) leads to

$$1 = \frac{\sigma}{1 - \sigma} \quad (43)$$

We should also consider case with $\lim_{y \rightarrow t} \frac{p(a(1-\sigma, y))}{p(a(\sigma, y))} = 0$. We simply repeat steps (39) through (43) with ρ and $1 - \rho$ interchanged and for $\sigma < 1/2$. The result is the same, namely $\sigma = 1/2$.

Since we supposed $\sigma \neq 1/2$ and have arrived at a contradiction, we conclude that $\sigma = 1/2$.

This proves the lemma.

5. Proof of the Riemann Hypothesis

Let us consider integral representation (21). Let us define function $\chi(s)$ according to

$$\chi(s) = \zeta(s)\Gamma(s) = \int_0^{\infty} \left(\frac{1}{e^p - 1} - \frac{1}{p} \right) p^{s-1} dp \quad (44)$$

We notice that gamma function $\Gamma(s)$ is finite and nonvanishing in the critical strip. We hence conclude that functions $\chi(s)$ and $\zeta(s)$ have common zeros.

Hence chi function zeros $\rho = \sigma + it$ are situated symmetrically with respect to point $s = 1/2$. Thus it satisfies condition **A**.

Both $\Gamma(s)$ and $\zeta(s)$ are analytic in critical strip. Hence we conclude that $\chi(s)$ is analytic in critical strip.

We also notice that (44) represents a Mellin transform for s in critical strip. Hence $\chi(s)$ satisfies condition **B**.

Further, $p^\sigma (1 + \log^2 p) \frac{d}{dp} \left(\frac{1}{e^p - 1} - \frac{1}{p} \right) = p^\sigma (1 + \log^2 p) \left(\frac{1}{p^2} + \frac{1}{2} \frac{1}{1 - \cosh(p)} \right) = 0$ as $p \rightarrow 0^+$ for any $0 < \sigma < 1$, hence for any zero $\rho = \sigma + it$ of function $\chi(s)$. This is so because by L'Hospital rule $\frac{p^2}{2(1 - \cosh(p))} = -1$ as $p \rightarrow 0^+$. This way we've checked the truth of condition **C** for $\chi(s)$ as defined by (44).

Therefore we conclude that $\chi(s)$ belongs to family \mathcal{F} .

Function $\chi(s)$ being in \mathcal{F} , we conclude that nontrivial zeros $\rho = \sigma + it$ of $\chi(s)$, and therefore the very same nontrivial zeros ρ of $\zeta(s)$, satisfy lemma, i.e. $\sigma = 1/2$. This proves the Riemann hypothesis.

6. Introduction to 3-relativity

The invention of 3-relativity was motivated by the fact that Klein-Gordon equation is a second order differential equation. Since Klein-Gordon equation being co-ordinate representation of on-shell relation $E^2 = p^2 + m^2$, we are linearizing the on-shell relation.

One way to linearize on-shell relation is to do it the Dirac way. Result is Dirac equation. However, Dirac equation describes fermions only. Therefore, this paper finds

another way to linearize on-shell condition. This linearization turns to be suitable for bosons as well as for fermions.

Let us begin by first considering virtual photons. To do so, let us pay attention to Bremsstrahlung where single particle emits a photon. Let the particle of rest-mass m move with 3-impulse p and let it emit a photon. Let the particle continue to move with some 3-impulse q after emission of a photon.

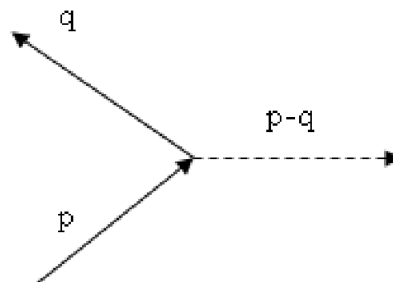


Fig. 1 *Bremsstrahlung diagram*

The conservation of impulse dictates photon to carry 3-impulse $p - q$. Such photon should have energy $p - q$ as measured in natural units. On the other hand, conservation of energy demands photon to have energy $\sqrt{p^2 + m^2} - \sqrt{q^2 + m^2}$. Therefore, if energy being conserved, emitted photon must lack some energy given its impulse. Such photon is called a virtual photon. The explanation is that virtual particles in general happen to exist only on account of borrowing the lacking energy from Dirac sea i.e. from quantum vacuum. Namely, if virtual particle borrows energy ΔE from vacuum, it may exist approximately in the mean for only $\Delta t \approx \frac{\hbar}{\Delta E}$ seconds by the virtue of Heisenberg uncertainty relation. This is quite a realistic scenario if virtual particle is absorbed by another particle while still being in existence during period of Δt . Hence, virtual particles exit Feynman diagrams and re-enter another Feynman diagram within Δt seconds. Virtual particles have to enter another diagram within Δt seconds – otherwise they would sink back into Dirac sea and thus violate conservation of 4-impulse.

Let us return back to Bremsstrahlung. Consider a particle emitting a photon. This photon exits this diagram depicting Bremsstrahlung and is therefore virtual. It is bound to enter another diagram within Δt seconds. The crucial argument is – in reality, in experiment, emitted photons do not have to interact for infinitely long time.

Consider this fact – all photons have been emitted by some particle. Every single photon in Universe once left some diagram. Therefore, all the photons are virtual. Having all the photons virtual, we are facing the enormously important question – how can electromagnetic forces be long-ranged? All the experiments done during last 200 years conform with the fact that photons have infinite range. The magnificent Maxwell theory, being a reference for every other theory in physics, predicts that photons are long-ranged. Maxwell theory implied conclusions of special theory of relativity. And yet, the very same special theory of relativity predicts that photon should be short ranged.

Let us consider photo-effect now. Photo-effect is really Bremsstrahlung with CT

inversion. Therefore, it is easy to notice that conservation of energy requires incoming photon to be virtual, whatever the nature of charged particle absorbing it.

The solution suggested by contemporary QFT to this problem is to require photo-effect to be followed by Bremsstrahlung, Fig. 2.

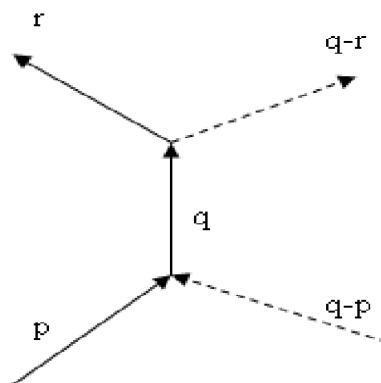


Fig. 2 Photo-effect followed by Bremsstrahlung

This way, one can blame all the virtuality on electron, and not on photons.

The way out of this puzzle was to claim the existence of real photons. These are the ones that do not enter any diagram and are free. However, when we measure a real photon, the very act of measuring the real photon is in fact interaction of a real photon with apparatus. Hence, real photon enters a diagram at the moment of measuring. Entering the diagram, it has to be virtual in order not to violate conservation of energy. So, it seems that there are no real photons. If there were real photons, then the electrons in apparatus should be virtual for a while. Virtual particles are undetectable per definitionem. The question arising is – what are we measuring, then? And what is it that enables measuring? Certainly not the undetectable virtual apparatus.

The situation is quite different if the mediator is massive. All the relativistic calculations still demand massive particle to be virtual in order to have energy conserved, but only in Bremsstrahlung as depicted in Fig. 1. The difference with respect to photons is that massive mediators are short-ranged. Massive mediators cannot originate from Bremsstrahlung depicted in Fig. 2 because such mediators are long-ranged since being real and not virtual. Therefore, massive mediators are short-ranged only with Bremsstrahlung as in Fig. 1.

These arguments are compelling enough to motivate us to find the answer to this puzzle – the puzzle of long-ranged virtual photons. This paper shows that there is another way to define special theory of relativity. We define special theory of relativity in 3-dimensional euclidean space, in contrast to 4-dimensional Minkowski space-time. This 3-relativity predicts the same results as 4-relativity as long as there are no interactions. This may seem trivial. However, the difference between 3-relativity and 4-relativity shows when considering interactions – and therefore when considering virtual particles. The results of 3-relativity show that photons do not violate energy conservation and indeed are long-ranged. This result does not stand for massive mediators, so the theory of strong

and weak interactions stands, as viewed in 3-relativity. So let us introduce 3-relativity.

