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Plane Symmetric Viscous Fluid Universe in Lyra Geometry

Pratima Singh* and Pawan Kumar Rai†

Department of Mathematics, P. G. College, Ghazipur - 233 001, India

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Abstract: A new class of plane-symmetric homogeneous cosmological models for viscous fluid distribution is obtained in the context of Lyra's geometry. We have obtained two types of solutions by considering the uniform as well as time dependent displacement field. To get the deterministic solutions of Einstein's modified field equations, the free gravitational field is assumed to be of type D which is of the next order in the hierarchy of Petrov classification. It has been found that the displacement vector β behaves like cosmological term Λ in the normal gauge treatment and the solutions are consistent with the observations. The displacement vector $\beta(t)$ affects entropy. Some physical and geometric properties of the models are discussed.

Keywords: Plane Symmetric Universe, Viscous Fluid Models, Lyra's Geometry
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1. Introduction and Motivations

In 1917 Einstein introduced the cosmological constant into his field equations of general relativity in order to obtain a static cosmological model since, as is well known, without the cosmological term his field equations admit only non-static solutions. After the discovery of the red-shift of galaxies and explanation thereof, Einstein regretted for the introduction of the cosmological constant. Recently, there has been much interest in the cosmological term in context of quantum field theories, quantum gravity, supergravity theories, Kaluza-Klein theories and the inflationary-universe scenario. Shortly after Einstein's general theory of relativity, Weyl [1] suggested the first so-called unified field theory based on a generalization of Riemannian geometry. With its backdrop, it would seem more appropriate to call Weyl's theory a geometrized theory of gravitation

* pratimasingh_100@yahoo.in
† p_rai01@yahoo.com

and electromagnetism (just as the general theory was a geometrized theory of gravitation only), instead a unified field theory. It is not clear as to what extent the two fields have been unified, even though they acquire (different) geometrical significance in the same geometry. The theory was never taken seriously in as much as it was based on the concept of non-integrability of length transfer; and, as pointed out by Einstein, this implies that spectral frequencies of atoms depend on their past histories and therefore have no absolute significance. Nevertheless, Weyl's geometry provides an interesting example of non-Riemannian connections, and Folland [2] has given a global formulation of Weyl manifolds clarifying considerably many of Weyl's basic ideas thereby.

In 1951, Lyra [3] proposed a modification of Riemannian geometry by introducing a gauge function into the structure-less manifold, as a result of which the cosmological constant arises naturally from the geometry. This bears a remarkable resemblance to Weyl's geometry. But in Lyra's geometry, unlike that of Weyl, the connection is metric preserving as in Remannian; in other words, length transfers are integrable. Lyra also introduced the notion of a gauge and in the "normal" gauge the curvature scalar is identical to that of Weyl. In consecutive investigations Sen [4], Sen and Dunn [5] proposed a new scalar-tensor theory of gravitation and constructed an analogue of the Einstein field equations based on Lyra's geometry. It is, thus, possible [4] to construct a geometrized theory of gravitation and electromagnetism much along the lines of Weyl's "unified" field theory, however, without the inconvenience of non-integrability length transfer.

Halford [6] has pointed out that the constant vector displacement field ϕ_i in Lyra's geometry plays the role of cosmological constant Λ in the normal general relativistic treatment. It is shown by Halford [7] that the scalar-tensor treatment based on Lyra's geometry predicts the same effects within observational limits as the Einstein's general theory. Several authors Sen and Vanstone [8], Bhamra [9], Karade and Borikar [10], Kalyanshetti and Wagmode [11], Reddy and Innaiah [12], Beesham [13], Reddy and Venkateswarlu [14], Soleng [15], have studied cosmological models based on Lyra's manifold with a constant displacement field vector. However, this restriction of the displacement field to be constant is merely one for convenience and there is no *a priori* reason for it. Beesham [16] considered FRW models with time dependent displacement field. He has shown that by assuming the energy density of the universe to be equal to its critical value, the models have the $k = -1$ geometry. Singh and Singh [17]− [20], Singh and Desikan [21] have studied Bianchi-type I, III, Kantowaski-Sachs and a new class of cosmological models with time dependent displacement field and have made a comparative study of Robertson-Walker models with constant deceleration parameter in Einstein's theory with cosmological term and in the cosmological theory based on Lyra's geometry. Soleng [22] has pointed out that the cosmologies based on Lyra's manifold with constant gauge vector ϕ will either include a creation field and be equal to Hoyle's creation field cosmology [20]− [24] or contain a special vacuum field, which together with the gauge vector term, may be considered as a cosmological term. In the latter case the solutions are equal to the general relativistic cosmologies with a cosmological term.

Recently, Pradhan et al. [25], Casama et al. [26], Rahaman et al. [27], Bali and Chandnani [28], Kumar and Singh [29], Singh [30] and Rao, Vinutha and Santhi [31] have studied cosmological models based on Lyra's geometry in various contexts. With these motivations, in this paper, we have obtained exact solutions of Einstein's field equations for viscous fluid distribution in plane symmetric homogeneous space-time within the frame work of Lyra's geometry for uniform and time varying displacement vector. This paper is organized as follows. In Section 1 the motivation for the present work is discussed. The metric and the field equations are presented in Section 2, in Section 3 the solution of field equations. The Section 4 describes the solution of the first model. The Subsections 4.1 and 4.2 deal with the solutions for uniform displacement field ($\beta = \beta_0$, constant) and time varying displacement field ($\beta = \beta(t)$). Subsections $4.2.1, 4.2.2$ and 4.2.3 describe the solutions of Empty Universe, Zeldovich Universe and Radiating Universe respectively whereas Subsection 4.3 deals with the physical and geometric aspects of the first model. The Section 5 describes the solution of the second model. The Subsections 4.1 and 4.2 deal with the solutions for uniform displacement field ($\beta = \beta_0$, constant) and time varying displacement field ($\beta = \beta(t)$) of the second model. Subsections $5.2.1, 5.2.2$ and 5.2.3 describe the solutions of Empty Universe, Zeldovich Universe and Radiating Universe respectively whereas Subsection 5.3 deals with the physical and geometric aspects of the second model. Finally, in Section 6 discussion and concluding remarks are given.