7. 3-relativity

Consider following manipulation

$$E^2 = p^2 + m^2 = (|p| + im)(|p| - im) \quad (45)$$

This factorization suggests that energy E should be rewritten as

$$E = |p| + im \quad (46)$$

If we choose to have $p, m \in R$, then energy E in Eq. (46) is complex and Eq. (1) becomes

$$|E|^2 = E\bar{E} = p^2 + m^2 = (|p| + im)(|p| - im) \quad (47)$$

Suppose that we could do this for each component of 3-impulse p . Let indices $i, j, k, l = 1, 2, 3$ label spatial components of a vector. Let us define Eq. (44) to be

$$\mathcal{E}_k = |p_k| + im_k \quad (48)$$

with \mathcal{E}_k denoting the k -component of complex energy 3-vector \mathcal{E} . This equation is linearization of Klein-Gordon equation.

Let vectors p , \mathcal{E} and m be spanned by quantities q_k that satisfy $\{q_k, q_l\} = 2\delta_{kl}$. For instance, quantities q_k may be cartesian unit vectors e_i . So the energy 3-vector as defined in Eq. (48) is

$$q_k(\mathcal{E}_k - |p_k| - im_k) = 0 \quad (49)$$

assuming summation over dummy index $k = 1, 2, 3$. This equation turns into Dirac equation for $q_k = \sigma_k$ with σ^k being Pauli matrices. It also turns into an equation similar to Dirac equation, but describing not fermions but bosons, as soon as we choose mutually commuting vectors q_k , such that $[q_k, q_l] = 0$. We will discuss this in a short while. Let us for the moment mimic Dirac equation and suppose that operator for conjugate wave function similar to one given in Eq. (49) is

$$q_l(\bar{\mathcal{E}}_l - |p_l| + im_l) = 0 \quad (50)$$

with $\{q_k, q_l\} = 2\delta_{kl}$.

Multiply Eq. (50) by Eq. (49) from the left, multiply Eq. (49) by Eq. (50) from the right, and sum these two products together. The result is

$$\frac{1}{2}\{q_k, q_l\}(\mathcal{E}_k - |p_k| - im_k)(\bar{\mathcal{E}}_l - |p_l| + im_l) = 0 \quad (51)$$

where we used the fact that \mathcal{E}_k , $\bar{\mathcal{E}}_k$, p_k and m_k all mutually commute. Since $\{q_k, q_l\} = 2\delta_{kl}$ we conclude from Eq. (51)

$$(\mathcal{E}_k - |p_k| - im_k)(\bar{\mathcal{E}}_k - |p_k| + im_k) = 0 \quad (52)$$

with summation over dummy index k . When multiplied through, Eq. (52) results in

$$\bar{\mathcal{E}}_k \mathcal{E}_k - p_k p_k - m^2 = 0 \quad (53)$$

with summation over dummy k assumed and with definition

$$m_k m_k = m^2 \quad (54)$$

with summation over dummy k .

We notice that Eq. (53) represents on-shell relation

$$E^2 - p_k p_k - m^2 = 0 \quad (55)$$

with

$$\bar{\mathcal{E}}_k \mathcal{E}_k = |\mathcal{E}|^2 = E^2 \quad (56)$$

with E denoting energy as defined in Einstein's 4-relativity.

So we conclude that we found another way to define special theory of relativity. The advantage of this approach is that we are able to express energy-impulse relation in linear form now, Eq. (48). In Einstein's formulation there was only one energy E and three impulses p_k and any statement about energy E had impact on all of impulses p_k . Now, we have defined a portion of energy \mathcal{E}_k for each impulse component p_k . This allows more detailed analysis of energy-impulse relationship than the use of 4-impulse. On the other hand, we have concealed conserved stationary quantities – 4-energy E and rest mass m , and thus have lost explicit form of these constants of motion.

We can linearize not only the energy-impulse on-shell relation, but also the time-space on-shell relation – the metric. For as soon as we define complex time 3-vector \mathcal{T}_k and substitute $\mathcal{E}_k \rightarrow \mathcal{T}_k$ and $p_k \rightarrow x_k$ and $m_k \rightarrow \tau_k$ with $\tau_k \tau_k = \tau^2$ in Eqs. (48) through (53) with τ being proper time, we find that

$$\mathcal{T}_k = |x_k| + i\tau_k \quad (57)$$

leads to

$$\mathcal{T}_k \bar{\mathcal{T}}_k = x_k x_k + \tau^2 \quad (58)$$

with summation over dummy index k . Eq. (58) is the metric 4-element as soon as we define

$$\mathcal{T}_k \bar{\mathcal{T}}_k = |\mathcal{T}|^2 = t^2 \quad (59)$$

with t denoting time as defined in Einstein's 4-relativity.

We notice that metric (58) is invariant to any transformation of 3-time \mathcal{T}_k and 3-space x_k as long as Eq. (57) stands invariant. Any such transformation is Lorentz transformation for 3-relativity.

It is interesting to notice appearance of absolute values $|p|$ and $|x|$ in Eqs. (48) and (57) describing relativistic particle in 3-relativity, as well as in Eq. (11) describing quantum chaotic oscillator.

8. Virtual Particles in 3-relativity

I would like to show that there is a difference between 3-relativity and 4-relativity when describing a virtual photon.

Let us for simplicity use only one spatial dimension, say x . Consider a particle of impulse p_i and rest mass m scattering with outgoing impulse p_f whilst emitting a photon of impulse f .

In 4-relativity impulses balance out according to

$$p_i = p_f + f \quad (60)$$

Energy of incoming particle is $\sqrt{p_i^2 + m^2}$, energy of outgoing particle is $\sqrt{p_f^2 + m^2}$ and energy of a photon is f . Energy balance therefore reads

$$\sqrt{p_i^2 + m^2} = \sqrt{p_f^2 + m^2} + f \quad (61)$$

Eqs. (60) and (61) are not solvable simultaneously, so the photon must tunnel through the Dirac sea borrowing extra energy ϵ from the vacuum according to uncertainty principle

$$\epsilon \Delta t \approx \hbar \quad (62)$$

The tunneling allows photon to be in existence only for Δt seconds. Such virtual photon can traverse only a distance $\Delta x = c\Delta t$.

This scenario is unphysical. This mechanism of production of a virtual photon is not realistic because electromagnetic forces are of infinite range and are not limited to ranges of Δx .

Let us describe this situation in 3-relativity. Impulses balance as in Eq. (60). Energy of an incoming particle is $|p_i| + im$. Energy of an outgoing particle is $|p_f| + im$. Energy of a photon is $|f|$. Hence,

$$|p_i| + im = |p_f| + im + |f| \quad (63)$$

Eqs. (60) and (63) are actually identities as soon as $sign(p_i) = sign(p_f) = sign(f)$. We notice that photon is no longer virtual. Real photon now has infinite range as it should.

The situation changes if emitted particle being massive. Let the emitted particle have impulse f and mass n . Then energy should balance according to

$$|p_i| + im = |p_f| + im + |f| + in \quad (64)$$

Eqs. (60) and (64) are not solvable simultaneously because of extra mass in , so the emitted massive particle has to be virtual thus being short-ranged. Since in Standard model massive mediators should be short-ranged, we conclude that 3-relativity predicts physically acceptable results.

9. Chaotic Oscillator and Gravity

Consider a particle of rest mass m in potential $V(x)$. Its energy H in 3-relativity can be written as

$$H = p + im + V(x) \quad (65)$$

Chaotic oscillator is also a one-particle system in some potential $V(x)$ and is completely described given Hamiltonian $H = px$. Product px already contains all information on potential $V(x)$, although potential $V(x)$ does not appear in it explicitly.

Hamiltonian $H = px$ is not 3-relativistic because it does not take rest mass energy into account. It's 3-relativistic form is

$$H = px + im \quad (66)$$

This represents particle's energy and is constant of motion. Hamiltonian $H = px + im$ obviously has complex eigenvalues and is no longer hermitean. Since rest mass m being constant, quantity

$$K \equiv H - im = px \quad (67)$$

is also a constant of motion for a given orbit, and is represented by hermitean operator.

Equations (65) and (66) say that

$$px = p + V(x) \quad (68)$$

Potential $V(x)$ is therefore

$$V(x) = px - p = K - \frac{K}{x} \quad (69)$$

This is one-dimensional gravitational potential plus some constant potential K being dependent on rest mass m and is the kinetic part of energy assigned to a given orbit.

Since any Hamiltonian can be split into a sum of Hamiltonians, we are free to interpret Eq. (66) as radial part of energy of a particle in potential $V(r)$, identifying x with radial distance r in spherical co-ordinates. Thus, potential in which a particle is moving chaotically as expressed by Hamiltonian's kinetic radial part $H_r = K = p_r r$ is

$$V(r) = -\frac{K}{r} \quad (70)$$

up to constant potential K . This is gravitational potential. Therefore, gravity makes particle move chaotically following unstable orbits as dictated by Hamiltonian $H_r = p_r r$.

This is quite interesting conclusion, since it links gravity with Riemann zeta function zeros and chaos. Constant K is not a global constant, but is constant only along given orbit. This shows that this result cannot be the complete information about the given system. The point we wish to stress here is not the complete theory, but rather single interesting detail about gravity and chaotic oscillator. Complete treatment demands more thorough analysis, of course.

10. Conclusions

This paper showed that chaotic system described by Hamiltonian $H = px$ has imaginary parts of Riemann zeta function nontrivial zeros as eigenenergies, although there is still left to prove that spectrum of H consists of all of the Riemann zeta function nontrivial zeros. By proving rigorously the Riemann hypothesis by analytic methods, this paper shows that Hamiltonian $H = px$ is very likely the Hamiltonian Hilbert and Polya wrote about, namely the Hamiltonian that might serve to prove the Riemann hypothesis without purely analytic and number-theoretical methods. We further introduced reformulation of special theory of relativity by describing it over three-dimensional space rather than in Minkowski four-dimensional space-time. In this relativistic representation, the potential driving a particle to move chaotically with energy $H = px$ proves to behave as gravitational potential, thus linking theory of chaos to gravity and to Riemann zeta function nontrivial zeros.

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Gaussian Delay Models for Light Broadenings and Redshifts

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Abstract: In astronomy light emission is characterized by a frequency $\omega_0/2\pi$, a redshift z (sometimes a blueshift), a FWHM (Full Width Half Maximum) and an EW (Equivalent Width). ω_0 relates to the nature of the concerned atom or molecule, z allows to determine the speed and the distance of the body through the Hubble law, FWHM measures the wave spectral width, and EW defines a kind of SNR (Signal-to-Noise Ratio). In this paper, we show that Gaussian time delays on pure waves can theoretically explain the width of emission lines, any redshift and a floor noise which can be matched to any EW.

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1. Introduction

A star spectrogram shows a continuum with emission and absorption lines following the composition of the star and of the medium crossed by the light. A particular line is defined by a wavelength λ_0 (or a frequency $f_0 = c/\lambda_0 = \omega_0/2\pi$), a redshift parameter z (more rarely it is a blueshift), a FWHM (Full Width Half Maximum) and an EW (Equivalent Width). Roughly speaking, the quotient EW/FWHM is like the quotient between the height of the line and the value of the continuum. Then, it is similar to a SNR (Signal-to-Noise Ratio) according to signal theory.

We show in this paper that the line in a spectrogram can be modelled by two random processes **A** and **B**, the first one for the change from a pure monochromatic wave to a wave with a given bandwidth, and the second for the propagation between the transmitter body to the receiver. The last one will explain a part of the redshift and the continuum. Both

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processes refer to media with very different physical properties, for the temperatures, the pressures, densities, electrical or magnetic fields. The first process is linked to a very turbulent medium for instance the neighbourhood or the interior of a star, and the second models the transmitter-receiver way, a very long path, which can cross perturbed regions.

Then, the model takes into account the real random processes $\mathbf{A} = \{A(t), t \in \mathbb{R}\}$ and $\mathbf{B} = \{B(t), t \in \mathbb{R}\}$, which define the transmitted process $\mathbf{Z}_e = \{Z_e(t), t \in \mathbb{R}\}$ and the observed process $\mathbf{Z}_r = \{Z_r(t), t \in \mathbb{R}\}$ where

$$Z_e(t) = e^{i\omega_0(t-A(t))} \quad Z_r(t) = Z_e(t - B(t)) \quad (1)$$

$f_0 = \omega_0/2\pi$ is the emitted frequency. We assume that the processes \mathbf{A}, \mathbf{B} are stationary Gaussian processes, zero-mean (the mean values are insignificant), independent, with regular power spectra $s_A(\omega), s_B(\omega)$, standard deviations σ_A, σ_B and autocorrelation functions

$$K_A(\tau) = E[A(t)A(t-\tau)], \quad K_B(\tau) = E[B(t)B(t-\tau)]$$

such as (for \mathbf{A}) [2], [11]

$$\begin{cases} K_A(\tau) = \sigma_A^2 \rho_A(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} s_A(\omega) d\omega \\ \rho_A(\tau) = 1 - \frac{\tau^2}{2\gamma_A^2} + o(\tau^2) \end{cases} \quad (2)$$

where $E[.]$ is for the mathematical expectation (or ensemble mean). The regularity of the correlation functions $\rho_A(\tau)$ and $\rho_B(\tau)$ near the origin point implies the existence of the m.s (mean-square) derivatives $A'(t), B'(t)$. Furthermore, the processes \mathbf{A}', \mathbf{B}' are Gaussian [2]. We note that four parameters are used, $\sigma_A, \sigma_B, \gamma_A, \gamma_B$, but we will see that only orders of magnitude are necessary to characterize the processes.

Random processes like \mathbf{Z}_e and \mathbf{Z}_r were theoretically studied in [5], and used to model varied schemes of propagation, in the acoustics or in the electromagnetic field, from audio-frequencies to visible frequencies [6], [7], [8], [9], [10], and particularly to fit varied shapes of power spectra. But, in the cited papers, only one random process was sufficient to fit the physical studied problems, and not two as in the present paper.

The pure monochromatic wave $e^{i\omega_0 t}$ summarizes the emission of a large number of particles of the same kind with different locations and celerities. Most authors give the consequences of this synthetic formula without references to the physical reasons which lead to it. The model is justified because the consequences agree with properties which we have to explain. In this paper, the studied models are not justified by considerations about the particles which create the wave, or by properties of the crossed medium, but a posteriori by the adequacy of the observed spectra. Moreover, these models highlight the obvious need of a separation in two consecutive operations, the first one for the emitted wave (a widened version of $e^{i\omega_0 t}$), the second one for the propagation. Both used processes \mathbf{A} and \mathbf{B} have the same properties, with a sufficient flexibility which allow to explain what is observed in devices. As explained above, they are able to explain many other phenomena.

In the next section, we give the properties of randomly delayed processes. After that, we will explain the properties to give to \mathbf{A} and \mathbf{B} in order to obey the following properties. \mathbf{Z}_e is a widening of the line at ω_0 in the reference fixed to the transmitter. The propagation of this wave ends in \mathbf{Z}_r , which is a version redshifted of \mathbf{Z}_e , accompanied by a continuum. When both systems are in relative motion, the usual Doppler redshift (or blueshift) has to be added.

2. Clock Changes Properties

1) Assume that the independent processes $\mathbf{C} = \{C(t), t \in \mathbb{R}\}$, $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ are characterized by (\mathbf{C} is a real process)

$$\begin{cases} K_Z(\tau) = \sigma_Z^2 \rho_Z(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} s_Z(\omega) d\omega \\ \psi_C(\omega) = E[e^{-i\omega C(t)}] & \phi_C(\tau, \omega) = E[e^{-i\omega[C(t)-C(t-\tau)]] \end{cases} \quad (3)$$

$K_Z(\tau) = E[Z(t)Z^*(t-\tau)]$ is the autocorrelation function, and the superscript $*$ is for the complex conjugate. $s_Z(\omega)$ is the spectral density of \mathbf{Z} , and possibly defined in a distribution sense (when pure spectral lines appear). Actually, $\psi_C(\omega)$ and $\phi_C(\tau, \omega)$ are the characteristic functions (in the probability sense) of the random variables (r.v) $C(t)$ and $C(t) - C(t - \tau)$. They define perfectly the probability laws of these r.v. Their shape implies a stationarity for \mathbf{C} stronger than the second order (i.e. the stationarity of the correlation functions).