2. The metric and field equations

We consider the metric in the form of Marder [32]

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2, \tag{1}$$

where the metric potentials A, B and C are functions of t alone. This ensures the model to be spatially homogeneous. This is a transform form of the metric of Bianchi type I spacetime in comoving coordinates which has been studied by a number of authors e.g. (Heckmann and Schucking [33], Thorne [34] and Roy and Prakash [35]).

The energy-momentum tensor for a viscous fluid distribution is given by Landau and Lifshitz [36]

$$T_i^j = (\rho + p)v_i v^j + p g_i^j - \eta(v_{i:}^j + v_{;i}^j + v^j v^l v_{i;l} + v_i v^l v_{;l}^j)$$
$$- \left(\xi - \frac{2}{3}\eta\right) v_{;l}^l (g_i^j + v_i v^j). \tag{2}$$

Here ρ, p, η and ξ are energy density, isotropic pressure, the coefficient of shear viscosity and bulk viscous coefficient respectively and v^i is the flow vector satisfying the relation

$$g_{ij}v^i v^j = -1. \tag{3}$$

The semicolon (;) indicates covariant differentiation. We choose the coordinates to be comoving, so that $v^1 = v^2 = v^3 = 0$ and $v^4 = \frac{1}{A}$.

The field equations, in normal gauge for Lyra's manifold, obtained by Sen [4] as

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -8\pi G T_{ij}, \tag{4}$$

where ϕ_i is the displacement field vector defined as

$$\phi_i = (0, 0, 0, \beta), \tag{5}$$

where β is either a constant or a function of t. The other symbols have their usual meaning as in Riemannian geometry.

For the line-element (1), the field Eq. (4) with Eqs. (2) and (5) lead to the following system of equations

$$\frac{1}{A^2}\left[\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}\right] - \frac{3}{4}\beta^2 = 8\pi\left[p - 2\eta\frac{\dot{A}}{A^2} - \left(\xi - \frac{2}{3}\eta\right)v^l_{;l}\right], \tag{6}$$

$$\frac{1}{A^2}\left[\frac{\dot{A}^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{C}}{C}\right] - \frac{3}{4}\beta^2 = 8\pi\left[p - 2\eta\frac{\dot{B}}{AB} - \left(\xi - \frac{2}{3}\eta\right)v^l_{;l}\right], \tag{7}$$

$$\frac{1}{A^2}\left[\frac{\dot{A}^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B}\right] - \frac{3}{4}\beta^2 = 8\pi\left[p - 2\eta\frac{\dot{C}}{AC} - \left(\xi - \frac{2}{3}\eta\right)v^l_{;l}\right], \tag{8}$$

$$\frac{1}{A^2}\left[\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right] + \frac{3}{4}\beta^2 = 8\pi\rho. \tag{9}$$

Here, and also in the following expressions a dot indicates ordinary differentiation with respect to t.

3. Solutions of the Field Equations

Equations (6)-(9) are four equations in six unknowns A, B, C, p, ρ, and β. The coefficients of viscosity η and ξ are taken as constants. Equations (6)-(9) are not independent, but are related by the contracted Bianchi identities. In the present case they lead to the single condition

$$\frac{d\rho}{dt} + (p+\rho)\ln(ABC) - \left(\rho - \frac{2}{3}\eta\right)\frac{1}{A}\left(\frac{d}{dt}\ln(ABC)\right)^2 -$$
$$\frac{2\eta}{A}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) = 0. \tag{10}$$

For complete solutions of equations (6)-(9), we need two extra conditions. The research on exact solutions is based on some physically reasonable restrictions used to simplify the Einstein equations. However we proceed from a different consideration. Although the distribution of matter at each point determines the nature of expansion in the model, the latter is also affected by the free gravitational field through its effect on the expansion, vorticity and shear in the fluid flow. A prescription of such a field may therefore be made on an *a priori* basis. The cosmological models of Friedman Robertson Walker, as well as the universe of Einstein and de Sitter, have vanishing free gravitational fields. Here we choose the free gravitational field to be of type D which is of the next order in the hierarchy of Petrov classification. This requires that either

$$(a) \quad C^{12}{}_{12} = C^{13}{}_{13},$$

or

$$(b) \quad C^{12}{}_{12} = C^{23}{}_{23}.$$

Conditions (a) and (b) are identically satisfied if $B = C$ and $A = C$ respectively. However, we shall assume A, B, C to be unequal on account of the supposed anisotropy.

From equations (6) and (7) we obtain

$$\frac{d}{dt}\left(\frac{\dot{A}}{A}\right) + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{\ddot{B}}{B} - \frac{\dot{B}\dot{C}}{BC} = 16\pi\eta A\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right). \tag{11}$$

Also from equations (7) and (8) we obtain

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = 16\pi\eta A\left(\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right). \tag{12}$$

In the following Sections 4 and 5, we have derived two models of the universe based on the above two conditions (a) and (b) respectively.

4. The First Model

The condition

$$C^{12}{}_{12} = C^{13}{}_{13} \tag{13}$$

leads to

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + 2\frac{\dot{A}}{A}\left(\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right) = 0. \tag{14}$$

Equations (12) and (14) lead to

$$A = \frac{1}{8\pi\eta t + a}, \tag{15}$$

where a is a constant of integration. From equations (14) and (15) we obtain

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = -\frac{16\pi\eta}{(8\pi\eta t + a)}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \tag{16}$$

which on integration gives

$$\dot{B}C - B\dot{C} = \frac{b}{(8\pi\eta t + a)^2}, \tag{17}$$

where b is an integrating constant. From equations (11) and (15) we get

$$\left(\frac{8\pi\eta}{8\pi\eta t + a}\right)^2 + \frac{8\pi\eta}{(8\pi\eta t + a)}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{16\pi\eta}{(8\pi\eta t + a)}\frac{\dot{B}}{B} + \frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} = 0. \tag{18}$$

From equations (17) and (18) we obtain

$$B = \frac{K(Lt + M)^{\frac{1}{2} + \frac{b}{2L}}}{(8\pi\eta t + a)}, \tag{19}$$

and

$$C = \frac{(Lt + M)^{\frac{1}{2} - \frac{b}{2L}}}{K(8\pi\eta t + a)}, \tag{20}$$

where K, L and M are constants of integration.