If we define the new process $\mathbf{U} = \{U(t), t \in \mathbb{R}\}$ by

$$U(t) = Z(t - C(t))$$

we have the decomposition $\mathbf{U} = \mathbf{G} + \mathbf{V}$ where \mathbf{G} and \mathbf{V} are stationary, uncorrelated and (see the appendix and [5])

$$\begin{cases} U(t) = G(t) + V(t) \\ s_G(\omega) = [|\psi_C|^2 s_Z](\omega) \\ K_V(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} [\phi_C(\tau, \omega) - |\psi_C(\omega)|^2] s_Z(\omega) d\omega. \end{cases} \quad (4)$$

The process \mathbf{G} is defined as the output of a LIF (Linear Invariant Filter) with input \mathbf{Z} and with complex gain (or transfer function or spectral response) $\psi_C(\omega)$. The second equality is a consequence of the wellknown Wiener-Lee theorem [11]. \mathbf{V} and \mathbf{Z} are uncorrelated and then so are \mathbf{V} and \mathbf{G} . The last equality of (4) allows to compute $s_V(\omega)$ by a Fourier transform inversion. The equations in (4) are true even if \mathbf{Z} is a pure monochromatic line. In this case, \mathbf{G} is a weakened and delayed version of \mathbf{Z} . In all cases, \mathbf{V} will have a finite spectral density when (rapidly enough)

$$\lim_{\tau \rightarrow \infty} \phi_C(\tau, \omega) = |\psi_C(\omega)|^2.$$

Practically, this means that the r.v. $C(t)$ and $C(t - \tau)$ become independent when the lag τ is large enough. The random delay \mathbf{C} leads to a linear filtering \mathbf{G} of \mathbf{Z} , and creates an uncorrelated “noise” \mathbf{V} . The spectra of both components and the relative powers depend on the probability laws defining \mathbf{C} , through the functions $\psi_C(\omega)$ and $\phi_C(\tau, \omega)$. We note that the sum of the powers of \mathbf{G} and \mathbf{V} remains equal to $K_Z(0)$, the power of \mathbf{Z} . Then, the transformation of \mathbf{Z} in \mathbf{U} is done at constant power.

2) Now, assume that \mathbf{C} is a (derivable) Gaussian process. In this case, we have [2]

$$\begin{cases} \psi_C(\omega) = \exp[-im_C\omega - (\omega\sigma_C)^2/2] \\ \phi_C(\tau, \omega) = \exp[-\omega^2\sigma_C^2(1 - \rho_C(\tau))] \\ \sigma_C^2\rho_C(\tau) = \text{Cov}[C(t), C(t - \tau)] \\ \rho_C(\tau) = 1 - \frac{\tau^2}{2\gamma_C^2} + o(\tau^2) \end{cases} \quad (5)$$

$m_C, \sigma_C, \rho_C(\tau)$ are the mean, the standard deviation and the correlation coefficient of \mathbf{C} , and γ_C measures the celerity of variations of $C(t)$ because

$$\text{E}[C'^2(t)] = \left[\frac{\sigma_C}{\gamma_C}\right]^2 \quad (6)$$

where $C'(t)$ is the m.s (mean-square) derivative of $C(t)$. We apply these results to the processes \mathbf{Z}_e and \mathbf{Z}_r defined by (1).

3. The Process \mathbf{Z}_e

3.1 Decomposition of \mathbf{Z}_e

The transmitted wave \mathbf{Z}_e is defined in (1) by $Z_e(t) = e^{i\omega_0(t-A(t))}$. The decomposition in the section 2 can be written as

$$\mathbf{Z}_e = \mathbf{G}_e + \mathbf{V}_e$$

From (4) and (5), \mathbf{G}_e is a pure monochromatic wave with ($\delta(\omega)$ is the “Dirac function”)

$$s_{G_e}(\omega) = e^{-\omega_0^2\sigma_A^2}\delta(\omega - \omega_0).$$

To eliminate \mathbf{G}_e , which is purely monochromatic, it suffices that the exponent $\omega_0\sigma_A$ will be large enough (ω_0 is around 3.10^{15}s^{-1} for the visible light). In the same time, it is not difficult to prove that the \mathbf{V}_e -spectrum can be confused with a Gaussian (see for instance [8], [10]) with standard deviation $\omega_0\sigma_A/\gamma_A$:

$$s_{V_e}(\omega) \simeq \frac{\gamma_A}{\omega_0\sigma_A\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\gamma_A}{\omega_0\sigma_A}\right)^2(\omega - \omega_0)^2\right]. \quad (7)$$

Then, it is possible to give values to the parameters σ_A, γ_A so that \mathbf{Z}_e will be confused with a Gaussian line at ω_0 with an arbitrary width $\sigma_{Z_e} = \omega_0\sigma_A/\gamma_A$.

It is possible to adopt an opposite strategy, giving a small value to $\omega_0\sigma_A$ and a large value to $\omega_0\sigma_A/\gamma_A$ (firstly fitting σ_A and secondly γ_A). As a result, the pure monochromatic line would be retained, surrounding a process with a spectrum as flat as we want (even if the Gaussian character is questionable for small $\omega_0\sigma_A$).

Either way, from (4) and (5), we can deduce $K_{V_e}(\tau) = E[V_e(t) V_e^*(t - \tau)]$

$$\begin{cases} K_{V_e}(\tau) = e^{i\omega_0\tau - \omega_0^2\sigma_A^2} \left[e^{\omega_0^2\sigma_A^2\rho_A(\tau)} - 1 \right] \\ \omega_{V_e} = \omega_0 \quad \sigma_{V_e} = \frac{\omega_0\sigma_A}{\gamma_A} \left(1 - e^{-\omega_0^2\sigma_A^2} \right)^{-1/2} \end{cases}$$

where ω_{V_e} is the gravity centre of $s_{V_e}(\omega)$, and σ_{V_e} the half-width (i.e. the standard deviation of the probability law proportional to $s_{V_e}(\omega)$). Clearly, $\omega_0\sigma_A$ rules the behavior of the pure line \mathbf{G}_e , and $\omega_0\sigma_A/\gamma_A$ the spreading of the continuum \mathbf{V}_e .

3.2 Example

When lines are not too large, a Gaussian power spectrum in $f = \omega/2\pi$ (in frequency) is also Gaussian in $\lambda = c/f$ (in wavelength). Of course, it is an approximation. The relation which links the half-widths σ (the standard deviation of the normalized Gaussian) in ω , and σ' in λ , is

$$\sigma' = \frac{\lambda_0^2}{2\pi c} \sigma$$

where λ_0 is the wavelength (the centre of its power spectrum). Furthermore, in the Gaussian case, the FWHM is close to 2.35 times the standard deviation, because $\exp[-x^2/2] = 1/2 \iff x = 1.177$.

For the Balmer H_α , $\lambda_0 = 6563\text{\AA}$, $\omega_0 = 2872.10^{12}\text{s}^{-1}$. A Gaussian ray with a FWHM of 10\AA (in λ) is equivalent to a standard deviation of 19.10^{11}s^{-1} (in ω). If the pure line \mathbf{G}_e has disappeared, we have to verify, from (7)

$$\frac{\omega_0\sigma_A}{\gamma_A} = 19.10^{11}\text{s}^{-1}$$

or equivalently

$$\frac{\sigma_A}{\gamma_A} \simeq 65.10^{-5}.$$

The pure monochromatic line has disappeared for instance when $\omega_0\sigma_A \simeq 3$, which ends in $\gamma_A \simeq 16.10^{-13}\text{s}$, $\sigma_A \simeq 10^{-15}\text{s}$ (half a period of H_α). For these values, the Gaussian part \mathbf{V}_e holds more than 99.98% of the power of \mathbf{Z}_e . The power of \mathbf{V}_e in an interval of width 0.2\AA around λ_0 (where is \mathbf{G}_e), is close to 0.02, which has to be compared with 0.00012, the power of \mathbf{G}_e . Clearly, \mathbf{G}_e is flooded in \mathbf{V}_e and cannot be seen. The contrast between both powers increases with σ_A , because $\exp[-\omega_0^2\sigma_A^2]$ decreases rapidly. When \mathbf{A} is a baseband process (its power spectrum is around the origin), $2/\gamma_A \simeq 13.10^{11}\text{s}^{-1}$ is an order of magnitude for the frequency bound of \mathbf{A} . Then, $A(t)$ has slow variations compared with the wave $Z_e(t)$.

The situation is very different when $\omega_0\sigma_A$ is smaller. For instance, when $\omega_0\sigma_A = 1$, the powers of \mathbf{G}_e and \mathbf{V}_e are equal to 0.37 and 0.63. For $\gamma_A < 14 \cdot 10^{-15}$ s, and assuming a Gaussian for $s_{V_e}(\omega)$, the power of \mathbf{V}_e in an interval of 2.2\AA around ω_0 is less than 0.24, and then the pure line will appear above the continuum.

4. The Process \mathbf{Z}_r

4.1 Decomposition of \mathbf{Z}_r

The received wave \mathbf{Z}_r is defined by $Z_r(t) = Z_e(t - B(t))$. We assume that \mathbf{Z}_e has a Gaussian spectrum profile of parameter $\sigma_{Z_e} = \omega_0\sigma_A/\gamma_A$ ($s_{Z_e}(\omega)$ is a Gaussian probability density with a standard deviation σ_{Z_e} such as $\text{FWHM}(\mathbf{Z}_e) = 2.35\sigma_{Z_e}$). Equivalently, we assume that, in the decomposition $\mathbf{Z}_e = \mathbf{G}_e + \mathbf{V}_e$, the pure monochromatic part \mathbf{G}_e has disappeared, and \mathbf{Z}_e is identified with \mathbf{V}_e (see the section 3 above). \mathbf{B} is a Gaussian process, with parameters m_B (which could be a propagation mean-time and which is useless), σ_B (which rules the range of \mathbf{B}), and γ_B (which gives insights for the celerity of variations of \mathbf{B}). Following the developments in the section 2, we write

$$\mathbf{Z}_r = \mathbf{G}_r + \mathbf{V}_r$$

and we study separately both processes \mathbf{G}_r and \mathbf{V}_r .

We have to show that \mathbf{B} can be defined so that \mathbf{G}_r becomes a redshifted version of \mathbf{Z}_e and so that \mathbf{V}_r represents a part of the continuum which accompanies the line \mathbf{Z}_e . \mathbf{V}_r results from the degradations due to the very long trajectory of \mathbf{Z}_e . We will see that this component contains the most part of the power, but on a very large frequency band, which likely contains other emitted lines than \mathbf{Z}_e . Consequently and reciprocally, the continuum is likely constituted by the sum of several processes like \mathbf{V}_r , due to the deteriorations of the spectral lines.

4.2 The Redshift

From (4), we have

$$s_{G_r}(\omega) = [|\psi_B|^2 s_{Z_e}](\omega) \quad (8)$$

where, using (5) and taking the spectra of \mathbf{Z}_e and \mathbf{V}_e to be the same

$$\begin{cases} \psi_B(\omega) = \exp[-im_B\omega - (\omega\sigma_B)^2/2] \\ s_{Z_e}(\omega) = \frac{1}{\sigma_{Z_e}\sqrt{2\pi}} \exp[-(\omega - \omega_0)^2/2\sigma_{Z_e}^2] \\ \sigma_{Z_e} = \omega_0\sigma_A/\gamma_A. \end{cases} \quad (9)$$

(8) and (9) lead to

$$\begin{cases} s_{G_r}(\omega) = \frac{1}{\sigma_{Z_e} \sqrt{2\pi}} \exp \left[-\omega_{G_r} \omega_0 \sigma_B^2 - (\omega - \omega_{G_r})^2 / 2\sigma_{G_r}^2 \right] \\ \omega_{G_r} = \omega_0 / (1 + 2\sigma_{Z_e}^2 \sigma_B^2) \quad \sigma_{G_r} = \sigma_{Z_e} / \sqrt{1 + 2\sigma_{Z_e}^2 \sigma_B^2} \end{cases} \quad (10)$$

where ω_{G_r} and σ_{G_r} are the centre of gravity (the mean) and the standard deviation (the half-width) of the probability density proportional to $s_{G_r}(\omega)$. We see that the power spectrum $s_{G_r}(\omega)$ of \mathbf{G}_r is still Gaussian but it is a weakened, shifted and narrowed version of $s_{Z_e}(\omega)$. Because $\omega_{G_r} < \omega_0$, \mathbf{G}_r appears as a redshifted version of \mathbf{Z}_e with parameter

$$z = \frac{\lambda_{G_r} - \lambda_0}{\lambda_0} = \frac{\omega_0 - \omega_{G_r}}{\omega_{G_r}} = 2\sigma_{Z_e}^2 \sigma_B^2. \quad (11)$$

Theoretically, z can reach any value, fitting the value(s) of σ_{Z_e} and/or σ_B . But, we know that \mathbf{G}_r is the result of a LIF with input \mathbf{Z}_e and complex gain $\psi_B(\omega)$ (see the section 2). The filter highlights the part of $s_{Z_e}(\omega)$ around ω_{G_r} . The accuracy of the formulae in (10) and (11) depends on the accuracy of (9). Obviously, the reliability of these formulae decreases when $\omega_0 - \omega_{G_r}$ increases, as the confidence with the formula (9) giving $s_{Z_e}(\omega)$ for large values of $|\omega - \omega_{G_r}|$. Today, large deviations are a part of the probability calculus which is particularly studied. This means that people gives a great importance to places of very weak probabilities. Nevertheless, it seems reasonable to only keep the formula (11) for low values of z . Even if the Gaussian character is questionable for the tails of distributions, it is obvious that these tails exist. They can be raised, for instance using a Cauchy law instead of a Gaussian law. Consequently, we can think that random delays allow large z , but with a value different of (11).

Because $\sigma_{G_r} < \sigma_{Z_e}$, the wave is narrowed. For low values of z , we have

$$\frac{\sigma_{G_r} - \sigma_{Z_e}}{\sigma_{Z_e}} \simeq -\frac{z}{2}$$

which seems beyond measurements. The weakening of the wave is mainly given in (10) by the coefficient

$$\exp \left[-\omega_{G_r} \omega_0 \sigma_B^2 \right] = \exp \left[-\frac{\omega_0^2 \sigma_B^2}{1 + 2\sigma_{Z_e}^2 \sigma_B^2} \right] = \exp \left[-\frac{\omega_0^2 \sigma_B^2}{1 + z} \right]. \quad (12)$$

For small z , it is a function strongly decreasing with σ_B . If \mathbf{B} is really a propagation time, it is reasonable to assume that σ_B varies linearly with the square root of the distance (it is the application of the Beer-Lambert law).

It is convenient to characterize the line \mathbf{Z}_e by its relative half-width $\mu = \sigma_{Z_e} / \omega_0$. μ measures the purity of the wave ($\mu = 0$ is for the ideal pure monochromatic wave). If we write (12) as

$$\exp \left[-\omega_{G_r} \omega_0 \sigma_B^2 \right] = \exp \left[-\frac{1}{2\mu^2} \frac{z}{1 + z} \right] \quad (13)$$

we see that too large values of z/μ^2 seem suppress \mathbf{G}_r which is not what we look for. But, the power of \mathbf{G}_r is compared to 1, the power of \mathbf{Z}_e . If we start from a larger power

for \mathbf{Z}_e , it can be possible to give to \mathbf{G}_r a non negligible power even if the weakening is strong. Moreover, what is important is the relative distribution of the powers of both components \mathbf{G}_r and \mathbf{V}_r . A weak power can be viewed if it is in the frequency band of the observer, and a strong power can be ignored in the opposite case, or when distributed over a too large frequency band. Finally, the power P_{G_r} of \mathbf{G}_r is obtained by integrating $s_{G_r}(\omega)$ on \mathbb{R} . We find

$$P_{G_r} = \frac{1}{\sqrt{1+z}} \exp \left[-\frac{1}{2\mu^2} \frac{z}{1+z} \right]. \quad (14)$$

4.3 Example

Let us consider a line H_α ($\lambda_0 = 6563\text{\AA}$, $\omega_0 = 2872.10^{12}\text{s}^{-1}$). A FWHM= 16\AA corresponds to $\sigma_{Z_e} = 3.10^{12}\text{s}^{-1}$ and $\mu = 10^{-3}$. The following table gives corresponding values of z , $10 \log P_{G_r}$ (P_{G_r} in dB) and σ_B (in 10^{-16}s).

$$\begin{cases} z & 10^{-6} & 5.10^{-6} & 10^{-5} & 5.10^{-5} & 10^{-4} & 5.10^{-4} & 10^{-3} \\ 10 \log P_{G_r} & -2.2 & -11 & -22 & -110 & -220 & -1100 & -2200 \\ \sigma_B & 2.4 & 5.3 & 7.5 & 17 & 24 & 53 & 75 \end{cases}$$

The extreme values of z correspond to celerities of 0.3km.s^{-1} and 300km.s^{-1} . We remark that 22.10^{-16}s is the period of the wave, and corresponds to the value of σ_B for $z = 10^{-4}$. Then, small random variations of transit times suffice for explaining non-Doppler redshifts. In dB, P_{G_r} is a linear function of z , and is related to a unit power of P_{Z_e} .