Hence, the geometry of the space time (1) takes the form

$$ds^2 = \frac{1}{(8\pi\eta t + a)^2}(dx^2 - dt^2) + \frac{K^2(Lt + M)^{1 + \frac{b}{L}}}{(8\pi\eta t + a)^2}dy^2 + \frac{(Lt + M)^{1 - \frac{b}{L}}}{K^2(8\pi\eta t + a)^2}dz^2. \tag{21}$$

For the specification of displacement vector β within the framework of Lyra geometry and for realistic models of physical importance, we consider following two cases:

4.1 When β is a constant i.e. $\beta = \beta_0$ (constant)

Using Eqs. (15), (19) and (20) in Eqs. (6) - (9), the expressions for pressure p and density ρ for the model (21) are given by

$$8\pi p = -192\pi^2\eta^2 + \frac{(L^2 - b^2)(8\pi\eta t + a)^2}{4(Lt + M)^2} + \frac{32\pi\eta L(8\pi\eta t + a)}{3(Lt + M)}$$

$$-8\pi\xi\left[24\pi\eta - \frac{(8\pi\eta t + a)L}{(Lt + M)}\right] - \frac{3}{4}\beta_0^2, \tag{22}$$

$$8\pi\rho = 192\pi^2\eta^2 - \frac{16\pi\eta L(8\pi\eta t + a)}{(Lt + M)} + \frac{(L^2 - b^2)(8\pi\eta t + a)^2}{4(Lt + M)^2} + \frac{3}{4}\beta_0^2. \tag{23}$$

From Eq. (23) it is observed that for $t > \frac{(a-m)}{(L-8\pi\eta)}$, the energy density decreases with time and is always positive.

The reality conditions (Ellis [37])

$$(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,$$

lead to

$$\frac{(L^2 - b^2)(8\pi\eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi\eta t + a)}{3(Lt + m)} > 192\pi^2\xi\eta, \tag{24}$$

and

$$\frac{(L^2-b^2)(8\pi\eta t+a)^2}{(Lt+M)^2}+\frac{8\pi L(3\xi+2\eta)(8\pi\eta t+a)}{(Lt+m)}$$

$$> 192\pi^2\eta(3\xi+2\eta)+\frac{3}{2}\beta_0^2, \tag{25}$$

respectively.

The dominant energy conditions (Hawking and Ellis [38])

$$(i)\rho-p\geq 0, \quad (ii)\rho+p\geq 0,$$

lead to

$$24\pi\eta(2\eta+\xi)+\frac{3}{16\pi}\beta_0^2\geq\frac{(\xi+6\eta)(8\pi\eta t+a)L}{(Lt+M)}, \tag{26}$$

and

$$\frac{(L^2-b^2)(8\pi\eta t+a)^2}{2(Lt+M)^2}+\frac{8\pi L(3\xi-2\eta)(8\pi\eta t+a)}{3(Lt+m)}\geq 192\pi^2\xi\eta, \tag{27}$$

respectively. The conditions (25) and (26) impose a restriction on β_0.

4.2 When β is a function of t i.e. $\beta=\beta(t)$

In this case to find the explicit value of displacement field $\beta(t)$, we assume that the fluid obeys an equation of state of the form

$$p=\gamma\rho, \tag{28}$$

where $\gamma(0\leq\gamma\leq 1)$ is a constant.

Using Eqs. (15), (19), (20) and (28) in Eqs. (6) - (9), we obtain the expressions for energy density ρ and displacement vector $\beta(t)$ for the model (21) as

$$8\pi(1+\gamma)\rho=\frac{(L^2-b^2)(8\pi\eta t+a)^2}{2(Lt+M)^2}+\frac{8\pi L(3\xi-2\eta)(8\pi\eta t+a)}{3(Lt+M)}-192\pi^2\xi\eta, \tag{29}$$

$$(1+\gamma)\beta^2(t)=\frac{(L^2-b^2)(8\pi\eta t+a)^2(1-\gamma)}{3(Lt+M)^2}+$$

$$\frac{32\pi L(8\pi\eta t+a)\{3\xi+2\eta(3\gamma+2)\}}{9(Lt+M)}-256\pi^2\eta\{\xi+\eta(1+\gamma)\}. \tag{30}$$

From Eqs. (29) and (30), we observe that for $t>\frac{(a-M)}{(L-8\pi\eta)}$, the energy density $\rho(t)$ and displacement vector $\beta(t)$ are decreasing function of time and are always positive.

4.2.1 Empty Universe

Let us consider $\gamma = 0$ in Eq. (28) which leads $p = 0$. Thus, from Eqs. (6) - (8) we obtain

$$\beta^2(t) = \frac{(L^2 - b^2)(8\pi\eta t + a)^2}{3(Lt + M)^2} + \frac{32\pi L(8\pi\eta t + a)(9\xi + 4\eta)}{3(Lt + M)} - 256\pi^2\eta^2(\xi + \eta). \quad (31)$$

Halford [6] has pointed out that the constant vector displacement field ϕ_i in Lyra's geometry plays the role of cosmological constant Λ in the normal general relativistic treatment. From Eq. (31), it is observed that for $t > \frac{(a-M)}{(L-8\pi\eta)}$, the displacement vector $\beta(t)$ is a decreasing function of time which is corroborated with Halford as well as with the recent observations [39, 40] leading to the conclusion that $\Lambda(t)$ is a decreasing function of t.

4.2.2 Zeldovich Universe

Let us consider $\gamma = 1$ in Eq. (28) which yields $p = \rho$. Therefore, in this case, the expressions for p, ρ and $\beta(t)$ are given by

$$16\pi p = 16\pi\rho = \frac{(L^2 - b^2)(8\pi\eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(8\pi\eta t + a)(3\xi - 2\eta)}{3(Lt + M)} - 192\pi^2\xi\eta, \quad (32)$$

$$\beta^2(t) = \frac{16\pi L(8\pi\eta t + a)(3\xi + 10\eta)}{9(Lt + M)} - 128\pi^2\eta(\xi + \eta), \quad (33)$$

From Eqs. (32) and (33), we observe that for $t > \frac{(a-M)}{(L-8\pi\eta)}$, the energy density $\rho(t)$ and displacement vector $\beta(t)$ are decreasing function of time and are always positive.