The relative half-width of the wave $\mu = \sigma_{Z_e}/\omega_0$ can be larger than 10^{-3} . Values in the order of 0.005 are common for Wolf-Rayet stars [4], and 0.01 for quasars [12]. In the last case, the table above is still correct, multiplying the values of z by 100. Then, it is possible to construct non-Doppler redshifts, with an appreciable power P_{G_r} (compared to 1). For instance we have together

$$\mu = 0.01, \quad z = 10^{-3}, \quad P_{G_r} = 0.0063$$

where the value of z corresponds to a celerity of 300km.s^{-1} .

4.4 The Blueshift

The power spectrum of the second part \mathbf{V}_r of \mathbf{Z}_r is defined by (using (4))

$$K_{V_r}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} [\phi_B(\tau, \omega) - |\psi_B(\omega)|^2] s_{Z_e}(\omega) d\omega.$$

where $K_{V_r}(\tau) = E[V_r(t) V_r^*(t - \tau)]$. Because the Gaussian character of \mathbf{B} , we have (see (5))

$$\begin{cases} K_{V_r}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau - \sigma_B^2 \omega^2} \left[e^{\sigma_B^2 \omega^2 \rho_B(\tau)} - 1 \right] s_{Z_e}(\omega) d\omega \\ s_{V_r}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K_{V_r}(\tau) e^{-i\omega\tau} d\tau. \end{cases} \quad (15)$$

If $\rho_B(\tau)$ is known, a Fourier transform inversion gives the spectrum $s_{V_r}(\omega)$. For large enough values of $\omega_0\sigma_B$, this spectrum can be identified with a Gaussian [8], [10]. In all cases, we have (using Fourier transform derivatives)

$$\begin{cases} \omega_{V_r} = -i \frac{K'_{V_r}(0)}{K_{V_r}(0)} \\ \sigma_{V_r}^2 = -\frac{K''_{V_r}(0)}{K_{V_r}(0)} - \omega_{V_r}^2 \end{cases}$$

where ω_{V_r} is the gravity centre and σ_{V_r} the half-width of $s_{V_r}(\omega)$ (i.e. the standard deviation of the probability law proportional to $s_{V_r}(\omega)$). Elementary algebra leads to ($\mu = \sigma_{Z_e}/\omega_0$ is the relative half-width of \mathbf{Z}_e)

$$\begin{cases} \omega_{V_r} = \omega_0 \frac{1-P_{G_r}(1+z)^{-1}}{1-P_{G_r}}, & P_{G_r} = \frac{1}{\sqrt{1+z}} \exp\left[-\frac{1}{2\mu^2} \frac{z}{1+z}\right] \\ \sigma_{V_r}^2 = \frac{1+\mu^2}{1-P_{G_r}} \left(\frac{\omega_0\sigma_B}{\gamma_B}\right)^2 + R \\ R = \frac{\omega_0^2}{1-P_{G_r}} \left[1 + \mu^2 - \frac{P_{G_r}}{1+z} \left(\frac{1}{1+z} + \mu^2\right)\right] - \omega_{V_r}^2 \end{cases} \quad (16)$$

$P_{G_r} = K_{G_r}(0) = 1 - K_{V_r}(0)$ is the power of \mathbf{G}_r and $P_{V_r} = 1 - P_{G_r}$ is the power of \mathbf{V}_r . From the first equality of (16), we deduce that $\omega_{V_r} > \omega_0$ and then \mathbf{V}_r is blueshifted by the quantity

$$z' = \frac{\omega_0 - \omega_{V_r}}{\omega_{V_r}} = -P_{G_r} \frac{z}{1+z} \quad (17)$$

The blueshift parameter z' is negligible when z and P_{G_r} are together small. If P_{G_r} and P_{V_r} are in the same order of magnitude, z and z' have the same property, and the following relation links the gravity centres

$$\omega_{G_r} P_{G_e} + \omega_{V_r} (1 - P_{G_r}) = \omega_0.$$

The term R in (16) is positive because it does not depend on γ_B which is a quantity as large as we want. Moreover, for small values of z, P_{G_r}, μ in front of 1, we have

$$\sigma_{V_r}^2 \cong \left(\frac{\omega_0\sigma_B}{\gamma_B}\right)^2 + \sigma_{Z_e}^2. \quad (18)$$

4.5 The SNR

γ_B does not intervene in the properties of \mathbf{G}_r , and allows to give arbitrary values to σ_{V_r} (above σ_{Z_e}). We know that γ_B is related to the celerity of the variations of \mathbf{B} by (see the section 2)

$$E[B'^2(t)] = \left[\frac{\sigma_B}{\gamma_B}\right]^2.$$

which can be adjusted to any value of σ_{V_r} above σ_{Z_e} . In particular, small values of γ_B lead to large values of σ_{V_r} , so that \mathbf{V}_r will have a very flat spectrum. Whatever the weakness of \mathbf{G}_r (but its half-width is smaller than this of \mathbf{Z}_e), it is possible to take γ_B

so that $s_{G_r}(\omega_{G_r}) > k s_{V_r}(\omega_{G_r})$, for any $k > 0$. In this circumstance, \mathbf{G}_r will appear as a line which dominates \mathbf{V}_r , and even when the ratio of the powers P_{G_r}/P_{V_r} is very small. To measure the visibility of \mathbf{G}_r , we can define a Signal-to-Noise Ratio (SNR) by

$$\text{SNR} = \frac{s_{G_r}(\omega_{G_r})}{s_{V_r}(\omega_{G_r})}$$

For small γ_B (to have large σ_{V_r}), we can take $s_{V_r}(\omega_{G_r})$ and $s_{V_r}(\omega_{V_r})$ to be the same, and then, for Gaussian $s_{V_r}(\omega)$ and small P_{G_r}, z , in front of 1

$$\text{SNR} \simeq \frac{\sigma_B}{\gamma_B \mu} P_{G_r}. \quad (19)$$

This formula is for highlighting a redshifted line \mathbf{G}_r in a surrounding “noise” \mathbf{V}_r . To do this, we see that it is possible to fit the parameters of the wave \mathbf{Z}_e and \mathbf{B} to any wanted value of SNR. The comparison between both spectra can be made through the Equivalent Width (EW). Approximately, we can admit that $\text{EW}/\text{FWHM} \approx \text{SNR}$ when $s_{V_r}(\omega)$ has slow variations in the neighbourhood of ω_{G_r} . Of course, it is possible that other processes, linked to other spectral lines or not, are added to \mathbf{V}_r , which will change the value of the SNR.

To observe a redshift, a sufficient SNR is required to highlight \mathbf{G}_r with respect to \mathbf{V}_r . The parameter γ_B allows to perform this operation. The opposite operation was done in the section 3, where \mathbf{V}_e was highlighted and \mathbf{G}_e neglected, to give a width to the transmitted wave. But the media which produce the wave and which propagate it have opposite properties.

4.6 Example

Assume that the emitted wave \mathbf{Z}_e is the C_{IV} at $\lambda_0 = 5808 \text{ \AA}$ ($\omega_0 = 3245.10^{12} \text{ s}^{-1}$) with $\text{FWHM} = 70 \text{ \AA}$, or $\mu = 0.005$ [4]. When $z = 10^{-4}$ (equivalent to 30 km.s^{-1}), we have $P_{G_r} = 0.135$, $P_{V_r} = 0.865$, $z' = -13.10^{-6}$ (equivalent to -4 km.s^{-1}). Because $P_{G_r}/\mu = 27$, it suffices to take σ_B/γ_B larger than few 0.05 to have $\text{SNR} > 1$, and then to highlight \mathbf{G}_r . But, for small values of σ_B/γ_B , the spectra of \mathbf{G}_r and \mathbf{V}_r will be blended so that the redshift (and the blueshift) will be not distinguishable. For $z = 3.10^{-4}$ (equivalent to 100 km.s^{-1}), $P_{G_r} = 0.0025$, which is a noticeable value, though $P_{V_r} \simeq 1$. Then, it seems possible to widen \mathbf{V}_r so that \mathbf{G}_r will be seen. Values in the order of $z = 10^{-3}$ (equivalent to 300 km.s^{-1}) seem difficult to reach, because the corresponding value $P_{G_r} = 2.10^{-9}$ is very small, but any value of z is theoretically possible though the formula (11) is questionable, because it depends on distribution tails. The power is in \mathbf{V}_r and can be spread or not, following the value of the parameter γ_B . So, we can highlight the line or not. If not, the random delay \mathbf{B} will transform the line in a floor noise (the lack of rays is a particularity of the BL-Lac. objects [1]). This example is favourable to the theory developed above, but the FWHM value is well adapted. In most cases, we will encounter lower values of FWHM, which are less favourable to high redshifts. Moreover, this source of spectral changes has to be added or subtracted to other phenomena.

5. Remarks

1) We may be interested in probability distributions of the processes $\mathbf{Z}_e, \mathbf{Z}_r \dots$. The Gaussian character of \mathbf{A} can be used to compute the laws. Moreover, we know that $\omega_0 A(t)$ (modulo 2π) is uniformly distributed when $\omega_0 \sigma_A$ is large enough (few units). If we want to tend to a Gaussian character, it suffices to consider a large enough sum of independent waves, so that a version of the central limit theorem could be applied. Indeed, the spectral properties which are developed in this paper are not modified.

2) \mathbf{Z}_r can be written in the shape

$$Z_r(t) = \exp [i\omega_0 \{t - B(t) - A(t - B(t))\}]$$

which is different from $\exp[i\omega_0 (t - A(t) - B(t))]$. This last process has the properties of \mathbf{Z}_e and not of \mathbf{Z}_r . It is a mixing of a pure spectral line at ω_0 , added to a process with a continuous spectrum centered at the same frequency. As soon noted, \mathbf{Z}_r models the physical result of two stages, the first one which starts from a pure wave and the second which modifies the result (and which may be due to a propagation). For having a realistic model, it seems that the parameters of the delays \mathbf{A} and \mathbf{B} have to be chosen in different ranges. For example, $\omega_0 \sigma_A$ has to be larger than few units, to eradicate the pure line, but $\omega_0 \sigma_A / \gamma_A$ gives the width of the wave which propagates (for instance corresponding to some Å). If the model has to end in a redshift, $\omega_0 \sigma_A \sigma_B$ defines its parameter z . The last parameter γ_B rules the signal-to-noise ratio, which has a meaning close to an equivalent width.

Moreover, it is not difficult to study non-Gaussian clock changes [5], which may lead to very different results. For instance, it is possible to multiply the number of pure rays, using discrete probability laws (like in the Stark and Zeeman effects).

3) Weakenings and redshifts can be explained directly from random delays on processes with continuous spectra [10], but other models can be built [13]. The model studied in this paper is able to explain various spectra of electromagnetic waves, encountered in laser propagation, radar backscattering, HF propagation, or in acoustics [6], [7], [8], [9]. In most cases, they have a common property coming from the application of the Beer-Lambert law. As a consequence, the characteristic function $\psi(\omega)$ of the delay has to belong to the class of infinitely divisible distributions [3].

Conclusion

This paper addresses the problem of modelling the behavior of a pure monochromatic line from the emission by an atom (for instance) up to its reception in some apparatus, taking into account changes due to firstly the medium surrounding the emitter, and the medium of propagation. Starting from a pure monochromatic wave $e^{i\omega_0 t}$, a Gaussian delay \mathbf{A} allows to create a wave \mathbf{Z}_e with an arbitrary spectral width, which value depends on the standard deviation σ_A / γ_A of \mathbf{A}' , the derivative of \mathbf{A} . Actually, we have

$\text{FWHM}(\mathbf{Z}_e) = 2.35\omega_0\sigma_A/\gamma_A$, when $\omega_0\sigma_A$ is large enough to suppress the pure part of the wave. The propagation of \mathbf{Z}_e is modelled by a second Gaussian delay \mathbf{B} . We show that a redshifted component can be highlighted in the result \mathbf{Z}_r , above a continuum (the second component). The respective values of both components are defined by a SNR (or a EW). Other situations can be modelled, for instance the creation of a blueshift, and also a blend of redshifted and blueshifted components, but these circumstances seem anecdotal. Though the mathematical model allows large values of the redshift (or blueshift) parameter, the reliability of the formulae is doubtful for strong deviations. Also, it is not difficult to modify the model by adding other spectral shifts (like Doppler) to reach larger values of z . Conversely, this paper shows that the conversion of a spectral shift in a celerity is more risky with broad lines than with narrow lines.

Appendix

With the notations of the section 2 and using conditional mathematical expectations, we have

$$\begin{cases} \mathbb{E}[U(t)U^*(t-\tau)] = \mathbb{E}\left[\int_{-\infty}^{\infty} e^{i\omega(\tau-C(t)+C(t-\tau))} s_Z(\omega) d\omega\right] = \dots \\ \dots \int_{-\infty}^{\infty} e^{i\omega\tau} \phi_C(\tau, \omega) s_Z(\omega) d\omega \\ \mathbb{E}[U(t)Z^*(t-\tau)] = \mathbb{E}\left[\int_{-\infty}^{\infty} e^{i\omega(\tau-C(t))} s_Z(\omega) d\omega\right] = \dots \\ \dots \int_{-\infty}^{\infty} e^{i\omega\tau} \psi_C(\omega) s_Z(\omega) d\omega. \end{cases} \quad (20)$$

If \mathbf{G} is the output of the LIF of complex gain $\psi_C(\omega)$ and input \mathbf{Z} , we have also, by the Wiener-Lee relations [11]

$$\mathbb{E}[G(t)Z^*(t-\tau)] = \int_{-\infty}^{\infty} e^{i\omega\tau} \psi_C(\omega) s_Z(\omega) d\omega.$$

Consequently, defining \mathbf{V} by $\mathbf{U}=\mathbf{G}+\mathbf{V}$, we deduce that $\mathbb{E}[V(t)Z^*(t-\tau)] = 0$, which implies $\mathbb{E}[V(t)G^*(t-\tau)] = 0$, whatever $\tau \in \mathbb{R}$. Then, both components \mathbf{V} and \mathbf{G} are uncorrelated and

$$\mathbb{E}[V(t)V^*(t-\tau)] = \int_{-\infty}^{\infty} e^{i\omega\tau} [\phi_C(\tau, \omega) - |\psi_C(\omega)|^2] s_Z(\omega) d\omega.$$

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Eigenfunctions of Spinless Particles in a One-dimensional Linear Potential Well

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Abstract: In the present paper, we work out the eigenfunctions of spinless particles bound in a one-dimensional linear finite range, attractive potential well, treating it as a time-like component of a four-vector. We show that the one-dimensional stationary Klein-Gordon equation is reduced to a standard differential equation, whose solutions, consistent with the boundary conditions, are the parabolic cylinder functions, which further reduce to the well-known confluent hypergeometric functions.

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1. Introduction

With ever-increasing applicability of relativistic wave equations in nuclear physics and other areas, the relativistic bound state solutions of the Klein-Gordon and Dirac equations for various potentials has drawn the attention of researchers. Setting apart the mathematical complexity and computational difficulty, there are certain unresolved questions in relativistic theory for treating a general potential. In fact, the way of incorporating a general potential is not unambiguously defined.

In literature, several authors [1 - 5] have addressed the bound states of various kinds of linear potential. While Chiu [6] has examined the quarkonium systems with the regulated linear plus Coulomb potential in momentum space, Deloff [7] has used a semi-spectral

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Chebyshev method for numerically solving integral equations and has applied the same to the quarkonium bound state problem in momentum space.