The reality condition (Ellis [37])

$$(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,$$

lead to

$$\frac{(L^2 - b^2)(8\pi\eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(8\pi\eta t + a)(3\xi - 2\eta)}{3(Lt + M)} > 192\pi^2\eta^2\xi\eta. \quad (34)$$

4.2.3 Radiating Universe

Let us consider $\gamma = \frac{1}{3}$ in Eq. (28) which gives $\rho = 3p$. Hence, in this case, the expressions for p, ρ and $\beta(t)$ are obtained as

$$8\pi\rho = 24\pi p = \frac{(L^2 - b^2)(8\pi\eta t + a)^2}{8(Lt + M)^2} + \frac{2\pi L(3\xi - 2\eta)(8\pi\eta + a)}{9(Lt + M)} - 48\pi^2\xi\eta, \quad (35)$$

$$\beta^2(t) = \frac{(L^2 - b^2)(8\pi\eta t + a)^2}{6(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi\eta + a)}{(Lt + M)} - 64\pi^2\eta(3\xi + 2\eta). \quad (36)$$

From Eqs. (35) and (36), it is observed that for $t > \frac{(a-M)}{(L-8\pi\eta)}$, the energy density $\rho(t)$ and the displacement vector $\beta(t)$ is decreasing function of time and always positive. Thus we see that $\beta(t)$ behaves like cosmological term Λ.

The reality conditions (Ellis [37])

$$(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,$$

and the dominant energy conditions (Hawking and Ellis [38])

$$(i)\rho - p \geq 0, \quad (ii)\rho + p \geq 0,$$

lead to

$$\frac{(L^2 - b^2)(8\pi\eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi\eta + a)}{9(Lt + M)} > 192\pi^2\xi\eta, \tag{37}$$

and

$$\frac{(L^2 - b^2)(8\pi\eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi\eta + a)}{9(Lt + M)} \geq 192\pi^2\xi\eta, \tag{38}$$

respectively.

4.3 Some Geometric Properties of First Model

We shall now give the expressions for kinematic quantities and components of conformal curvature tensor. With regard to the kinematical properties of the velocity vector v^i in the metric (21), a straightforward calculation leads to the expressions for expansion (θ), deceleration parameter q, proper volume V^3 and shear (σ_{ij}) of the fluid:

$$\theta = \frac{(8\pi\eta t + a)L}{(Lt + M)} - 24\pi\eta, \tag{39}$$

$$q = -1 - \frac{\frac{256\pi^2\eta^2}{(8\pi\eta t+a)^2} - \frac{L^2}{(Lt+M)^2}}{\frac{L^2}{(Lt+M)^2} + \frac{(32\pi\eta)^2}{(8\pi\eta t+a)^2} - \frac{64\pi\eta L}{(8\pi\eta t+a)(Lt+M)}}, \tag{40}$$

$$V^3 = \sqrt{-g} = \frac{(Lt + M)}{(8\pi\eta + a)^4}, \tag{41}$$

$$\sigma_{11} = -\frac{L(Lt + M)^{-1}}{3(8\pi\eta t + a)}, \tag{42}$$

$$\sigma_{22} = \frac{K^2(L + 3b)(Lt + M)^{\frac{b}{L}}}{6(8\pi\eta t + a)}, \tag{43}$$

$$\sigma_{33} = \frac{(L - 3b)(Lt + M)^{-\frac{b}{L}}}{6K^2(8\pi\eta t + a)}, \tag{44}$$

and other components of the shear tensor (σ_{ij}) being zero. Hence

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \left(\frac{(L^2 + 3b^2)}{12}\right)\left(\frac{8\pi\eta t + a}{Lt + M}\right)^2. \tag{45}$$

From Eqs. (39) and (45) we obtain

$$\frac{\sigma}{\theta} = \frac{\sqrt{\frac{(L^2+3b^2)}{12}}(8\pi\eta t + a)}{(8\pi\eta t + a)L - 24\pi\eta(Lt + M)}. \tag{46}$$

The non-vanishing components of the conformal curvature tensor are

$$C^{12}{}_{12} = C^{13}{}_{13} = -\frac{1}{2}C^{23}{}_{23} = \left(\frac{(L^2 - b^2)}{12}\right)\left(\frac{8\pi\eta t + a}{Lt + M}\right)^2 . \tag{47}$$

For large t, we find

$$C^{23}{}_{23} = -\frac{32}{3}\pi^2\eta^2\left(1 - \frac{b^2}{L^2}\right), \tag{48}$$

and

$$\sigma^2 = \frac{16}{3}\pi^2\eta^2\left(1 + \frac{3b^2}{L^2}\right). \tag{49}$$

Here we find

$$C^{12}{}_{12} + C^{13}{}_{13} + C^{23}{}_{23} = 0. \tag{50}$$

The rotation ω is identically zero. The rate of expansion H_i in the directions of x, y and z are given by

$$H_x = \frac{\dot{A}}{A} = -\frac{8\pi\eta}{(8\pi\eta t + a)}, \tag{51}$$

$$H_y = \frac{\dot{B}}{B} = \left(\frac{1}{2} + \frac{b}{2L}\right)\frac{L}{(Lt + M)} - \frac{8\pi\eta}{(8\pi\eta t + a)}, \tag{52}$$

$$H_z = \frac{\dot{C}}{C} = \left(\frac{1}{2} - \frac{b}{2L}\right)\frac{L}{(Lt + M)} - \frac{8\pi\eta}{(8\pi\eta t + a)}. \tag{53}$$

Now since

$$\int_{t_0}^{t}\frac{dt}{V(t)} = \int_{t_0}^{t}\frac{(8\pi\eta t + a)^{\frac{4}{3}}}{(Lt + M)^{\frac{1}{3}}}dt. \tag{54}$$

This is convergent integral hence particle horizon exists.

The models represent shearing, non-rotating and Petrov Type D universe in general, in which the flow is geodetic. It is also observed that the viscosity prevents the free gravitational field as well as the shear from withering away. It is also obvious from (39) that the effect of viscosity is to retard expansion of the model. Since $\lim_{t\to\infty}\frac{\sigma}{\theta} \neq 0$, the models do not approach isotropy for large values of t. It is observed from Eq. (40) which implies an accelerating model of the universe. Recent observations of type Ia supernovae [39, 40] reveal that the present universe is in accelerating phase and deceleration parameter lies somewhere in the range $-1 < q \leq 0$. It follows that our models of the universe are consistent with recent observations. For

$$256\pi^2\eta^2(Lt + M)^2 = L^2(8\pi\eta t + a)^2 \tag{55}$$

the deceleration parameter q approaches the value (-1) as in the case of de-Sitter universe.

5. The Second Model

The condition

$$C^{12}{}_{12} = C^{23}{}_{23}, \tag{56}$$

leads to

$$\frac{d}{dt}\left(\frac{\dot{A}}{A}\right) = \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right). \tag{57}$$

Equations (11), (12) and (57) lead to

$$\frac{\dot{B}}{B} = -8\pi\eta A. \tag{58}$$

From equations (12) and (58) we obtain

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = 2\frac{\dot{B}}{B}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \tag{59}$$

which on integration leads

$$C = B(k_1 - kt), \tag{60}$$

where k and k_1 are constants of integration. From equations (57) and (60) we get

$$\frac{d}{dt}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)\left(\frac{k}{k_1 - kt}\right), \tag{61}$$

which on integration gives

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{k_2}{k_1 - kt}, \tag{62}$$

where k_2 is an integrating constant. From equations (58) and (62) we obtain

$$A = \left[\frac{8\pi\eta(k_1 - kt)}{k_2 - k} + k_3(k_1 - kt)^{\frac{k_2}{k}}\right]^{-1}, \tag{63}$$

k_3 being a constant of integration. From equations (58) and (63) we obtain

$$B = k_4\left[\frac{(k_2 - k)k_3(k_1 - kt)^{\frac{k_2}{k} - 1}}{8\pi\eta + (k_2 - k)k_3(k_1 - kt)^{\frac{k_2}{n} - 1}}\right], \tag{64}$$

where k_4 is a constant of integration. Also, from equations (60) and (64) we obtain

$$C = k_4\left[\frac{(k_2 - k)k_3(k_1 - kt)^{\frac{k_2}{k}}}{8\pi\eta + (k_2 - k)k_3(k_1 - kt)^{\frac{k_2}{n} - 1}}\right]. \tag{65}$$

By a suitable transformation of coordinates the metric of this model can be put into the form

$$ds^2 = \left(\frac{8\pi\eta}{\alpha - 1}T + \ell T^{\alpha}\right)^{-2}\left[dX^2 - dT^2 + T^{2\alpha}dY^2 + T^{2(\alpha+1)}dZ^2\right], \tag{66}$$

where α and ℓ are arbitrary constants.

For the specification of displacement vector β within the framework of Lyra geometry and for realistic models of physical importance, we consider following two cases:

5.1 When β is a constant i.e. $\beta = \beta_0$(constant)

Using Eqs. (63), (64) and (65) in Eqs. (6) and (9) the expressions for pressure p and density ρ for the model (66) are given by

$$8\pi p = 64\pi^2\eta^2\left(\frac{2-\alpha}{\alpha-1}\right) + 16\pi\eta\ell T^{\alpha-1} + \alpha(\alpha-1)\ell^2 T^{2(\alpha-1)}$$

$$-\left(\frac{8\pi\eta}{\alpha-1} + \alpha\ell T^{\alpha-1}\right)^2 - \frac{16}{3}\pi\eta(\alpha+2)\left(\frac{8\pi\eta}{\alpha-1} + \alpha\ell T^{\alpha-1}\right)$$

$$+8\pi\xi[(\alpha-1)\ell T^{\alpha-1} - 16\pi\eta] - \frac{3}{4}\beta_0^2, \tag{67}$$

$$8\pi\rho = 64\pi^2\eta^2\left[1 - \frac{\alpha}{(\alpha-1)^2}\right] + 16\pi\eta\ell\left(1 - \frac{\alpha^2}{\alpha-1}\right)T^{\alpha-1}$$

$$-\alpha\ell^2 T^{2(\alpha-1)} + \frac{3}{4}\beta_0^2. \tag{68}$$

It is observed from Eq. (68) that for $\alpha < 0$, the energy density ρ is a decreasing function of t and is always positive.

The reality conditions (Ellis [37])

$$(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,$$

lead to

$$\left[\xi\ell(\alpha-1) + \frac{2\eta\ell\alpha(\alpha-2)}{(\alpha-1)}\right]T^{\alpha-1} - \frac{\alpha\ell^2}{4\pi}T^{2(\alpha-1)}$$

$$> 16\pi\xi\eta + \frac{8\pi\eta^2(2\alpha^2+5\alpha-4)}{3(\alpha-1)^2} \tag{69}$$

and

$$\left[36\xi\ell(\alpha-2) - \frac{2\eta\ell(\alpha^2+4\alpha-2)}{(\alpha-1)}\right]T^{\alpha-1} - \frac{3\alpha^2\ell^2}{8\pi}T^{2(\alpha-1)}$$

$$> 48\pi\xi\eta - \frac{32\pi\eta^2(2\alpha-3)}{(\alpha-1)^2} - \frac{3}{16\pi}\beta_0^2, \tag{70}$$

respectively.

The dominant energy conditions (Hawking and Ellis [38])

$$(i)\rho - p \geq 0, \quad (ii)\rho + p \geq 0,$$

lead to

$$\left[\frac{10\eta\ell(5\alpha^2-4\alpha+2)}{(\alpha-1)} - \xi\ell(\alpha-1)\right] \geq -\frac{64}{3}\pi\eta^2 - 16\pi\xi\eta - \frac{3}{16\pi}\beta_0^2, \tag{71}$$

and

$$\left[\xi\ell(\alpha-1) + \frac{2\eta\ell\alpha(\alpha-2)}{(\alpha-1)}\right]T^{\alpha-1} - \frac{\alpha\ell^2}{4\pi}T^{2(\alpha-1)}$$

$$\geq 16\pi\xi\eta + \frac{8\pi\eta^2(2\alpha^2+5\alpha-4)}{3(\alpha-1)^2}, \tag{72}$$

respectively. The conditions (70) and (71) impose a restriction on β_0.