In recent years, Rao and Kagali [8 - 10] have analysed the bound states of spin-half and spin-zero particles in a screened Coulomb potential, having a linear behaviour near the origin and shown the existence of genuine bound states. Very recently, Rao and Kagali[11] have reported on the bound states of a non-relativistic particle in a finite, short-range linearly rising potential, envisaged as a quark-confining potential. In the present paper, we explore the relativistic bound states of spinless particles in the one-dimensional linear potential by considering the celebrated Klein-Gordon equation.

2. The Klein-Gordon Equation with the Linear Potential

Since the early days of quantum mechanics, the relativistic investigation of various one-dimensional systems is considered to be important. The Klein-Gordon equation which essentially describes spin zero particles like the pions and kaons, is a second order wave equation in space and time and indeed a Lorentz invariant. Presently, we explore the solutions of the stationary Klein-Gordon equation with the linear potential well, treating it as a time-like component of a four-vector.

The one-dimensional time-independent form of the Klein-Gordon equation for a free particle of mass ' m' ', is

$$\left[\frac{d^2}{dx^2} + \frac{E^2 - m^2 c^4}{c^2 \hbar^2} \right] \psi(x) = 0 \quad (1)$$

For a general potential $V(x)$, treated as the fourth component of a Lorentz-vector, this equation takes the form[12]

$$\left[\frac{d^2}{dx^2} + \frac{(E - V(x))^2 - m^2 c^4}{c^2 \hbar^2} \right] \psi(x) = 0. \quad (2)$$

Thus for the potential $V(x)$, in the vector-coupling scheme, the above equation may be written as

$$\left[\frac{d^2}{dx^2} + \frac{E^2 - 2EV(x) + V^2(x) - m^2 c^4}{c^2 \hbar^2} \right] \psi = 0. \quad (3)$$

Interestingly, the above equation may be rewritten in the Schrodinger form, with an effective energy and effective potential as

$$\left[\frac{d^2}{dx^2} + (E_{eff} - V_{eff}) \right] \psi = 0 \quad (4)$$

$$\text{with } E_{eff} = \frac{E^2 - m^2 c^4}{c^2 \hbar^2} \quad \text{and} \quad V_{eff} = \frac{2EV(x) - V^2(x)}{c^2 \hbar^2}.$$

Since E_{eff} and V_{eff} are non-linear in E and V , some novel results may be expected. As in the non-relativistic case, the allowed free-particle solution, outside the potential boundry, would yield

$$\psi_1(x) = C_1 e^{\alpha x} \quad -\infty < x \leq -a \quad (5)$$

$$\psi_4(x) = D_1 e^{-\alpha x} \quad a \leq x < \infty, \quad (6)$$

consistent with the requirement $\psi(x)$ vanishes as $|x| \rightarrow \infty$. Here $\alpha^2 = -E_{eff}$ is implied.

To discuss the nature of the solution within the potential region, $-a < x < a$, we consider a simple linear rising, finite range potential of the form [11]

$$V(x) = -\frac{V_0}{a}(a - |x|) \quad (7)$$

in which the well depth V_0 and range $2a$ are positive and adjustable parameters. Owing to its shape, this potential could also be called the triangular potential well. The linear, finite-ranged potential so constructed, serves as a good model to describe the energy specrum of particles, both relativistically and non-relativistically. We have recently reported that this potential has a rich set of solutions and can bind non-relativistic particles. Presently, we study the bound states of spin zero particles with this linear potential, treating it as a Lorentz vector.

Introducing the potential in Eqn.(3) and on simplification, we obtain, for $x > 0$,

$$\left[\frac{d^2}{dx^2} + \frac{1}{c^2 \hbar^2} \left\{ \frac{V_0^2}{a^2} x^2 - (2EV_0 + 2V_0^2) \frac{x}{a} + (E + V_0)^2 - m^2 c^4 \right\} \right] \psi = 0 \quad (8)$$

This equation may be written as

$$\left[\frac{d^2}{dx^2} + \frac{A}{a^2} \left(\frac{x^2}{a^2} \right) + \frac{B}{a^2} \left(\frac{x}{a} \right) + \frac{C}{a^2} \right] \psi = 0 \quad (9)$$

where $A = \bar{V}_0^2$, $B = -2\bar{E}\bar{V}_0 - 2\bar{V}_0^2$ and $C = (\bar{E} + \bar{V}_0)^2 - \bar{m}^2$.

Here $\bar{V}_0 = \frac{V_0}{\hbar c/a}$, $\bar{E} = \frac{E}{\hbar c/a}$ and $\bar{m} = \frac{mc^2}{\hbar c/a}$.

It is trivial to note that \bar{V}_0 , \bar{E} and \bar{m} are all dimensionless quantities.

Defining a new variable

$$y = \frac{x}{a},$$

Eqn.(9) transforms into a standard form [13]

$$\frac{d^2 \psi}{dy^2} + (Ay^2 + By + C) \psi = 0, \quad (10)$$

whose solutions are the well-known Parabolic Cylinder Functions.

Further, with the substitution $z = 2\sqrt{A} \left(y + \frac{B}{2A} \right)$, the above equation takes the form

$$4A \frac{d^2 \psi}{dz^2} + \left(\frac{z^2}{4} - D \right) \psi = 0 \quad (11)$$

where $D = \frac{B^2 - 4AC}{4A}$ is implied.

Using the transformation $\rho^2 = \frac{z^2}{\sqrt{4A}}$, we obtain

$$\frac{d^2\psi}{d\rho^2} + \left(\frac{\rho^2}{4} - b\right)\psi = 0 \quad (12)$$

with $b = \frac{B^2 - 4AC}{(4A)^{\frac{3}{2}}}$.

It is straightforward to check that $b = \frac{\bar{m}^2}{2V_0} > 0$, since both \bar{m} and \bar{V}_0 are positive. The eigenfunctions of spinless particles, which are the solutions of Eqn.(12) may be written in terms of the confluent hypergeometric functions as

$$\psi = N \exp\left(-\frac{1}{4}\rho^2 e^{\frac{i\pi}{2}}\right) M\left(\frac{-ib}{2} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\rho^2 e^{\frac{i\pi}{2}}\right) \quad (13)$$

Physically admissible solutions require finiteness and normalizability and as is evident from the above equation, we see that the wavefunction vanishes as $|x| \rightarrow \infty$, and thus being square integrable, represents genuine bound states.

3. Results and Discussion

In relativistic quantum mechanics, it is well-known that a general potential can be introduced in the wave equation in two different ways following the *minimal coupling scheme*. While in vector coupling, the potential $V(x)$ is treated as the fourth component of a four vector field, in scalar coupling, it is added to the invariant mass. Whereas the vector interaction is charge dependent and acts differently on particles and antiparticles, the scalar interaction is independent of the charge of the particle and has the same effect on both particles and antiparticles. Hence, it is interesting to study the quantum dynamics of relativistic particles for various interactions using different coupling schemes, with a view to decide on the appropriate prescription for a given potential.

The linear potential, envisaged as a quark-confining potential, is central in particle physics. Our investigation concerning the boundstates of spinless particle in the one-dimensional linear, finite-range potential, is seemingly interesting. It is trivial to note that the Klein Gordon equation can be reduced to a Schrodinger-like equation with an effective energy E_{eff} and an effective potential V_{eff} . The illuminating relation between the Klein Gordon equation and the Schrodinger equation with an equivalent energy dependent potential has a number of applications. If the potential is weak enough to ignore the V^2 term, the relativistic formalism becomes equivalent to the non-relativistic formalism. More importantly, in situations where the Klein-Gordon equation is not exactly solvable, the Schrodinger form of the KG equation sheds some light on the problem as it could be reduced to a solvable eigenvalue problem.

In the present work, we show that the one-dimensional Klein Gordon equation for the linear potential in the vector-coupling scheme is reduced to a standard differential equation, whose solutions, consistent with the boundary condition are the parabolic cylinder

functions, which on further simplification yield the confluent hypergeometric functions. Apart from being elegant, the vector coupling prescription is particularly significant in the sense that it preserves gauge invariance. Such studies, apart from being pedagogical in nature, are potentially exciting and significant.

The linear potential well so described, has potential applications in electronics[14]. It would be interesting to study the Dirac bound states of such a linearly rising potential of finite range, which would serve as a good model to describe the quarkonia.

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