5.2 When β is a function of T i.e. $\beta = \beta(T)$

In this case to find the explicit value of displacement field $\beta(t)$, we assume that the fluid obeys an equation of state given by (28). Using Eqs. (63) - 65) and (28) in Eqs. (6) - (9), we obtain ρ and β for the model (66)

$$8\pi(1+\gamma)\rho = -2\alpha\ell^2 T^{2(\alpha-1)} - \frac{8\pi\ell}{3(\alpha-1)}\left[2n(\alpha^2+4\alpha+1)+3\xi(\alpha-1)^2\right]T^{\alpha-1}$$

$$-\frac{128\pi^2\eta^2(\alpha^2+\alpha+1)}{3(\alpha-1)^2} - 128\pi^2\xi\eta, \tag{73}$$

and

$$\frac{3}{4}(1+\gamma)\beta^2 = 2\alpha\ell^2[4\pi\xi(1+\gamma)-1]T^{2(\alpha-1)} + \frac{8\pi\ell}{(\alpha-1)}\Big[\xi(\alpha-1)^2 -$$

$$2\eta(\alpha+1)\Big]T^{\alpha-1} - \frac{64\pi^2\eta^2(2\alpha^2-\alpha+2)}{3(\alpha-1)^2} - 64\pi^2\eta\{\eta(1+\gamma)+2\xi\}. \tag{74}$$

We consider three following cases of physical interest.

5.2.1 Empty Universe

In this case $p = 0$. Hence, we obtain the expression for β as

$$\frac{3}{4}\beta^2 = 2\alpha\ell^2(4\pi\xi-1)T^{2(\alpha-1)} + +\frac{8\pi\ell}{(\alpha-1)}\Big[\xi(\alpha-1)^2 -$$

$$2\eta(\alpha+1)\Big]T^{\alpha-1} - \frac{64\pi^2\eta^2(2\alpha^2-\alpha+2)}{3(\alpha-1)^2} - 64\pi^2\eta(\eta+2\xi). \tag{75}$$

From above equation it is observed that the displacement vector β is a decreasing function of time for $\alpha < 0$.

5.2.2 Zeldovice Universe

In this case we have $\rho = p$. Therefore, in this case, the expressions for $\beta(t)$ is given by

$$\frac{3}{2}\beta^2 = 2\alpha\ell(8\pi\xi-1)T^{2(\alpha-1)} + \frac{8\pi\ell}{(\alpha-1)}\Big[\xi(\alpha-1)^2 - 2\eta(\alpha+1)\Big]T^{\alpha-1}$$

$$-\frac{64\pi^2\eta^2(2\alpha^2-\alpha+2)}{3(\alpha-1)^2} - 128\pi^2\eta(\eta+\xi). \tag{76}$$

From Eq. (76), it is observed that displacement vector β is decreasing function of time for $\alpha < 1$. The expressions for pressure p and energy density ρ are given by

$$8\pi p = 8\pi\rho = -2\alpha\ell^2 T^{2(\alpha-1)} - \frac{8\pi\ell}{3(\alpha-1)}\Big[2n(\alpha^2+4\alpha+1)+3\xi(\alpha-1)^2\Big]T^{\alpha-1}$$

$$-\frac{128\pi^2\eta^2(\alpha^2+\alpha+1)}{3(\alpha-1)^2} - 128\pi^2\xi\eta. \tag{77}$$

The reality conditions (Ellis [37])

$$(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,$$

lead to

$$-\alpha\ell^2 T^{2(\alpha-1)} - \frac{4\pi\ell}{3(\alpha-1)}\left[2n(\alpha^2+4\alpha+1)+3\xi(\alpha-1)^2\right]T^{\alpha-1} > \frac{64\pi^2\eta^2(\alpha^2+\alpha+1)}{3(\alpha-1)^2} + 64\pi^2\xi\eta. \quad (78)$$

5.2.3 Radiating Universe

In this case we have $\rho = 3p$. From Eqs. (6) - (9), the expressions for ρ, p and $\beta(t)$ are obtained as

$$\rho = 3p = -\frac{3\alpha\ell^2}{16\pi}T^{2(\alpha-1)} - \frac{\ell}{4(\alpha-1)^2}\left[2n(\alpha^2+4\alpha+1)+3\xi(\alpha-1)^2\right]T^{\alpha-1} - \frac{4\pi\eta^2(\alpha^2+\alpha+1)}{(\alpha-1)^2} - 12\pi\xi\eta, \quad (79)$$

$$\beta^2 = \frac{2\alpha\ell^2}{3}(16\pi\xi-3)T^{2(\alpha-1)} + \frac{8\pi\ell}{(\alpha-1)}\left[\xi(\alpha-1)^2 - 2\eta(\alpha+1)\right]T^{\alpha-1} - \frac{64\pi^2\eta^2(2\alpha^2-\alpha+2)}{3(\alpha-1)} - \frac{128\pi^2\eta}{3}(2\eta+3\xi). \quad (80)$$

From Eq. (80), it is observed that displacement vector β is decreasing function of time for $\alpha < 1$. For $\alpha < 0$ and $\ell < 0$, the energy density decreases with time and is always positive.

The reality conditions (Ellis [37])

$$(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,$$

and the dominant energy conditions (Hawking and Ellis [38])

$$(i)\rho - p \geq 0, \quad (ii)\rho + p \geq 0,$$

lead to

$$-\frac{3\alpha\ell^2}{4\pi}T^{2(\alpha-1)} - \frac{\ell}{(\alpha-1)^2}\left[2n(\alpha^2+4\alpha+1)+3\xi(\alpha-1)^2\right]T^{\alpha-1} > \frac{16\pi\eta^2(\alpha^2+\alpha+1)}{(\alpha-1)^2} + 48\pi\xi\eta, \quad (81)$$

and

$$-\frac{3\alpha\ell^2}{4\pi}T^{2(\alpha-1)} - \frac{ell}{(\alpha-1)^2}\left[2n(\alpha^2+4\alpha+1)+3\xi(\alpha-1)^2\right]T^{\alpha-1} \geq \frac{16\pi\eta^2(\alpha^2+\alpha+1)}{(\alpha-1)^2} + 48\pi\xi\eta, \quad (82)$$

respectively.

5.3 Some Geometric Properties of Second Model

The expressions for the expansion θ, Hubble parameter H, the magnitude of shear σ^2, deceleration parameter q and proper volume V^3 for the model (66) are given by

$$\theta = 3H = \ell(\alpha-1)T^{\alpha-1} - 16\pi\eta, \tag{83}$$

$$\sigma^2 = \frac{1}{3}(\alpha^2+\alpha+1)\Big[\frac{8\pi\eta}{\alpha-1} + \ell T^{\alpha-1}\Big]^2. \tag{84}$$

$$q = -1 - \frac{1}{\left[\frac{(2\alpha+1)}{3T} - \frac{2\ell(\alpha-1)T^{\alpha-1}}{3\{8\pi\eta+\ell(\alpha-1)T^{\alpha}\}}\right]^2}\times$$
$$\left[-\frac{(2\ell+1)}{3T^2} + \frac{2\ell(\alpha-1)^2(8\pi\eta T^{\alpha-2}+\ell\alpha T^{\alpha-1})}{3\left\{8\pi\eta T^{\alpha-1}+\ell(\alpha-1)T^{\alpha}\right\}^2}\right], \tag{85}$$

$$V^3 = \sqrt{-g} = \left[\frac{8\pi\eta}{(\alpha-1)}T + \ell T^{\alpha}\right]^{-2} T^{2\alpha+1}. \tag{86}$$

The non-vanishing components of the conformal curvature tensor are

$$C^{12}{}_{12} = C^{23}{}_{23} = -\frac{1}{2}C^{13}{}_{13} = -\frac{1}{3}\alpha\Big[\frac{8\pi\eta}{\alpha-1} + \ell T^{\alpha-1}\Big]. \tag{87}$$

Here we also find

$$C^{12}{}_{12} + C^{13}{}_{13} + C^{23}{}_{23} = 0. \tag{88}$$

The rotation ω is identically zero.

The rate of expansion H_i in the directions of x, y and z are given by

$$H_x = \frac{\dot{A}}{A} = -\left[\frac{8\pi\eta + \alpha\ell(\alpha-1)T^{\alpha-1}}{8\pi\eta T + \ell(\alpha-1)T^{\alpha}}\right], \tag{89}$$

$$H_y = \frac{\dot{B}}{B} = \frac{\alpha}{T} - \left[\frac{8\pi\eta + \alpha\ell(\alpha-1)T^{\alpha-1}}{8\pi\eta T + \ell(\alpha-1)T^{\alpha}}\right], \tag{90}$$

$$H_z = \frac{\dot{C}}{C} = \frac{(\alpha+1)}{T} - \left[\frac{8\pi\eta + \alpha\ell(\alpha-1)T^{\alpha-1}}{8\pi\eta T + \ell(\alpha-1)T^{\alpha}}\right]. \tag{91}$$

The models represent shearing, non-rotating and Petrov Type D universe in general, in which the flow is geodetic. For this model too, it is observed that the effect of viscosity prevents the shear and the free gravitational field from withering away for large value of T. It also retards expansion of the model. Since $\lim_{T\to\infty}\frac{\sigma}{\theta} \neq 0$, the models do not approach isotropy for large values of T. It is observed from Eq. (85) which implies an accelerating model of the universe. It follows that our models of the universe are consistent with recent observations [39, 40]. For the critical time T_c given by

$$\frac{(2\ell+1)}{3T_c^2} = \frac{2\ell(\alpha-1)^2(8\pi\eta T_c^{\alpha-2}+\ell\alpha T_c^{\alpha-1})}{3\left\{8\pi\eta T_c^{\alpha-1}+\ell(\alpha-1)T_c^{\alpha}\right\}^2}, \tag{92}$$

the deceleration parameter q approaches the value (-1) as in the case of de-Sitter universe.

We also find

$$\int_{t_0}^{t} \frac{dt}{V(t)} = \int_{t_0}^{t} \frac{\left[\frac{8\pi\eta}{(\alpha-1)}T + \ell T^{\alpha}\right]^{\frac{2}{3}}}{T^{\frac{2\alpha+1}{3}}} dt. \tag{93}$$

It is the convergent integral hence particle horizon exits.

The metric (66) is conformal to the metric

$$ds^2 = dX^2 - dT^2 + T^{2\alpha} dY^2 + T^{2(\alpha+1)} dZ^2. \tag{94}$$

The universe (94) represents a viscous fluid cosmological model in which kinematic viscosity η_0 is $-\frac{\alpha}{8\pi T}$ and the pressure p_0 and the density ρ_0 are given by

$$8\pi p_0 = 8\pi\xi \left(\frac{2\alpha+1}{T}\right) - \left[\frac{\alpha(5\alpha+1)}{3T^2}\right] - \Lambda, \tag{95}$$

$$8\pi\rho_0 = \frac{\alpha(\alpha+1)}{T^2} + \Lambda. \tag{96}$$

It is also remarkable that the space-time (94) becomes flat when α is zero. The corresponding model

$$ds^2 = (\beta - 8\pi\eta T)^{-2}(dX^2 - dT^2 + dY^2 + T^2 dZ^2) \tag{97}$$

represents a conformally flat viscous fluid cosmological model.

Discussion and Concluding Remarks

In this paper, we have presented a new class of exact solutions of Einstein's field equations for plane-symmetric space-time with bulk viscous fluid distribution within the framework of Lyra's geometry both for uniform and time dependent displacement field. Generally, the models represent shearing, non-rotating and Petrov type D universe in which the flow vector is geodetic. In all these models, we observe that they do not approach isotropy for large values of time.

It is possible to discuss entropy in our universe. In thermodynamics the expression for entropy is given by

$$TdS = d(\rho V^3) + \bar{p}(dV^3), \tag{98}$$

where $V^3 = A^2BC$ is the proper volume in our case and $\bar{p}$ is the effective pressure given by

$$\bar{p} = p - (\xi - \frac{2}{3}\eta)v^i_{;i}. \tag{99}$$

To solve the entropy problem of the standard model, it is necessary to treat $dS > 0$ for at least a part of evolution of the universe. Hence Eq. (98) reduces to

$$TdS = \dot{\rho} + (\rho + \bar{p})\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) > 0. \tag{100}$$

The conservation equation $T^{j}_{i:j} = 0$ for (1) leads to

$$\dot{\rho} + (\rho + \bar{p})\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{3}{2}\beta\dot{\beta} + \frac{3}{2}\beta^2\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0. \tag{101}$$

Therefore, Eqs. (100) and (101) lead to

$$\frac{3}{2}\beta\dot{\beta} + \frac{3}{2}\beta^2\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) < 0. \tag{102}$$

which gives to $\beta < 0$. Thus, the displacement vector $\beta(t)$ affects entropy because for entropy $dS > 0$ leads to $\beta(t) < 0$.

It is observed that the displacement vector $\beta(t)$ coincides with the nature of the cosmological constant Λ which has been supported by the work of several authors as discussed in the physical behaviour of the model in Sections 4 and 5. In recent time Λ-term has attracted theoreticians and observers for many a reason. The nontrivial role of the vacuum in the early universe generates a Λ-term that leads to inflationary phase. Observationally, this term provides an additional parameter to accommodate conflicting data on the values of the Hubble constant, the deceleration parameter, the density parameter and the age of the universe (for example, see Refs. [41] and [42]). In recent past there is an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology [43, 44, 45]. Therefore the study of cosmological models in Lyra's geometry may be relevant for inflationary models. There seems a good possibility of Lyra's geometry to provide a theoretical foundation for relativistic gravitation, astrophysics and cosmology. However, the importance of Lyra's geometry for astrophysical bodies is still an open question.

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References

[1] H. Weyl, Sber. Preuss. Akad. Wiss. Berlin, 465 (1918).

[2] G. Folland, J. Diff. Geom. **4**, 145 (1970).

[3] G. Lyra, Math. Z. **54**, 52 (1951).

[4] D. K. Sen, Z. Phys. **149**, 311 (1957).

[5] D. K. Sen and K. A. Dunn, J. Math. Phys. **12**, 578 (1971).

[6] W. D. Halford, Austr. J. Phys. **23**, 863 (1970).

[7] W. D. Halford, J. Math. Phys. **13**, 1399 (1972).

[8] D. K. Sen and J. R. Vanstone, J. Math. Phys. **13**, 990 (1972).

[9] K. S. Bhamra, Austr. J. Phys. **27**, 541 (1974).

[10] T. M. Karade and S. M. Borikar, Gen. Rel. Gravit. **9**, 431 (1978).

[11] S. B. Kalyanshetti and B. B. Waghmode, Gen. Rel. Gravit. **14**, 823 (1982).

[12] D. R. K. Reddy and P. Innaiah, Astrophys. Space Sci. **123**, 49 (1986).

[13] A. Beesham, Astrophys. Space Sci. **127**, 189 (1986).

[14] D. R. K. Reddy and R. Venkateswarlu, Astrophys. Space Sci. **136**, 191 (1987).

[15] H. H. Soleng, Gen. Rel. Gravit. **19**, 1213 (1987).

[16] A. Beesham, Austr. J. Phys. **41**, 833 (1988).

[17] T. Singh and G. P. Singh, J. Math. Phys. **32**, 2456 (1991).

[18] T. Singh and G. P. Singh, Il. Nuovo Cimento **B106**, 617 (1991).

[19] T. Singh and G. P. Singh, Int. J. Theor. Phys. **31**, 1433 (1992).

[20] T. Singh, and G. P. Singh, Fortschr. Phys. **41**, 737 (1993).

[21] G. P. Singh and K. Desikan, Pramana-journal of physics, **49**, 205 (1997).

[22] F. Hoyle, Monthly Notices Roy. Astron. Soc. **108**, 252 (1948).

[23] F. Hoyle and J. V. Narlikar, Proc. Roy. Soc. London Ser. A, **273**, 1 (1963).

[24] F. Hoyle and J. V. Narlikar, Proc. Roy. Soc. London Ser.A, **282**, 1 (1964).

[25] A. Pradhan, I. Aotemshi and G. P. Singh, Astrophys. Space Sci. **288**, 315 (2003).
A. Pradhan and A. K. Vishwakarma, J. Geom. Phys. **49**, 332 (2004).
A. Pradhan, L. Yadav and A. K. Yadav, Astrophys. Space Sci. **299**, 31 (2005).
A. Pradhan, V. Rai and S. Otarod, Fizika B, **15**, 23 (2006).
A. Pradhan, K. K. Rai and A. K. Yadav, Braz. J. Phys. **37**, 1084 (2007).
A. Pradhan, J. Math. Phys. **50**, 022501 (2009).
A. Pradhan, Commun. Theor. Phys. **51**, 378 (2009).

[26] R. Casama, C. Melo and B. Pimentel, Astrophys. Space Sci. **305**, 125 (2006).

[27] F. Rahaman, B. Bhui and G. Bag, Astrophys. Space Sci. **295**, 507 (2005).
F. Rahaman, S. Das, N. Begum, M. Hossain, Astrophys. Space Sci. **295**, 507 (2005).

[28] R. Bali and N. K. Chandnani, J. Math. Phys. **49**, 032502 (2008).

[29] S. Kumar and C. P. Singh, Int. Mod. Phys. A **23**, 813 (2008).

[30] J. K. Singh, Astrophys. Space Sci. **314**, 361 (2008).

[31] V. U. M. Rao, T. Vinutha and M. V. Santhi, Astrophys. Space Sci. **314**, 213 (2008).

[32] L. Marder, Proc. Roy. Soc. A **246**, 133 (1958).

[33] O. Heckmann and E. Schucking: *Gravitation: An Introduction to Current Research*, Ed. L. Witten (New York, 1962), p. 438.

[34] K. S. Thorne: Astrophys. J. **148** (1967) 51.

[35] S. R. Roy and S. Prakash: Ind. J. Phys. **52** B (1978) 47.

[36] L. D. Landau and E. M. Lifshitze, *Fluid Mechanics* (Oxford: Pergamon, 1937).

[37] G, F, R. Ellis, *General Relativity and Cosmology*, ed. R. K. Sachs (Clarendon Press, 1973), p. 117.

[38] S. W. Hawking and G, F, R. Ellis, *The Large-scale Structure of Space Time* (Cambridge University Press, Cambridge, 1973), p. 94.

[39] S. Perlmutter et al., Astrophys. J. **483**, 565 (1997).
S. Perlmutter et al., Nature **391**, 51 (1998).
S. Perlmutter et al., Astrophys. J. **517**, 565 (1999).

[40] A. G. Reiss et al., Astron. J. **116**, 1009 (1998).
A. G. Reiss et al., Astron. J. **607**, 665 (2004).

[41] J. Gunn and B. M. Tinsley, Nature, **257**, 454 (1975).

[42] E. J. Wampler and W. L. Burke, in *New Ideas in Astronomy*, Eds. F. Bertola, J. W. Sulentix and B. F. Madora (Cambridge University Press, 1988), p. 317.

[43] G. F. R. Ellis, *Standard and Inflationary Cosmologies*, Preprint SISSA, Trieste, 176/90/A (1990).

[44] D. La and P. J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989).

[45] J. D. Barrow, Phys. Lett. B **235**, 40 (1990).